PROPERTIES AND APPLICATIONS OF A CERTAIN OPERATOR ASSOCIATED WITH THE KONTOROVICH–LEBEDEV TRANSFORM†

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(Received 21 August, 1974)

1. Introduction. The integral

$$Q(\tau, m) = \int_0^\infty K_{i\tau}(\beta_1 r) K_{im}(\beta_2 r) \frac{dr}{r} \tag{1.1}$$

arises in problems of scalar wave propagation in welded elastic wedges. In (1.1), $K_{im}(\beta_1 r)$ is the modified Bessel function of the second kind and m, τ are real. It is shown that $Q(\tau, m)$ is a generalized function that includes a complex shift operator. We shall investigate the properties of this operator and establish a new integral transform based on the kernel $Q(\tau, m)$.

A summation formula based on $Q(\tau, m)$ is derived, which facilitates the evaluation of sums involving the Jacobi polynomials. Finally, $Q(\tau, m)$ is used to obtain a new multiplication theorem for the MacDonald functions.

2. Representation of delta functions via the Kontorovich–Lebedev (K–L) transform. The *K–L* transform of a function $f(r)$, $0 < r < \infty$, is given by the relation

$$F(\tau) = \int_0^\infty f(r) K_{i\tau}(\beta r) \frac{dr}{r}, \tag{2.1}$$

where τ is real and β is a complex constant, [**1**, **4**]. If $f(r)$ is such that $\frac{f(r)}{r}$ is continuously differentiable and both $rf(r)$ and $r \frac{d}{dr}\left\{\frac{f(r)}{r}\right\}$ are absolutely integrable over the positive real axis, the inversion formula assumes the form [**5**],

$$f(r) = \frac{2}{\pi^2} \int_0^\infty F(\tau) K_{i\tau}(\beta r) \tau \operatorname{sh} \pi\tau \, d\tau. \tag{2.2}$$

This pair of reciprocal formulas can be combined to yield the integral theorem

$$f(r) = \frac{2}{\pi^2} \int_0^\infty \tau \operatorname{sh} \pi\tau \, K_{i\tau}(\beta r) d\tau \int_0^\infty f(\xi) K_{i\tau}(\beta \xi) \frac{d\xi}{\xi}. \tag{2.3}$$

Writing (2.3) in the form $f(r) = \int_0^\infty f(\xi) \delta^+(r - \xi) d\xi$, where $\delta^+(x) = 2H(x)\delta(x)$ is the unit

† This research has been sponsored by the Cambridge Laboratories (AFCRL), United States Air Force under grant AFOSR–73–2528A.

impulse function ($\delta(x)$ is the usual Dirac function and $H(x)$ is the Heaviside unit step function), we obtain the representation

$$\delta^{+}(r-r_0)=\frac{2}{\pi^2 r_0}\int_0^\infty \tau \operatorname{sh}\pi\tau\, K_{i\tau}(\beta r)K_{i\tau}(\beta r_0)d\tau. \tag{2.4}$$

Similarly, from (2.1) and (2.2),

$$F(\tau)=\frac{2}{\pi^2}\int_0^\infty K_{i\tau}(\beta r)\frac{dr}{r}\int_0^\infty F(m)K_{im}(\beta r)m \operatorname{sh}\tau m\, dm \tag{2.5}$$

and therefore

$$\delta(\tau+m)+\delta(\tau-m)=\frac{2}{\pi^2}\tau \operatorname{sh}\pi\tau\int_0^\infty K_{i\tau}(\beta r)K_{im}(\beta r)\frac{dr}{r}, \quad \beta>0. \tag{2.6}$$

Furthermore, since [5],

$$K_{i\tau}(y)=\int_0^\infty e^{-y \operatorname{ch}\xi}\cos(\tau\xi)d\xi \tag{2.7}$$

and

$$\pi\delta(\tau-m)=\int_0^\infty \cos\xi(\tau-m)d\xi, \tag{2.8}$$

it follows that

$$\pi\delta(\tau-m)=K_{i(\tau-m)}(0). \tag{2.9}$$

Consequently, (2.6) can be recast in the form

$$Q_0(\tau, m)=\int_0^\infty K_{i\tau}(\beta r)K_{im}(\beta r)\frac{dr}{r}=\frac{\pi}{2}\frac{K_{i(\tau+m)}(0)+K_{i(\tau-m)}(0)}{\tau \operatorname{sh}\pi\tau}. \tag{2.10}$$

The integral $\int_{-\infty}^{\infty} K_{i\tau}(x)d\tau=\pi e^{-x}$ verifies that the normalization constant in (2.9) is correct. We may generalize the concept of the Dirac delta function to include complex arguments in the following sense: consider the identity [3, p. 67],

$$\begin{aligned}\frac{\partial K_{i(m-\tau)}(y)}{\partial y} &= -\tfrac{1}{2}[K_{i(m-\tau)+1}(y)+K_{i(m+\tau)-1}(y)]\\ &= -\operatorname{Re}\{K_{i[m-(\tau-i)]}(y)\} = -\int_0^\infty e^{-y\operatorname{ch}\xi}\operatorname{ch}\xi\cos\xi(m-\tau)d\xi.\end{aligned}$$

If we interpret

$$K_{i(m-\tau)+1}(0)=\pi\delta[m-(\tau\pm i)]$$

then, for any entire function $f(m)$

$$\int_{-\infty}^{\infty}\left[\frac{\partial K_{i(m-\tau)}(y)}{\partial y}\right]_{y=0} f(m)dm=-\pi \operatorname{Re}[f(\tau+i)]. \tag{2.11}$$

