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JENSEN TYPE INEQUALITIES FOR Q-CLASS FUNCTIONS

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Abstract

Some inequalities of Jensen type for Q-class functions are proved. More precisely, a refinement of the inequality $f((1/P)\sum_{i=1}^n p_i x_i) \le P\sum_{i=1}^n (f(x_i)/p_i)$ is given in which $p_1, \ldots, p_n$ are positive numbers, $P = \sum_{i=1}^n p_i$ and f is a Q-class function. The notion of the jointly Q-class function is introduced and some Jensen type inequalities for these functions are proved. Some Ostrowski and Hermite–Hadamard type inequalities related to Q-class functions are presented as well.

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1. Introduction and preliminaries

Assume that $p_1, \ldots, p_n$ are positive numbers and $P = \sum_{i=1}^n p_i$. If $f : J \to \mathbb{R}$ is a convex function and $x_i \in J$ $(i = 1, \ldots, n)$, then the following so-called Jensen inequality holds:

$$f\Big(\frac{1}{P}\sum_{i=1}^n p_i x_i\Big) \le \frac{1}{P}\sum_{i=1}^n p_i f(x_i). \tag{1.1}$$

A real-valued function f on an interval J is said to be of Q-class or to be a Q-class function if

$$f(\lambda x + (1-\lambda)y) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$

for all $x, y \in J$ and all $\lambda \in (0, 1)$. This notion was first introduced by Godunova and Levin [12]. It is easy to see that this class of functions contains all nonnegative monotone increasing functions and nonnegative convex functions. Also, all such functions are nonnegative [11]. Many other properties of such functions can be found in [8, 9, 17, 18]. Mitrinović and Pečarić [15] proved the following inequality similar to (1.1) for Q-class functions:

$$f\Big(\frac{1}{P}\sum_{i=1}^n p_i x_i\Big) \le P\sum_{i=1}^n \frac{f(x_i)}{p_i}. \tag{1.2}$$

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Dragomir and Pearce [8] obtained some Jensen type inequalities for Q-class functions. Some new properties of Q-class functions are proved in [17, 18].

Recall that a function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be jointly convex if

$$f(\lambda(x, y) + (1 - \lambda)(x', y')) \leq \lambda f(x, y) + (1 - \lambda) f(x', y')$$

for all $x, x' \in [a, b], y, y' \in [c, d]$ and $\lambda \in [0, 1]$. Klaričić Bakula and Pečarić [14] proved some inequalities related to jointly convex functions.

If $f : I \subseteq [0, \infty) \to \mathbb{R}$ is differentiable on I°, the interior of $I, a, b \in I, a < b, f$ is integrable on $[a, b]$ and $|f'(x)| \leq M$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \frac{M}{b-a} \left(\frac{(b-x)^2 + (x-a)^2}{2} \right).$$

This is known as the Ostrowski inequality.

If $f : I \subseteq [0, \infty) \to \mathbb{R}$ is a convex function and $a, b \in I, a < b$, then the following so-called Hermite–Hadamard inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{2}.$$

Dragomir *et al.* [9] obtained a Hermite–Hadamard type inequality for Q-class functions. Recently some authors have established some results concerned with the Ostrowski inequality and the Hermite–Hadamard inequality. In [3], some Ostrowski type inequalities for functions, whose derivatives are s-convex in the second sense, are proved. Dragomir and Agarwal [7] proved some Hermite–Hadamard type inequalities for differentiable convex functions. A series of other results can be found in [1, 2, 4–6, 10, 13].

In Section 2 we give a refinement of (1.2) and state some of its applications. In Section 3 we introduce the notion of a jointly Q-class function and investigate some of its properties. We also present some inequalities related to the Jensen inequality for such functions. In the last section, we prove some Ostrowski and Hermite–Hadamard type inequalities for functions whose first or second derivatives are Q-class.

2. Refinement of the Jensen inequality

In this section we aim to establish some Jensen type inequalities. If $p_1 > 0$ and $p_i < 0$ $(i = 2, \ldots, n)$ are such that $P = \sum_{i=1}^n p_i > 0$, then

$$f\left(\frac{1}{P} \sum_{i=1}^n p_i x_i\right) \geq P \sum_{i=1}^n \frac{f(x_i)}{p_i}.$$

To see this, set $p_1 = P$, $x_1 = (1/P)\sum_{i=1}^n p_i x_i$ and $p_i = -p_i$ $(i = 2, \ldots, n)$ in (1.2). Now let X be a real vector space and $f : X \to \mathbb{R}$ be a Q-class function. Let $p_1, \ldots, p_n$ be positive numbers such that $\sum_{i=1}^n p_i = 1$. Let J be a nonempty proper subset

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of $\{1, \dots, n\}$. Assume that $\bar{J} = \{1, \dots, n\} - J$, $P_J = \sum_{i \in J} p_j$ and $\bar{P}_J = P_{\bar{J}} = 1 - P_J$. For $x_i \in X$ $(i = 1, \dots, n)$ we put $x = (x_1, \dots, x_n)$. Set

$$\Omega(f, J, P, x) := \frac{1}{P_J} f\Big(\frac{1}{P_J} \sum_{i \in J} p_i x_i\Big) + \frac{1}{\bar{P}_J} f\Big(\frac{1}{\bar{P}_J} \sum_{i \in \bar{J}} p_i x_i\Big).$$

We obtain the following refinement of (1.2).

Theorem 2.1. *With the notation as above:*

(i)
$$f\Big(\sum_{i=1}^n p_i x_i\Big) \le \Omega(f, J, P, x) \le \sum_{i=1}^n \frac{f(x_i)}{p_i}; \tag{2.1}$$

(ii)
$$\sum_{i=1}^n \frac{f(x_i)}{p_i} - f\Big(\sum_{i=1}^n p_i x_i\Big) \ge \sum_{i \in J} \frac{f(x_i)}{p_i} - \frac{1}{P_J} f\Big(\frac{1}{P_J} \sum_{i \in J} p_i x_i\Big) \ge 0. \tag{2.2}$$

In particular,

$$\begin{aligned} \sum_{i=1}^n \frac{f(x_i)}{p_i} - f\Big(\sum_{i=1}^n p_i x_i\Big) &\ge \max_{1 \le k, \ell \le n} \Big\{ \frac{f(x_k)}{p_k} + \frac{f(x_\ell)}{p_\ell} - \frac{1}{p_k + p_\ell} f\Big(\frac{p_k x_k + p_\ell x_\ell}{p_k + p_\ell}\Big)\Big\} \\ &\ge 0. \end{aligned} \tag{2.3}$$

Proof. (i) It follows from (1.2) that

$$\begin{aligned} \Omega(f, J, P, x) &= \frac{1}{P_J} f\Big(\frac{1}{P_J} \sum_{i \in J} p_i x_i\Big) + \frac{1}{\bar{P}_J} f\Big(\frac{1}{\bar{P}_J} \sum_{i \in \bar{J}} p_i x_i\Big) \\ &\le \frac{1}{P_J}\Big(P_J \sum_{i \in J} \frac{f(x_i)}{p_i}\Big) + \frac{1}{\bar{P}_J}\Big(\bar{P}_J \sum_{i \in \bar{J}} \frac{f(x_i)}{p_i}\Big) \\ &= \sum_{i=1}^n \frac{f(x_i)}{p_i}, \end{aligned}$$

which is the second inequality of (2.1). To get the first inequality of (2.1), note that $P_J + P_{\bar{J}} = 1$. We have

$$\begin{aligned} \Omega(f, J, P, x) &= \frac{1}{P_J} f\Big(\frac{1}{P_J} \sum_{i \in J} p_i x_i\Big) + \frac{1}{\bar{P}_J} f\Big(\frac{1}{\bar{P}_J} \sum_{i \in \bar{J}} p_i x_i\Big) \\ &\ge f\Big(P_J\Big(\frac{1}{P_J} \sum_{i \in J} p_i x_i\Big) + \bar{P}_J\Big(\frac{1}{\bar{P}_J} \sum_{i \in \bar{J}} p_i x_i\Big)\Big) \quad \text{(since } f \text{ is } Q\text{-class)} \\ &= f\Big(\sum_{i \in J} p_i x_i + \sum_{i \in \bar{J}} p_i x_i\Big) \\ &= f\Big(\sum_{i=1}^n p_i x_i\Big). \end{aligned}$$

(ii) By (1.2), the second inequality of (2.2) holds. Also

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) = f\Big(\sum_{i\in\bar{J}} p_i x_i + P_J\Big(\frac{1}{P_J}\sum_{i\in J} p_i x_i\Big)\Big)$$
$$\leq \sum_{i\in\bar{J}} \frac{f(x_i)}{p_i} + \frac{1}{P_J} f\Big(\frac{1}{P_J}\sum_{i\in J} p_i x_i\Big)$$

since $\sum_{i\in\bar{J}} p_i + P_J = 1$ and f is Q-class. Therefore

$$\sum_{i=1}^{n} \frac{f(x_i)}{p_i} - f\Big(\sum_{i=1}^{n} p_i x_i\Big) \geq \sum_{i=1}^{n} \frac{f(x_i)}{p_i} - \sum_{i\in\bar{J}} \frac{f(x_i)}{p_i} - \frac{1}{P_J} f\Big(\frac{1}{P_J}\sum_{i\in J} p_i x_i\Big)$$
$$= \sum_{i\in J} \frac{f(x_i)}{p_i} - \frac{1}{P_J} f\Big(\frac{1}{P_J}\sum_{i\in J} p_i x_i\Big) \geq 0.$$

Putting $J = \{k, \ell\}$ for some $k, \ell \in \{1, \ldots, n\}$ we obtain (2.3). □

Let f, X and P be as above. If $J = \{k\}$ for some $k \in \{1, \ldots, n\}$ we obtain the following consequence.

Corollary 2.2. *Let f be a Q-class function on a real vector space X and suppose that $x_1, \ldots, x_n \in X$. If $p_1, \ldots, p_n$ are positive numbers such that $\sum_{i=1}^{n} p_i = 1$, then*

$$f\Big(\sum_{i=1}^{n} p_i x_i\Big) \leq \min_{1\leq k\leq n}\left\{\frac{f(x_k)}{p_k} + \frac{f\Big(\frac{\sum_{i\neq k} p_i x_i}{\sum_{i\neq k} p_i}\Big)}{\sum_{i\neq k} p_i}\right\}$$
$$\leq \max_{1\leq k\leq n}\left\{\frac{f(x_k)}{p_k} + \frac{f\Big(\frac{\sum_{i\neq k} p_i x_i}{\sum_{i\neq k} p_i}\Big)}{\sum_{i\neq k} p_i}\right\}$$
$$\leq \sum_{i=1}^{n} \frac{f(x_i)}{p_i}.$$

Now we give two illustrations describing our work.