The same result holds for

$$\int_{-\infty}^{\infty}\left[\frac{K_{i(m-\tau)}(y)}{y}\right]_{y=0} f(m)dm.$$

For example

$$\int_{-\infty}^{\infty} K_{i(m-\tau)}(y)\,\mathrm{ch}\,\eta m\,dm = \int_{-\infty}^{\infty} K_{ix}(y)\,\mathrm{ch}\,\eta(\tau+x)dx$$

$$= \mathrm{ch}\,\eta\tau\int_{-\infty}^{\infty} K_{ix}(y)\,\mathrm{ch}\,\eta x\,dx = \pi\,\mathrm{ch}\,\eta\tau\, e^{-y\cos\eta}$$

$$\eta \leqq \frac{\pi}{2} \quad [4, \text{p. } 8].$$

In the limit $y \to 0$

$$\left[\frac{d}{dy}\int_{-\infty}^{\infty} K_{i(m-\tau)}(y)\,\mathrm{ch}\,\eta m\,dm\right]_{y=0} = -\pi\cos\eta\,\mathrm{ch}\,\eta\tau = -\pi\,\mathrm{Re}[\mathrm{ch}\,\eta(\tau+i)].$$

Clearly (2.11) can be extended to algebraic and differential operators of higher order.

3. An integral of Titchmarsh. We shall next use a result of Titchmarsh [7]. Consider the Hankel transform pair

$$f(x) = \int_0^{\infty} J_\nu(xt)\sqrt{xt}\,F(t)dt,\quad F(x) = \int_0^{\infty} J_\nu(xt)\sqrt{xt}\,f(t)dt \quad (\nu \geqq -1/2) \tag{3.1}$$

and let $g(x), G(x)$ be similarly related. Assuming that f^2 and g^2 are integrable over $(0, \infty)$, we invoke Parseval's formula

$$\int_0^{\infty} F(x)G(x)dx = \int_0^{\infty} f(x)g(x)dx \tag{3.2}$$

for the particular case

$$f(x) = x^{\lambda+\nu+\frac{1}{2}}K_\lambda(ax),\quad g(x) = x^{\mu+\nu+\frac{1}{2}}K_\mu(bx). \tag{3.3}$$

The inverse Hankel transforms of these functions are

$$F(x) = 2^{\lambda+\nu}a^\lambda x^{\nu+\frac{1}{2}}\Gamma(\lambda+\nu+1)(a^2+x^2)^{-\lambda-\nu-1}$$

$$G(x) = 2^{\mu+\nu}b^\mu x^{\nu+\frac{1}{2}}\Gamma(\mu+\nu+1)(b^2+x^2)^{-\mu-\nu-1}. \tag{3.4}$$

Thus the application of (3.2) yields directly

$$\int_0^{\infty} x^{\lambda+\mu+2\nu+1}K_\lambda(ax)K_\mu(bx)dx$$

$$= 2^{\lambda+\mu+2\nu}a^\lambda b^\mu\Gamma(\lambda+\nu+1)\Gamma(\mu+\nu+1)\int_0^{\infty}\frac{x^{2\nu+1}dx}{(a^2+x^2)^{\lambda+\nu+1}(b^2+x^2)^{\mu+\nu+1}}. \tag{3.5}$$

The integral on the right is evaluated by putting $x = b\tan\theta$ and expanding in powers of $\varepsilon = 1 - a^2/b^2 \geqq 0$, for $b \geqq a$. Hence the Titchmarsh integral

$$Q(\lambda, \mu; \rho) = \int_0^\infty K_\lambda(ax)K_\mu(bx)x^{\rho-1}dx$$
$$= \frac{2^{\rho-3}a^\lambda}{\Gamma(\rho)b^{\lambda+\rho}}\Gamma\left(\frac{\rho+\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda+\mu}{2}\right)\Gamma\left(\frac{\rho-\lambda-\mu}{2}\right)\Gamma\left(\frac{\rho+\lambda+\mu}{2}\right)$$
$$\times {}_2F_1\left(\frac{\rho+\lambda-\mu}{2}, \frac{\rho+\lambda+\mu}{2}; \rho; \varepsilon\right), \tag{3.6}$$

where $\rho = 2\nu+\lambda+\mu+2$ and ${}_2F_1$ is the hypergeometric function. It can easily be demonstrated that (3.6) reduces to (2.10) if $a = b = \beta > 0$, $\lambda = i\tau$, $\mu = im$ and $\rho \to 0$. Indeed

$$Q_0(\tau, m) = \int_0^\infty K_{i\tau}(\beta r)K_{im}(\beta r)\frac{dr}{r}$$
$$= \lim_{\rho\to 0}\frac{\Gamma\left(\frac{\rho+i\{\tau-m\}}{2}\right)\Gamma\left(\frac{\rho-i\{\tau-m\}}{2}\right)\Gamma\left(\frac{\rho+i\{\tau+m\}}{2}\right)\Gamma\left(\frac{\rho-i\{\tau+m\}}{2}\right)}{8\Gamma(\rho)}. \tag{3.7}$$

If $\tau = m$ or $\tau = -m$, the expression on the right of (3.7) varies like $\Gamma(\rho)$ and therefore tends to infinity. If $|\tau| \neq |m|$, this expression varies like $\frac{1}{\Gamma(\rho)}$ which tends to zero. Using the Mellin–Barnes integral [9],

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)ds = \frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)} \tag{3.8}$$

with

$$\alpha = \gamma = \tfrac{1}{2}(\rho+i\tau), \quad \beta = \delta = \tfrac{1}{2}(\rho-i\tau), \quad s = \tfrac{1}{2}im$$

and the relations

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin\pi z}, \quad \lim_{z\to 0}\frac{\Gamma(z)}{\Gamma(2z)} = 2, \tag{3.9}$$

we obtain, from (3.7), (3.8) and (3.9),

$$\int_{-\infty}^{\infty} Q_0(\tau, m)d\tau = \frac{\pi^2}{\tau\,\mathrm{sh}\,\pi\tau} \tag{3.10}$$

in accordance with (2.6).