Example 2.3. Let $p_1, \ldots, p_n$ be positive numbers and $\sum_{i=1}^{n} p_i = 1$. Consider the function $f(t) = \log t$ on $[1, \infty)$. Then f is Q-class. Hence

$$\log\Big(\sum_{i=1}^{n} p_i x_i\Big) \leq \frac{\log(x_k)}{p_k} + \frac{\log\Big(\frac{\sum_{i\neq k} p_i x_i}{\sum_{i\neq k} p_i}\Big)}{\sum_{i\neq k} p_i} \leq \sum_{i=1}^{n} \frac{\log(x_i)}{p_i}.$$

Therefore

$$\log\Big(\sum_{i=1}^{n} p_i x_i\Big) \leq \log\Big({x_k}^{1/p_k}\Big(\frac{\sum_{i\neq k} p_i x_i}{\sum_{i\neq k} p_i}\Big)^{1/(1-p_k)}\Big) \leq \log\Big(\prod_{i=1}^{n} {x_i}^{1/p_i}\Big),$$

whence

$$\Big(\prod_{i=1}^{n} {x_i}^{p_i}\Big) \leq \Big(\sum_{i=1}^{n} p_i x_i\Big) \leq \Big({x_k}^{1/p_k}\Big(\frac{\sum_{i\neq k} p_i x_i}{\sum_{i\neq k} p_i}\Big)^{1/(1-p_k)}\Big) \leq \Big(\prod_{i=1}^{n} {x_i}^{1/p_i}\Big). \tag{2.4}$$

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Note that the first inequality of (2.4) is the arithmetic–geometric means inequality. Applying (2.3), we get

$$\sum_{i=1}^{n}\frac{\log(x_i)}{p_i}-\log\Big(\sum_{i=1}^{n}p_ix_i\Big)\geq\frac{\log(x_k)}{p_k}+\frac{\log(x_\ell)}{p_\ell}-\frac{1}{p_k+p_\ell}\log\Big(\frac{p_kx_k+p_\ell x_\ell}{p_k+p_\ell}\Big)\geq 0.$$

Hence

$$\frac{\prod_{i=1}^{n}x_i^{1/p_i}}{\sum_{i=1}^{n}p_ix_i}\geq\frac{x_k^{1/p_k}x_\ell^{1/p_\ell}}{\left(\frac{p_kx_k+p_\ell x_\ell}{p_k+p_\ell}\right)^{1/(p_k+p_\ell)}}\geq 1.$$

EXAMPLE 2.4. Let $p_1,\dots,p_n$ be positive numbers and $\sum_{i=1}^{n}p_i=1$. For $f(t)=\sqrt{t}$ on $[0,\infty)$,

$$\sum_{i=1}^{n}p_i\sqrt{x_i}\leq\sqrt{\sum_{i=1}^{n}p_ix_i}\leq\frac{\sqrt{x_k}}{p_k}+\frac{\sqrt{\left(\frac{\sum_{i\neq k}p_ix_i}{\sum_{i\neq k}p_i}\right)}}{\sum_{i\neq k}p_i}\leq\sum_{i=1}^{n}\frac{\sqrt{x_i}}{p_i} \tag{2.5}$$

and

$$\sum_{i=1}^{n}\frac{\sqrt{x_i}}{p_i}-\sqrt{\sum_{i=1}^{n}p_ix_i}\geq\frac{\sqrt{x_k}}{p_k}+\frac{\sqrt{x_\ell}}{p_\ell}-\frac{1}{p_k+p_\ell}\sqrt{\frac{p_kx_k+p_\ell x_\ell}{p_k+p_\ell}}\geq 0.$$

Note that if $n=3$, $p_i=\frac{1}{3}$ and $x_i=(i+r-1)^2$ $(i=1,2,3)$ for some $r\in\mathbb{R}^+$, then all inequalities above are strict. For example, with $k=1$ we have from (2.5) that

$$r+1<\sqrt{r^2+2r+\tfrac{5}{3}}<3\Big(r+\tfrac{1}{2}\sqrt{r^2+3r+\tfrac{5}{2}}\Big)<9(r+1).$$

COROLLARY 2.5. *Let f, X and x_i $(i=1,\dots,n)$ be as above. Let $p_i=(1/n)$ $(i=1,\dots,n)$ and $J=\{1,\dots,m\}$ for some $m<n$. Then*

$$f\Big(\sum_{i=1}^{n}\frac{x_i}{n}\Big)\leq\frac{n}{m}f\Big(\frac{1}{m}\sum_{i=1}^{m}x_i\Big)+\frac{n}{n-m}f\Big(\frac{1}{n-m}\sum_{i=m+1}^{n}x_i\Big)\leq n\sum_{i=1}^{n}f(x_i).$$

COROLLARY 2.6. *Let X, f, p_i and J be as in Theorem 2.1. Let $y_i\in X$ and $x_i=y_i-\sum_{j=1}^{n}p_jy_j$ $(i=1,\dots,n)$. Then*