4. Properties of $Q(\tau, m)$. The integral (1.1) is an even function of both τ and m. It is a special case of the Titchmarsh integral for the parameters

$$\beta_1 = a < b = \beta_2, \quad \lambda = i\tau, \quad \mu = im, \quad \varepsilon = 1-(\beta_1/\beta_2)^2 \geqq 0 \tag{4.1}$$

at the limit $\rho \to 0$. From the explicit expression (3.6), we deduce that $Q(\tau, m)$ diverges at $\tau = \pm m$. It can be shown that in the neighbourhood of $\rho = 0$ it behaves like

$$\frac{\pi}{4\tau \operatorname{sh} \pi\tau} \Gamma(\rho/2) \cos\{\tau \ln(\beta_1/\beta_2)\}.$$

Certain results from the theory of the hypergeometric function ${}_2F_1$ can be applied in order to recast (3.6) in other convenient forms. Invoking the relation [4, p. 38]

$$\lim_{\rho \to 0} \frac{1}{\Gamma(\rho)} {}_2F_1(A-1, B-1; \rho; \varepsilon) = (A-1)(B-1)\varepsilon\, {}_2F_1(A, B; 2; \varepsilon), \tag{4.2}$$

where

$$A = 1 + \frac{i}{2}(\tau + m), \quad B = 1 + \frac{i}{2}(\tau - m), \tag{4.3}$$

and using $\Gamma(ix)\Gamma(-ix) = \dfrac{\pi}{x \operatorname{sh} \pi x}$, (3.6) yields, for $|\tau| \neq |m|$,

$$Q(\tau, m) = \varepsilon \frac{\pi^2}{4} e^{ik\tau} \frac{{}_2F_1(A, B; 2; \varepsilon)}{\operatorname{ch} \pi m - \operatorname{ch} \pi\tau}, \quad k = \ln(\beta_1/\beta_2). \tag{4.4}$$

The fact that $Q(\tau, m)$ is real and even in τ is not obvious from (4.4). Using however the transformation of [3, p. 47],

$$\begin{aligned} {}_2F_1(A, B; 2; \varepsilon) = {} & \Gamma(-i\tau) \frac{{}_2F_1(A, B; 1+i\tau; 1-\varepsilon)}{\Gamma(\bar{A})\Gamma(\bar{B})} \\ & + e^{-2ik\tau}\Gamma(i\tau) \frac{{}_2F_1(\bar{A}, \bar{B}; 1-i\tau; 1-\varepsilon)}{\Gamma(A)\Gamma(B)} \quad (\tau \neq iN, \quad N = 0, 1, 2, \ldots), \end{aligned} \tag{4.5}$$

where $\bar{A}$ denotes the complex conjugate of A, etc. Thus

$$Q(\tau, m) = \varepsilon \frac{\pi^2}{4} \frac{L + \bar{L}}{\operatorname{ch} \pi m - \operatorname{ch} \pi\tau}$$

$$L = e^{ik\tau} \frac{\Gamma(-i\tau)}{\Gamma(\bar{A})\Gamma(\bar{B})} {}_2F_1(A, B; 1+i\tau; 1-\varepsilon) \tag{4.6}$$

and clearly

$$Q(\tau, m) = Q(-\tau, -m) = Q(\tau, -m) = Q(-\tau, m).$$

In order to examine the behaviour of $Q(\tau, m)$ at $m = +\tau$ and $m = -\tau$ on the real m axis for all values of ε, we shall make use of the multiplication theorem of the modified Bessel functions [3, p. 130]

$$K_{i\tau}(az) = a^{i\tau} \sum_{n=0}^{\infty} \frac{1}{n!} \{z^n K_{i\tau+n}(z)[\tfrac{1}{2}(1-a^2)]^n\}. \tag{4.7}$$

Choosing $a = \beta_1/\beta_2$, $z = \beta_2 r$ we have

$$K_{i\tau}(\beta_1 r) = e^{ik\tau}K_{i\tau}(\beta_2 r) + e^{ik\tau}\sum_{n=1}^{\infty}\frac{1}{n!}(\varepsilon\beta_2 r/2)^n K_{i\tau+n}(\beta_2 r). \tag{4.8}$$

But since the left-hand side of (4.8) is real and even in τ,

$$K_{i\tau}(\beta_1 r) = \cos k\tau K_{i\tau}(\beta_2 r) + \mathrm{Re}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{1}{n!}\left[\frac{\varepsilon\beta_2 r}{2}\right]^n K_{i\tau+n}(\beta_2 r)\right\}. \tag{4.9}$$

Multiplying both sides of (4.9) by $r^{-1}K_{im}(\beta_2 r)$, integrating with respect to r over $(0, \infty)$ and using (3.6) with $\rho = n$, $a = b = \beta_2$, $\lambda = im$, $\mu = i\tau + n$, yields

$$Q(t, m) = \frac{\pi^2\cos k\tau}{2\tau\,\mathrm{sh}\,\pi\tau}[\delta(\tau+m)+\delta(\tau-m)] + \varepsilon\frac{\pi^2}{4}e^{ik\tau}\frac{{}_2F_1(A, B; 2; \varepsilon)}{\mathrm{ch}\,\pi m - \mathrm{ch}\,\pi\tau}, \tag{4.10}$$

where $e^{ik\tau}{}_2F_1(A, B; 2; \varepsilon)$ is real and even in τ and m according to (4.5). Thus $Q(\tau, m)$ is a *generalized function* and the representation (4.10) is valid for all real values of τ and m.

To expose further the nature of $Q(\tau, m)$, we shall consider integrals of the form

$$U(\tau) = \int_{-\infty}^{\infty} Q(\tau, m)g(m)dm, \tag{4.11}$$

where $g(m)$ is an entire even function in the complex m-plane. Substituting from (4.10), this integral is evaluated by the method of residues: half a residue at $m = \tau$ and $m = -\tau$ and a full residue at $m = \pm\tau + 2in$ $(n = 1, 2, 3, \dots)$. It turns out that the half-residues at $m = \pm\tau$ cancel each other, thus indicating that the singularities of $Q(\tau, m)$ on the real m-axis are fully accounted for by the delta-function terms in (4.10). The contribution of the poles above the real axis amounts to

$$\varepsilon\frac{\pi^2}{4}e^{ik\tau}\int_{-\infty}^{\infty}\frac{{}_2F_1(A, B; 2; \varepsilon)}{\mathrm{ch}\,\pi m - \mathrm{ch}\,\pi\tau}g(m)dm$$

$$= \frac{i\varepsilon\pi^2}{2\,\mathrm{sh}\,\pi\tau}e^{ik\tau}\sum_{n=0}^{\infty}\{g(\tau+is)\,{}_2F_1(2+n, -n+i\tau; 2; \varepsilon) - g(-\tau+is)\,{}_2F_1(-n, 2+n+i\tau; 2; \varepsilon)\}$$

$$(s = 2(n+1)), \tag{4.12}$$

where $g(m)$ is such that the integral over the infinite arc vanishes (see Appendix B).

However, [3, p. 212]

$${}_2F_1(-n, 2+n+i\tau; 2; \varepsilon) = \frac{1}{n+1}P_n^{(1, i\tau)}(1-2\varepsilon) \tag{4.13}$$

$${}_2F_1(n+2, -n+i\tau; 2; \varepsilon) = \frac{e^{-2ik\tau}}{n+1}P_n^{(1, -i\tau)}(1-2\varepsilon) \tag{4.14}$$

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where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial by Szegö definition [6]. Hence

$$\varepsilon\frac{\pi^2}{4}e^{ik\tau}\int_{-\infty}^{\infty}\frac{{}_2F_1(A,B;2;\varepsilon)}{\operatorname{ch}\pi m-\operatorname{ch}\pi\tau}g(m)dm=\frac{\varepsilon\pi^2}{\operatorname{sh}\pi\tau}\operatorname{Im}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}g(\tau-2in)\right\}. \tag{4.15}$$

The special case $g\equiv 1$ is important. A straightforward integration yields

$$\int_{-\infty}^{\infty}Q(\tau,m)dm=2\int_0^{\infty}K_{i\tau}(\beta_1 r)\frac{dr}{r}\int_0^{\infty}K_{im}(\beta_2 r)dm$$

$$=\pi\int_0^{\infty}K_{i\tau}(\beta_1 r)e^{-\beta_2 r}\frac{dr}{r}=\pi^2\frac{\cos\Omega\tau}{\tau\operatorname{sh}\pi\tau}, \tag{4.16}$$

where

$$\Omega=\operatorname{ch}^{-1}\frac{\beta_2}{\beta_1}=\log\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right),\qquad \cos\theta=\frac{\beta_1}{\beta_2}.$$

Letting $\varepsilon\to 0$ in (4.10), we find that

$$\lim_{\varepsilon\to 0}Q(\tau,m)=Q_0(\tau,m)=\frac{\pi^2}{2}\left\{\frac{\delta(\tau+m)+\delta(\tau-m)}{\tau\operatorname{sh}\pi\tau}\right\} \tag{4.17}$$

in accordance with (3.10).

Moreover, in the light of (4.10) and (4.15), we may represent $Q(\tau,m)$ for $\varepsilon\neq 0$ by the operator

$$Q(\tau,m)=\frac{\pi^2\cos k\tau}{2\tau\operatorname{sh}\pi\tau}\{\delta(m-\tau)+\delta(m+\tau)\}$$

$$+\frac{\varepsilon\pi^2}{\operatorname{sh}\pi\tau}\operatorname{Im}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}\delta(m-\tau+2in)\right\} \tag{4.18}$$

in the sense that

$$\int_{-\infty}^{\infty}Q(\tau,m)g(m)dm=\frac{\pi^2\cos k\tau}{\tau\operatorname{sh}\pi\tau}g(\tau)$$

$$+\frac{\varepsilon\pi^2}{\operatorname{sh}\pi\tau}\operatorname{Im}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}g(\tau-2in)\right\}$$

provided that $g(m)$ is such that the sum converges. In particular, if $g(\tau)$ is periodic with a period of $2in$, then

$$\int_{-\infty}^{\infty}Q(\tau,m)g(m)dm=g(\tau)\frac{\pi^2\cos\Omega\tau}{\tau\operatorname{sh}\pi\tau}. \tag{4.19}$$

Further properties are:

$$\lim_{\tau\to 0}\{\tau\operatorname{sh}\pi\tau\, Q(\tau,m)\}=\pi^2\delta(m) \tag{4.20}$$

$$Q_0(\tau,m)=Q_0(m,\tau). \tag{4.21}$$

But

$$Q(\tau, m; \beta_1; \beta_2) = Q(m, \tau; \beta_2; \beta_1), \tag{4.22}$$

since the interchange of τ and m has the same effect as the interchange of β_1 with β_2.