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{p_i}f\Big(y_i-\sum_{j=1}^{n}p_jy_j\Big)\\&\quad\geq\frac{1}{P_J}f\Big(\frac{1}{P_J}\sum_{i\in J}p_i\Big(y_i-\sum_{j=1}^{n}p_jy_j\Big)\Big)+\frac{1}{\bar{P}_J}f\Big(\frac{1}{\bar{P}_J}\sum_{i\in\bar{J}}p_i\Big(y_i-\sum_{j=1}^{n}p_jy_j\Big)\Big)\\&\quad\geq f\Big(\sum_{i\in J}p_i\Big(y_i-\sum_{j=1}^{n}p_jy_j\Big)+\sum_{i\in\bar{J}}p_i\Big(y_i-\sum_{j=1}^{n}p_jy_j\Big)\Big)=f(0).\end{aligned}$$

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The next result is a Levinson type inequality [16] for monotone increasing functions.

Theorem 2.7. *Let $p_1, \ldots, p_n$ be positive numbers, $P = \sum_{i=1}^n p_i$ and f be a nonnegative monotone increasing function on $[a, b]$. Let $x_i, y_i \in [a, b]$ $(i = 1, \ldots, n)$ such that $x_i + y_i = m$ for some fixed real number m and $\max_{1 \le i \le n} y_i \le \min_{1 \le i \le n} x_i$. Then*

$$P \sum_{i=1}^{n} \frac{f(y_i)}{p_i} - f\Big(\frac{1}{P} \sum_{i=1}^{n} p_i y_i\Big) \le P \sum_{i=1}^{n} \frac{f(x_i)}{p_i} - f\Big(\frac{1}{P} \sum_{i=1}^{n} p_i x_i\Big).$$

Proof. We have to show that

$$f\Big(\frac{1}{P} \sum_{i=1}^{n} p_i x_i\Big) - f\Big(\frac{1}{P} \sum_{i=1}^{n} p_i y_i\Big) \le P \sum_{i=1}^{n} \frac{f(x_i)}{p_i} - P \sum_{i=1}^{n} \frac{f(y_i)}{p_i}$$

or equivalently

$$f\Big(\frac{1}{P} \sum_{i=1}^{n} p_i x_i\Big) - f\Big(m - \frac{1}{P} \sum_{i=1}^{n} p_i x_i\Big) \le P \sum_{i=1}^{n} \frac{f(x_i)}{p_i} - P \sum_{i=1}^{n} \frac{f(m - x_i)}{p_i}$$

To this end, we apply (1.2) to the Q-class function $g(x) = f(x) - f(m - x)$. □

3. Jointly Q-class functions

In this section, we define a jointly Q-class function, analogous to joint convexity.

Definition 3.1. A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is said to be jointly Q-class if

$$f(\lambda(x, y) + (1 - \lambda)(x', y')) \le \frac{f(x, y)}{\lambda} + \frac{f(x', y')}{1 - \lambda}$$

for all $x, x' \in [a, b]$, $y, y' \in [c, d]$ and all $\lambda \in (0, 1)$.

Let $f : [0, \infty) \to \mathbb{R}$ be a Q-class function. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be n-tuples of positive numbers. Set

$$\Delta_f(x, y) := \sum_{i=1}^{n} \frac{1}{x_i} f\Big(\frac{y_i}{x_i}\Big).$$

We have the following theorem.

Theorem 3.2. *Let $f : [0, \infty) \to [0, \infty)$ be a function and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be n-tuples of positive numbers. Then:*

(i) *if f is Q-class, then $\Delta_f(x, y)$ is jointly Q-class in x and y;*
(ii) $0 \le (1/X) f(\sum_{i=1}^n y_i / \sum_{i=1}^n x_i) \le \Delta_f(x, y)$, *for every $x, y \in (0, \infty)$, where $X = \sum_{i=1}^n x_i$.*

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Proof. (i) Let $\lambda \in (0,1)$ and let $x = (x_1, \ldots, x_n)$, $x' = (x'_1, \ldots, x'_n)$, $y = (y_1, \ldots, y_n)$ and $y' = (y'_1, \ldots, y'_n)$ be n-tuples of positive numbers. Then

$$\begin{aligned}\Delta_f(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') &= \sum_{i=1}^n \frac{1}{\lambda x_i + (1-\lambda)x'_i} f\Big(\frac{\lambda y_i + (1-\lambda)y'_i}{\lambda x_i + (1-\lambda)x'_i}\Big) \\ &= \sum_{i=1}^n \frac{1}{\lambda x_i + (1-\lambda)x'_i} f\Big(\frac{\frac{\lambda}{x'_i}\frac{y_i}{x_i} + \frac{1-\lambda}{x_i}\frac{y'_i}{x'_i}}{\frac{\lambda}{x'_i} + \frac{1-\lambda}{x_i}}\Big) \\ &= \sum_{i=1}^n \frac{1}{\lambda x_i + (1-\lambda)x'_i} f\Big(\mu\frac{y_i}{x_i} + (1-\mu)\frac{y'_i}{x'_i}\Big),\end{aligned}$$

where

$$\mu = \frac{\frac{\lambda}{x'_i}}{\frac{\lambda}{x'_i} + \frac{1-\lambda}{x_i}} = \frac{\lambda x_i}{\lambda x_i + (1-\lambda)x'_i}.$$