Therefore, if $\hat{\varepsilon} = \varepsilon/(\varepsilon-1)$, then

$$Q(m, \tau) = \frac{\pi^2 \cos k\tau}{2\tau \operatorname{sh} \pi\tau}[\delta(m-\tau)+\delta(m+\tau)]$$

$$+\frac{\hat{\varepsilon}\pi^2}{\operatorname{sh}\pi\tau}\operatorname{Im}\left\{e^{-ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(1-2\hat{\varepsilon})}{n}\delta(m-\tau+2in)\right\} = \hat{Q}(\tau, m). \tag{4.23}$$

In Appendix A we have presented a second proof which throws light on this formula from a different angle.

5. The Q-transform and Jacobi sums. We define the Q-transform of $f(x)$ by means of the integral

$$f(\tau) = T\{g(x)\} = \frac{2}{\pi^2}\tau \operatorname{sh}\pi\tau \int_0^\infty Q(\tau, x)g(x)dx. \tag{5.1}$$

The derivation of the inversion formula is obtained by writing (2.6) with the aid of (2.4) as follows:

$$\begin{aligned}
&\delta(\tau+x)+\delta(\tau-x)\\
&= \frac{2}{\pi^2}\tau \operatorname{sh}\pi\tau \int_0^\infty K_{i\tau}(\beta_1 r)K_{ix}(\beta_1 r)\frac{dr}{r}\\
&= \frac{2}{\pi^2}\tau \operatorname{sh}\pi\tau \int_0^\infty K_{i\tau}(\beta_1 r)\frac{dr}{r}\int_0^\infty K_{ix}(\beta_1 u)\delta^+(r-u)du\\
&= \frac{4}{\pi^4}\tau \operatorname{sh}\pi\tau \int_0^\infty K_{i\tau}(\beta_1 r)\frac{dr}{r}\int_0^\infty K_{ix}(\beta_1 u)\frac{du}{u}\int_0^\infty K_{i\sigma}(\beta_2 r)K_{i\sigma}(\beta_2 u)\sigma \operatorname{sh}\pi\sigma\, d\sigma\\
&= \frac{4}{\pi^4}\tau \operatorname{sh}\pi\tau \int_0^\infty \sigma \operatorname{sh}\pi\sigma\, d\sigma\int_0^\infty K_{i\tau}(\beta_1 r)K_{i\sigma}(\beta_2 r)\frac{dr}{r}\int_0^\infty K_{ix}(\beta_1 u)K_{i\sigma}(\beta_2 u)\frac{du}{u}
\end{aligned} \tag{5.2}$$

that is

$$\delta(\tau+x)+\delta(\tau-x) = \frac{4}{\pi^4}\tau \operatorname{sh}\pi\tau \int_0^\infty Q(\tau, \sigma)Q(x, \sigma)\sigma \operatorname{sh}\pi\sigma\, d\sigma. \tag{5.3}$$

Similarly,

$$\delta(\tau+x)+\delta(\tau-x) = \frac{4}{\pi^4}\tau \operatorname{sh}\pi\tau \int_0^\infty Q(\sigma, \tau)Q(\sigma, x)\sigma \operatorname{sh}\pi\sigma\, d\sigma, \tag{5.4}$$

since in (5.2) we can start with the argument $(\beta_2 r)$ and express $\delta^+(r-u)$ in terms of functions of the argument $(\beta_1 r)$.

Now, from (5.1) and (5.4),

$$\frac{2}{\pi^2} x \operatorname{sh} \pi x \int_0^\infty Q(\tau, x) f(\tau) d\tau = \frac{4}{\pi^4} x \operatorname{sh} \pi x \int_0^\infty Q(\tau, x) \tau \operatorname{sh} \pi\tau \, d\tau \int_0^\infty Q(\tau, \sigma) g(\sigma) d\sigma$$
$$= \int_0^\infty g(\sigma) d\sigma \left[\frac{4}{\pi^4} x \operatorname{sh} \pi x \int_0^\infty Q(\tau, \sigma) Q(\tau, x) \tau \operatorname{sh} \pi\tau \, d\tau \right]$$
$$= \int_0^\infty g(\sigma) [\delta(\sigma - x) + \delta(\sigma + x)] d\sigma = g(x).$$

Hence we have the transform-pair

$$f(\tau) = T\{g(x)\} = \frac{2}{\pi^2} \tau \operatorname{sh} \pi\tau \int_0^\infty Q(\tau, x) g(x) dx \tag{5.5}$$

$$g(x) = T^{-1}\{f(\tau)\} = \frac{2}{\pi^2} x \operatorname{sh} \pi x \int_0^\infty Q(\tau, x) f(\tau) d\tau, \tag{5.6}$$

where

$$T^{-1} T\{g(x)\} = T T^{-1}\{g(x)\} = g(x). \tag{5.7}$$

Equation (5.6) can be written as

$$g(\tau) = \frac{2}{\pi^2} \tau \operatorname{sh} \pi\tau \int_0^\infty \hat{Q}(\tau, x) f(x) dx, \tag{5.8}$$

where $\hat{Q}(\tau, x)$ is given by (4.23). A collection of simple Q-transforms is given in Table 1. These were derived by using the Kontorovich–Lebedev transform pairs given in the literature [e.g. **1**, **4**].