Since f is Q-class,

$$\begin{aligned}&\sum_{i=1}^n \frac{1}{\lambda x_i + (1-\lambda)x'_i} f\Big(\mu\frac{y_i}{x_i} + (1-\mu)\frac{y'_i}{x'_i}\Big) \\ &\quad \le \sum_{i=1}^n \frac{1}{\lambda x_i + (1-\lambda)x'_i}\Big(\frac{1}{\mu} f\Big(\frac{y_i}{x_i}\Big) + \frac{1}{1-\mu} f\Big(\frac{y'_i}{x'_i}\Big)\Big) \\ &\quad = \sum_{i=1}^n \frac{1}{\lambda x_i + (1-\lambda)x'_i}\Big(\frac{\lambda x_i + (1-\lambda)x'_i}{\lambda x_i} f\Big(\frac{y_i}{x_i}\Big) + \frac{\lambda x_i + (1-\lambda)x'_i}{(1-\lambda)x'_i} f\Big(\frac{y'_i}{x'_i}\Big)\Big) \\ &\quad = \frac{1}{\lambda}\sum_{i=1}^n \frac{1}{x_i} f\Big(\frac{x_i}{y_i}\Big) + \frac{1}{1-\lambda}\sum_{i=1}^n \frac{1}{x'_i} f\Big(\frac{x'_i}{y'_i}\Big) = \frac{1}{\lambda}\Delta_f(x,y) + \frac{1}{1-\lambda}\Delta_f(x',y').\end{aligned}$$

(ii) Note that every Q-class function is nonnegative [11], so the first inequality is valid. For the second inequality,

$$f\Big(\frac{\sum_{i=1}^n y_i}{\sum_{i=1}^n x_i}\Big) = f\Big(\frac{\sum_{i=1}^n x_i \frac{y_i}{x_i}}{\sum_{i=1}^n x_i}\Big) \le \sum_{i=1}^n \frac{\sum_{k=1}^n x_k}{x_i} f\Big(\frac{y_i}{x_i}\Big) = X\Delta_f(x,y).$$

This concludes the proof. □

Corollary 3.3. *Let f be a Q-class function. Let x_{ij}, y_{ij}, p_j $(1 \le i \le m,\ 1 \le j \le n)$ be positive numbers such that $\sum_{j=1}^n p_j = 1$. Then*

$$\sum_{i=1}^m \frac{1}{\sum_{j=1}^n p_j x_{ij}} f\Big(\frac{\sum_{j=1}^n p_j y_{ij}}{\sum_{j=1}^n p_j x_{ij}}\Big)$$

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$$\leq P_J \sum_{i=1}^{m} \frac{1}{\sum_{j\in J} p_j x_{ij}} f\Big(\frac{\sum_{j\in J} p_j y_{ij}}{\sum_{j\in J} p_j x_{ij}}\Big) + \bar{P}_J \sum_{i=1}^{m} \frac{1}{\sum_{j\in \bar{J}} p_j x_{ij}} f\Big(\frac{\sum_{j\in \bar{J}} p_j y_{ij}}{\sum_{j\in \bar{J}} p_j x_{ij}}\Big)$$

$$\leq \sum_{j=1}^{n} \sum_{i=1}^{m} \frac{1}{x_{ij} p_j} f\Big(\frac{y_{ij}}{x_{ij}}\Big).$$

Proof. This follows from Theorems 2.1 and 3.2. □

The next theorem gives a lower bound for the function Δ_f.

Theorem 3.4. *Let f be a Q-class function. Let $x = (x_1, \dots, x_n)$ and $p = (p_1, \dots, p_n)$ be n-tuples of positive numbers such that $\sum_{i=1}^{n} p_i = 1$. Let J be a nonempty proper subset of $\{1, \dots, n\}$ and $\bar{J} = \{1, \dots, n\} - J$. Put $X_J = \sum_{i\in J} x_i$ and $P_J = \sum_{i\in J} p_i$. Then*

$$0 \leq f\Big(\sum_{i=1}^{n} x_i\Big) \leq \frac{1}{P_J} f\Big(\frac{X_J}{P_J}\Big) + \frac{1}{1-P_J} f\Big(\frac{\sum_{i\in \bar{J}} x_i}{1-P_J}\Big) \leq \Delta_f(p, x).$$

Proof. The left inequality follows from Theorem 2.1. The proof of the right inequality is similar to that of Theorem 3.2 and we omit it. □

Example 3.5. Let $f(t) = \log t$, and $x_1, \dots, x_n$ and $p_1, \dots, p_n$ be positive numbers such that $x_i \geq p_i$ and $\sum_{i=1}^{n} p_i = 1$. Let $J = \{k\}$ for some $k \in \{1, \dots, n\}$. It follows from Theorem 3.4 that

$$0 \leq \log\Big(\sum_{i=1}^{n} x_i\Big) \leq \frac{1}{p_k} \log\Big(\frac{x_k}{p_k}\Big) + \frac{1}{1-p_k} \log\Big(\frac{\sum_{i\neq k} x_i}{1-p_k}\Big) \leq \sum_{i=1}^{n} \frac{1}{p_i} \log\Big(\frac{x_i}{p_i}\Big).$$

Therefore, it follows from the monotonicity of the function e^t that

$$1 \leq \sum_{i=1}^{n} x_i \leq \sqrt[p_k]{\frac{x_k}{p_k}} \sqrt[1-p_k]{\frac{\sum_{i\neq k} x_i}{1-p_k}} \leq \prod_{i=1}^{n} \sqrt[p_i]{\frac{x_i}{p_i}}.$$

The next theorem gives a Jensen type inequality for jointly Q-class functions.

Theorem 3.6. *Let $f : I \times J \to \mathbb{R}$ be a jointly Q-class function. Let $x_1, \dots, x_n \in I$ and $y_1, \dots, y_n \in J$. Let $p_i, q_i > 0$ $(i = 1, \dots, n)$, $P = \sum_{i=1}^{n} p_i$ and $Q = \sum_{i=1}^{n} q_i$. Put*

$$\bar{x} = \frac{1}{P} \sum_{i=1}^{n} p_i x_i \quad \text{and} \quad \bar{y} = \frac{1}{Q} \sum_{i=1}^{n} q_i y_i.$$

Then

$$f(\bar{x}, \bar{y}) \leq \frac{1}{2}\Big(P \sum_{i=1}^{n} \frac{f(x_i, \bar{y})}{p_i} + Q \sum_{j=1}^{n} \frac{f(\bar{x}, y_j)}{q_j}\Big) \leq PQ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f(x_i, y_j)}{p_i q_j}. \tag{3.1}$$

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PROOF. Since f is jointly Q-class, it is clearly Q-class in each of its variables, from which we get

$$f(x_i, \bar{y}) \le Q \sum_{j=1}^{n} \frac{f(x_i, y_j)}{q_j} \quad (i = 1, \ldots, n), \tag{3.2}$$

$$f(\bar{x}, y_j) \le P \sum_{i=1}^{n} \frac{f(x_i, y_j)}{p_i} \quad (j = 1, \ldots, n). \tag{3.3}$$

Multiplying both sides of (3.2) and (3.3) by $1/p_i$ and $1/q_j$, respectively, and summing over i and j, we obtain

$$\sum_{i=1}^{n} \frac{f(x_i, \bar{y})}{p_i} \le Q \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f(x_i, y_j)}{p_i q_j}, \tag{3.4}$$

$$\sum_{j=1}^{n} \frac{f(\bar{x}, y_j)}{q_j} \le P \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f(x_i, y_j)}{p_i q_j}. \tag{3.5}$$

Multiplying (3.4) and (3.5), respectively, by P and Q and adding both inequalities, we get the second inequality in (3.1). To get the first one, note that

$$f(\bar{x}, \bar{y}) = f\Big(\frac{1}{P} \sum_{i=1}^{n} p_i x_i, \bar{y}\Big) \le P \sum_{i=1}^{n} \frac{f(x_i, \bar{y})}{p_i},$$

$$f(\bar{x}, \bar{y}) = f\Big(\bar{x}, \frac{1}{Q} \sum_{j=1}^{n} q_j y_j\Big) \le Q \sum_{i=1}^{n} \frac{f(\bar{x}, y_j)}{q_j}.$$

These inequalities immediately yield the first inequality of (3.1). □

Let $f, P, Q, p_i, q_j, x_i, y_j, \bar{x}, \bar{y}$ be as in Theorem 3.6. Define a function $\Gamma_f : [0, 1] \times [0, 1] \to \mathbb{R}$ by

$$\Gamma_f(t, s) := PQ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f(tx_i + (1-t)\bar{x}, sy_j + (1-s)\bar{y})}{p_i q_j}.$$

THEOREM 3.7. *With notation as above:*

(i) Γ_f *is jointly Q-class;*

(ii) $0 \le f(\bar{x}, \bar{y}) \le \Gamma_f(t, s)$ *for each* $t, s \in [0, 1]$.

PROOF. (i) Let $\lambda \in (0, 1)$ and $t_1, t_2, s_1, s_2 \in [0, 1]$. Then

$$\begin{aligned}
&\Gamma_f(\lambda t_1 + (1-\lambda)t_2, \lambda s_1 + (1-\lambda)s_2)\\
&\quad = PQ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{p_i q_j} f((\lambda t_1 + (1-\lambda)t_2)x_i + (1 - (\lambda t_1 + (1-\lambda)t_2))\bar{x},\\
&\qquad (\lambda s_1 + (1-\lambda)s_2)y_j + (1 - (\lambda s_1 + (1-\lambda)s_2))\bar{y})\\
&\quad = PQ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{p_i q_j} f(\lambda(t_1 x_i + (1-t_1)\bar{x}, s_1 y_j + (1-s_1)\bar{y})
\end{aligned}$$

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$$
\begin{aligned}
&+ (1-\lambda)(t_2 x_i + (1-t_2)\bar{x}, s_2 y_j + (1-s_2)\bar{y})) \\
&\le \frac{PQ}{\lambda} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f((t_1 x_i + (1-t_1)\bar{x}, s_1 y_j + (1-s_1)\bar{y}))}{p_i q_j} \\
&\quad + \frac{PQ}{1-\lambda} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f((t_2 x_i + (1-t_2)\bar{x}, s_2 y_j + (1-s_2)\bar{y}))}{p_i q_j} \\
&= \frac{1}{\lambda}\Gamma_f(t_1, s_1) + \frac{1}{1-\lambda}\Gamma_f(t_2, s_2).
\end{aligned}
$$