Next we consider (5.5) as an *integral equation* in the unknown function $g(x)$, that is

$$\frac{2}{\pi^2} \tau \operatorname{sh} \pi\tau \int_0^\infty g(m) Q(\tau, m) dm = f(\tau). \tag{5.9}$$

By (4.18), it is equivalent to the *difference equation*

$$g(\tau) \cos k\tau + \varepsilon\tau \operatorname{Im} \left\{ e^{ik\tau} \sum_{n=1}^{\infty} \frac{P_{n-1}^{(1, i\tau)}(1-2\varepsilon)}{n} g(\tau - 2in) \right\} = f(\tau) \tag{5.10}$$

which, due to (4.23) and (5.8), has the solution

$$g(\tau) = f(\tau) \cos k\tau + \hat{\varepsilon}\tau \operatorname{Im} \left\{ e^{-ik\tau} \sum_{n=1}^{\infty} \frac{P_{n-1}^{(1, i\tau)}(1-2\hat{\varepsilon})}{n} f(\tau - 2in) \right\}. \tag{5.11}$$

Thus, both (5.8) and (5.11) are solutions of (5.9). One form requires the evaluation of an integral and the other the evaluation of an infinite sum, which we shall call a *Jacobi sum*. The connection between the two solutions is furnished by (4.18)

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$$\varepsilon\tau \operatorname{Im}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}g(\tau-2in)\right\}=\frac{2}{\pi^2}\tau \operatorname{sh}\pi\tau\int_0^{\infty}Q(\tau,x)g(x)dx-g(\tau)\cos k\tau, \quad (5.12)$$

where $g(\tau)$ is entire even function in τ and also is such that the sum converges (see Appendix B). With the aid of (5.12), one may evaluate Jacobi sums of the type $\sum_{n=1}^{\infty}\frac{z^n}{n}P_{n-1}^{(1,i\tau)}(1-2\varepsilon)g(\tau-2in)$, provided one can reproduce a function from its imaginary (or real) part. The techniques for doing this are however well-known.

In this sense (5.12) plays an analogous role to Poisson's summation formula in the Fourier integral theory.

Let us demonstrate its usefulness by means of an example. We choose $g(m) = m \operatorname{sh}\pi m\, K_{im}(\beta_2 r_0)$. Then, because of (2.4),

$$\int_{-\infty}^{\infty}Q(\tau,m)g(m)dm = 2\int_0^{\infty}K_{i\tau}(\beta_1 r)\frac{dr}{r}\int_0^{\infty}m \operatorname{sh}\pi m\, K_{im}(\beta_2 r)K_{im}(\beta_2 r_0)dm$$

$$= 2\int_0^{\infty}K_{i\tau}(\beta_1 r)\frac{dr}{r}\pi^2 r_0\delta^+(r-r_0) = \pi^2 K_{i\tau}(\beta_1 r_0).$$

Therefore by (4.18), with $a=\beta_1/\beta_2$, $\beta_2 r_1 = z$, $k = \ln a$, we obtain

$$K_{i\tau}(az) = \cos k\tau\, K_{i\tau}(z)+(1-a^2)\operatorname{Im}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(2a^2-1)}{n}(\tau-2in)K_{i\tau+2n}(z)\right\} \quad (5.13)$$

This is a *multiplication theorem* for the modified Bessel function of the second kind. To the best of the author's knowledge, this formula appears here for the first time.

In particular, for $\tau = 0$,

$$K_0(az) = K_0(z)+2(a^2-1)\sum_{n=1}^{\infty}P_{n-1}^{(1,0)}(2a^2-1)K_{2n}(z).$$

If we choose $g(m) = \operatorname{ch} m\theta\, K_{im}(\beta_1 r_0)$, we obtain the new relation

$$\frac{1}{\pi}\int_0^{\infty}K_{i\tau}(\beta_1 r)K_0\{\beta_2\surd(r^2+\rho_0^2+2r\rho_0\cos\theta)\}\frac{dr}{r} = \frac{\cos k\tau \operatorname{ch}\tau\theta}{\tau \operatorname{sh}\pi\tau}K_{i\tau}(\beta_1 r_0)$$

$$+\frac{\varepsilon}{\operatorname{sh}\pi\tau}\operatorname{Im}\left\{e^{ik\tau}\sum_{n=1}^{\infty}\frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}\operatorname{ch}\theta(\tau-2in)K_{i\tau+2n}(\beta_1 r_0)\right\} \quad (5.14)$$

$$(\rho_0 = (\beta_1/\beta_2)r_0).$$

APPENDIX A

A multiplication theorem of the Bessel functions has the form [8, p. 140]

$$J_\nu(az) = a^\nu J_\nu(z) + a^\nu \sum_{n=1}^{\infty} \frac{(\nu+2n)}{n!} \frac{\Gamma(n+\nu)}{\Gamma(1+\nu)} {}_2F_1(-n, n+\nu; 1+\nu; \beta_1^2/\beta_2^2) J_{\nu+2n}(z) \tag{A.1}$$