(ii) Using the Jensen inequality for the Q-class function f, we get

$$
\begin{aligned}
\Gamma_f(t,s) &= PQ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{f(tx_i + (1-t)\bar{x}, sy_j + (1-s)\bar{y})}{p_i q_j} \\
&= P \sum_{i=1}^{n} \frac{Q}{p_i} \sum_{j=1}^{n} \frac{f(tx_i + (1-t)\bar{x}, sy_j + (1-s)\bar{y})}{q_j} \\
&\ge P \sum_{i=1}^{n} \frac{1}{p_i} f\Big(tx_i + (1-t)\bar{x}, \frac{1}{Q} \sum_{j=1}^{n} q_j (sy_j + (1-s)\bar{y})\Big) \\
&\ge f\Big(\frac{t}{P} \sum_{i=1}^{n} p_i x_i + (1-t)\bar{x}, \frac{s}{Q} \sum_{j=1}^{n} q_j y_j + (1-s)\bar{y}\Big) \\
&= f(\bar{x}, \bar{y}).
\end{aligned}
$$

This concludes the proof. □

4. Ostrowski and Hermite–Hadamard type inequalities

We now turn our attention to the Ostrowski and Hermite–Hadamard inequalities. First, we state some Hermite–Hadamard inequalities related to Q-class functions. We need the following lemma for our next result.

Lemma 4.1 [2]. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differential mapping on I°, $a, b \in I$ with $a < b$ and f'' integrable on $[a, b]$. Then*

$$
\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx = \frac{(b-a)^2}{2} \int_0^1 t(1-t) f''(ta + (1-t)b)\,dt.
$$

Theorem 4.2. *Let f be a function as in Lemma 4.1. If $|f''|$ is Q-class, then*

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\,dx \right| \le \frac{(b-a)^2}{4} (|f''(a)| + |f''(b)|).
$$

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PROOF. It follows from Lemma 4.1 that

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f''(ta+(1-t)b)|\,dt\\ &\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)\Big(\frac{f''(a)}{t}+\frac{f''(b)}{1-t}\Big)\,dt\\ &= \frac{(b-a)^2}{2}\Big(\int_0^1 (1-t)|f''(a)|\,dt+\int_0^1 t|f''(b)|\,dt\Big)\\ &= \frac{(b-a)^2}{4}(|f''(a)|+|f''(b)|).\end{aligned}$$

This concludes the proof. □

THEOREM 4.3. *Let f be a function as in Lemma 4.1. If $|f''|^q$ is Q-class for some $q>1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le (b-a)^2\Big(\frac{1}{2}\Big)^{1+(1/q)}\Big(\frac{1}{6}\Big)^{1-(1/q)}(|f''(a)|^q+|f''(b)|^q)^{1/q}.$$

PROOF. It follows from Lemma 4.1 that

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\ &\quad\le \frac{(b-a)^2}{2}\int_0^1 t(1-t)|f''(ta+(1-t)b)|\,dt\\ &\quad\le \frac{(b-a)^2}{2}\Big(\int_0^1 t(1-t)\,dt\Big)^{1-(1/q)}\Big(\int_0^1 t(1-t)|f''(ta+(1-t)b)|^q\,dt\Big)^{1/q}\\ &\quad\le \frac{(b-a)^2}{2}\Big(\frac{1}{6}\Big)^{1-(1/q)}\Big(|f''(a)|^q\int_0^1 (1-t)\,dt+|f''(b)|^q\int_0^1 t\,dt\Big)^{1/q}\\ &\quad\le \frac{(b-a)^2}{2}\Big(\frac{1}{6}\Big)^{1-(1/q)}\Big(\frac{|f''(a)|^q+|f''(b)|^q}{2}\Big)^{1/q}\\ &\quad= (b-a)^2\Big(\frac{1}{2}\Big)^{1+(1/q)}\Big(\frac{1}{6}\Big)^{1-(1/q)}(|f''(a)|^q+|f''(b)|^q)^{1/q}.\end{aligned}$$

This concludes the proof. □

To prove the next two theorems, which give some Ostrowski type inequalities, we need the following lemma.

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Lemma 4.4 [1]. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differential mapping on I°, $a, b \in I$ with $a < b$ and f' integrable on $[a, b]$. Then*

$$f(x) - \frac{1}{b-a}\int_a^b f(u)\,du = (b-a)\Big(\int_0^m t f'(ta+(1-t)b)\,dt + \int_m^1 (1-t)f'(ta+(1-t)b)\,dt\Big),$$

where $m = (b-x)/(b-a)$.