Substituting $\varepsilon = 1-a^2$, $\nu = i\tau$, $a = \beta_1/\beta_2$, $z = \beta_2 r$ and invoking the definition [3, p. 212]

$$\frac{\Gamma(n+i\tau)}{n!\Gamma(1+i\tau)} {}_2F_1(-n, n+i\tau; 1+i\tau; 1-\varepsilon) = (-)^{n+1} \frac{\varepsilon}{n} P_{n-1}^{(1,i\tau)}(1-2\varepsilon), \tag{A.2}$$

where $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials by Szegö definition [6], we obtain the new multiplication theorem for the Bessel functions

$$J_{i\tau}(\beta_1 r) = e^{ik\tau} J_{i\tau}(\beta_2 r) + \varepsilon e^{ik\tau} \sum_{n=0}^{\infty} (-)^n \frac{i\tau+2n+2}{n+1} P_n^{(1,i\tau)}(1-2\varepsilon) J_{i\tau+2n+2}(\beta_2 r) \tag{A.3}$$

$$(k = \ln(\beta_1/\beta_2).)$$

Invoking the definitions of the modified Bessel function [3]

$$I_\nu(z) = e^{-\frac{1}{2}\pi i\nu} J_\nu(ze^{\pi i/2}); \quad K_\nu(x) = \frac{\pi}{2\sin\pi\nu}\{I_{-\nu}(x) - I_\nu(x)\}, \tag{A.4}$$

(5.5) yields

$$K_{i\tau}(\beta_1 r) = -\frac{\pi}{\operatorname{sh}\pi\tau} \operatorname{Im}[e^{ik\tau} I_{i\tau}(\beta_2 r)]$$

$$-\frac{\pi\varepsilon}{\operatorname{sh}\pi\tau} \operatorname{Im}\left\{e^{ik\tau} \sum_{n=1}^{\infty} \frac{i\tau+2n}{n} P_{n-1}^{(1,i\tau)}(1-2\varepsilon) I_{i\tau+2n}(\beta_2 r)\right\}. \tag{A.5}$$

However, from (2.6) and (A.4) we deduce the result

$$\int_0^\infty K_{i\tau}(\beta r) I_{im}(\beta r) \frac{dr}{r} = -\frac{\pi i}{2m}[\delta(\tau+m) + \delta(\tau-m)]. \tag{A.6}$$

Then, multiplying both sides of (A.5) by $K_{im}(\beta_2 r)\dfrac{dr}{r}$ and integrating over $(0, \infty)$, using (1.6), we arrive finally at the desired representation

$$Q(\tau, m) = \frac{\pi^2 \cos k\tau}{2\tau \operatorname{sh}\pi\tau}[\delta(m+\tau) + \delta(m-\tau)]$$

$$+\frac{\varepsilon\pi^2}{\operatorname{sh}\pi\tau} \operatorname{Im}\left\{\sum_{n=1}^{\infty} \frac{e^{ik\tau}}{n} P_{n-1}^{(1,i\tau)}(1-2\varepsilon)\delta(m-\tau+2in)\right\} \tag{A.7}$$

valid in the sense of (4.18).

APPENDIX B

From [3, p. 214] and [2, pp. 1040–1041] we deduce, after a few algebraic steps, that

$$S_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} P_{n-1}^{(1,i\tau)}(1-2\varepsilon) = \frac{1-e^{-iA\tau}}{i\tau\varepsilon} \qquad |z| \leqq 1, \tag{B.1}$$

where

$$A = \ln\frac{1+z+R}{2} = \tfrac{1}{2}\ln z + \ln\beta_1/\beta_2 + \mathrm{ch}^{-1}\{(1+z)\beta_2/2\sqrt{z}\beta_1\},$$

$$R = \sqrt{\{(1-z)^2+4z\varepsilon\}}.$$

Also

$$S_2(z) = \sum_{n=1}^{\infty} z^n P_{n-1}^{(1,i\tau)}(1-2\varepsilon) = z\frac{\partial}{\partial z}S_1(z) = \frac{2ze^{-iA\tau}}{R(1-z+R)}. \tag{B.2}$$

The particular case

$$z = e^{2i\eta}$$

$$A = i\eta + \ln(\beta_1/\beta_2) + \mathrm{ch}^{-1}(\beta_2/\beta_1 \cos\eta)$$

leads to the new sums

$$\sum_{n=1}^{\infty} \frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}\cos 2n\eta = \tfrac{1}{2}[S_1(z)+S_1(z^*)] = \frac{1-\mathrm{ch}\,\tau\eta\, e^{-i\omega\tau}}{i\tau\varepsilon} \tag{B.3}$$

$$\sum_{n=1}^{\infty} \frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}\sin 2n\eta = \tfrac{1}{2}[S_1(z)-S_1(z^*)] = \frac{\mathrm{sh}\,\tau\nu\, e^{-i\omega\tau}}{\tau\varepsilon} \tag{B.4}$$

$$\sum_{n=1}^{\infty} \frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}\,\mathrm{ch}\,\eta(\tau-2in) = \frac{\mathrm{ch}\,\eta\tau - e^{-i\omega\tau}}{i\tau\varepsilon} \tag{B.5}$$

$$\sum_{n=1}^{\infty} \frac{P_{n-1}^{(1,i\tau)}(1-2\varepsilon)}{n}\,\mathrm{sh}\,\eta(\tau-2in) = \frac{\mathrm{sh}\,\eta\tau}{i\tau\varepsilon} \tag{B.6}$$