Theorem 4.5. *Let f be a function as in Lemma 4.4. If $|f'|$ is Q-class, then*

$$\Big|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\Big| \le (b-a)\Big(|f'(a)|\Big(\log\Big|\frac{b-a}{b-x}\Big| + \frac{b+a-2x}{b-a}\Big) + |f'(b)|\Big(\log\Big|\frac{b-a}{x-a}\Big| + \frac{2x-a-b}{b-a}\Big)\Big).$$

Proof. By Lemma 4.4 we have

$$\begin{aligned}
\Big|f(x) - \frac{1}{b-a}\int_a^b f(u)du\Big| &\le (b-a)\Big(\int_0^m t|f'(ta+(1-t)b)|\,dt + \int_m^1 (1-t)|f'(ta+(1-t)b)|\,dt\Big)\\
&\le (b-a)\Big(\int_0^m t\Big(\frac{|f'(a)|}{t} + \frac{|f'(b)|}{1-t}\Big)dt + \int_m^1 (1-t)\Big(\frac{|f'(a)|}{t} + \frac{|f'(b)|}{1-t}\Big)dt\Big)\\
&= (b-a)\Big(|f'(a)|\int_0^m dt + |f'(b)|\int_0^m \frac{t}{1-t}\,dt\\
&\quad + |f'(a)|\int_m^1 \frac{1-t}{t}\,dt + |f'(b)|\int_m^1 dt\Big)\\
&= (b-a)\Big(|f'(a)|\Big(\log\Big|\frac{b-a}{b-x}\Big| + \frac{b+a-2x}{b-a}\Big) + |f'(b)|\Big(\log\Big|\frac{b-a}{x-a}\Big| + \frac{2x-a-b}{b-a}\Big)\Big).
\end{aligned}$$

This concludes the proof. □

Corollary 4.6. *Let f and f' be as in Theorem 4.5.*

$$\Big|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(u)\,du\Big| \le (b-a)\log 2(|f'(a)| + |f'(b)|).$$

Proof. Put $x = (a+b)/2$ in Theorem 4.5. □

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Theorem 4.7. *Let f be a function as in Lemma 4.4. If $|f'|^q$ is Q-class for some $q \geq 1$, then*

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|\\ &\quad\leq (b-a)\left(\frac{1}{2}\left(\frac{b-x}{b-a}\right)^{2(1-(1/q))}\left(|f'(a)|^q\frac{b-x}{b-a} + |f'(b)|^q\left(\log\left|\frac{b-a}{x-a}\right| - \frac{b-x}{b-a}\right)\right)^{1/q}\right)\\ &\qquad + (b-a)\left(\frac{2x-a-b}{2b-2a} + \frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\right)^{1-(1/q)}\\ &\qquad \times\left(|f'(b)|^q\frac{x-a}{b-a} + |f'(a)|^q\left(\log\left|\frac{b-a}{b-x}\right| - \frac{x-a}{b-a}\right)\right)^{1/q}.\end{aligned}$$

Proof. It follows from Lemma 4.4 that

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right|\\ &\quad\leq (b-a)\left(\int_0^m t|f'(ta+(1-t)b)|\,dt + \int_m^1 (1-t)|f'(ta+(1-t)b)|\,dt\right)\\ &\quad\leq (b-a)\left(\left(\int_0^m t\,dt\right)^{1-(1/q)}\left(\int_0^m t|f'(ta+(1-t)b)|^q\,dt\right)^{1/q}\right.\\ &\qquad\left. + \left(\int_m^1 (1-t)\,dt\right)^{1-(1/q)}\left(\int_m^1 (1-t)|f'(ta+(1-t)b)|^q\,dt\right)^{1/q}\right),\end{aligned}$$

by the power mean inequality. Since $|f'|^q$ is Q-class,

$$\begin{aligned}\int_0^m t|f'(ta+(1-t)b)|^q\,dt &\leq \int_0^m t\left(\frac{|f'(a)|^q}{t} + \frac{|f'(b)|^q}{1-t}\right)dt\\ &= |f'(a)|^q\int_0^m dt + |f'(b)|^q\int_0^m \frac{t}{1-t}\,dt\\ &= |f'(a)|^q\frac{b-x}{b-a} + |f'(b)|^q\left(\log\left|\frac{b-a}{x-a}\right| - \frac{b-x}{b-a}\right)\end{aligned}$$

and

$$\begin{aligned}\int_m^1 (1-t)|f'(ta+(1-t)b)|^q\,dt &\leq \int_m^1 (1-t)\left(\frac{|f'(a)|^q}{t} + \frac{|f'(b)|^q}{1-t}\right)dt\\ &= |f'(a)|^q\int_m^1 \frac{1-t}{t}\,dt + |f'(b)|^q\int_m^1 dt\\ &= |f'(b)|^q\frac{x-a}{b-a} + |f'(a)|^q\left(\log\left|\frac{b-a}{b-x}\right| - \frac{x-a}{b-a}\right).\end{aligned}$$

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Hence

$$\begin{aligned}
&\left|f(x) - \frac{1}{b-a}\int_a^b f(u)\,du\right| \\
&\quad\le (b-a)\left(\frac{1}{2}\left(\frac{b-x}{b-a}\right)^{2(1-(1/q))}\left(|f'(a)|^q\frac{b-x}{b-a} + |f'(b)|^q\left(\log\left|\frac{b-a}{x-a}\right| - \frac{b-x}{b-a}\right)\right)^{1/q}\right) \\
&\qquad + (b-a)\left(\frac{2x-a-b}{2b-2a} + \frac{1}{2}\left(\frac{b-x}{b-a}\right)^2\right)^{1-(1/q)} \\
&\qquad \times\left(|f'(b)|^q\frac{x-a}{b-a} + |f'(a)|^q\left(\log\left|\frac{b-a}{b-x}\right| - \frac{x-a}{b-a}\right)\right)^{1/q}.
\end{aligned}$$

This concludes the proof. □

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