$$k = \ln(\beta_1/\beta_2), \quad \mathrm{ch}\,\phi = \beta_2/\beta_1 \cos\eta,$$

$$\omega = \phi + k, \quad \mathrm{ch}\,\Omega = (\beta_2/\beta_1).$$

Then the use of (4.18), (B.3) and (B.4) enables us to evaluate the Q-transform

$$\int_0^{\infty} Q(\tau, m)\,\mathrm{ch}\,\eta m\, dm = \frac{\pi^2}{2\tau\,\mathrm{sh}\,\pi\tau}\cos\tau\phi, \qquad \int_{-\infty}^{\infty} Q(\tau, m)\,\mathrm{sh}\,\eta m\, dm = 0 \tag{B.7}$$

which is valid for $\eta \leqq \pi/2$. Note that the evaluation of the transform by means of integration (rather then summation), depends on the permissibility of interchanging the orders of integration over r and m. For example

$$\int_0^\infty Q(\tau, m)\,\mathrm{ch}\,\eta m\,dm = \int_0^\infty K_{i\tau}(\beta_1 r)\frac{dr}{r}\int_0^\infty K_{im}(\beta_2 r)\,\mathrm{ch}\,\eta m\,dm$$

$$= \frac{\pi}{2}\int_0^\infty K_{i\tau}(\beta_1 r)e^{-\beta_2 r\cos\eta}\frac{dr}{r} = \frac{\pi^2}{2\tau\,\mathrm{sh}\,\pi\tau}\cos\tau\phi$$

TABLE 1. Q-TRANSFORMS ($\lambda, \beta, \eta, \mu,$ real)

$g(x) = \frac{2}{\pi^2}x\,\mathrm{sh}\,\pi x\int_0^\infty Q(\tau, x)f(\tau)d\tau$	$f(\tau) = \frac{2}{\pi^2}\tau\,\mathrm{sh}\,\pi\tau\int_0^\infty Q(\tau, x)g(x)dx$
1	$\cos\Omega\tau \qquad \mathrm{ch}\,\Omega = \beta_2/\beta_1$
$g(x) = g(x-2in)$	$f(\tau)\cos\Omega\tau$
$g(x) = p(x)f(x) = p(x-2in)f(-x)$	$p(\tau)G(\tau)$ (provided $G(\tau)$ exists)
$x\,\mathrm{sh}\,\pi x\,K_{ix}(\beta_2 r_0)$	$\tau\,\mathrm{sh}\,\pi\tau\,K_{i\tau}(\beta_1 r_0)$
$\cos\eta x$	$\cos\phi\tau \qquad \mathrm{ch}\,\phi = \frac{\beta_2}{\beta_1}\mathrm{ch}\,\eta$
$\mathrm{ch}\,\eta x, \quad \eta \leqq \pi/2$	$\mathrm{ch}\,\phi\tau \qquad \cos\phi = \frac{\beta_2}{\beta_1}\cos\eta$
$x\sin\eta x$	$c_0\tau\sin\phi\tau \qquad c_0 = \frac{\beta_2\,\mathrm{sh}\,\eta}{\beta_1\,\mathrm{sh}\,\phi}$
$x\,\mathrm{sh}\,\eta x$	$A_0\tau\,\mathrm{sh}\,\phi\tau \qquad A_0 = \frac{\beta_2\sin\eta}{\beta_1\sin\phi}$
$\mathrm{ch}(\eta x)\,\mathrm{ch}(\mu x), \qquad \eta+\mu \leqq \pi/2$	$\mathrm{ch}\,\alpha\tau\,\mathrm{ch}\,\beta\tau$ $2\alpha = \cos^{-1}\left\{\frac{\beta_2}{\beta_1}(\eta-\mu)\right\} + \cos^{-1}\left\{\frac{\beta_2}{\beta_1}(\eta+\mu)\right\}$
$\mathrm{sh}(\eta x)\,\mathrm{sh}(\mu x)$	$\mathrm{sh}\,\alpha\tau\,\mathrm{sh}\,\beta\tau$ $2\beta = \cos^{-1}\left\{\frac{\beta_2}{\beta_1}(\eta-\mu)\right\} - \cos^{-1}\left\{\frac{\beta_2}{\beta_1}(\eta+\mu)\right\}$
$x\,\mathrm{sh}(\mu x)\,\mathrm{ch}(\eta x)$	$\tau\alpha\,\mathrm{sh}\,\alpha\tau\,\mathrm{ch}\,\beta\tau\frac{\partial\alpha}{\partial\mu} + \tau\beta\,\mathrm{ch}\,\alpha\tau\,\mathrm{sh}\,\beta\tau\frac{\partial\beta}{\partial\mu}$
$x\,\mathrm{sh}\,\pi x\,\Gamma(\lambda+ix)\Gamma(\lambda-ix)B^{\frac{1}{2}-\lambda}_{ix-\frac{1}{2}}(y)$, $\mathrm{Re}\,y > -1$ B = Legendre function	$A_0\tau\,\mathrm{sh}\,\pi\tau\,\Gamma(\lambda+i\tau)\Gamma(\lambda-i\tau)B^{\frac{1}{2}-\lambda}_{i\tau-\frac{1}{2}}(y\,\beta_2/\beta_1)$ $A_0 = (\beta_2/\beta_1)^\lambda\left\{\frac{y^2-1}{(y\beta_2/\beta_1)^2-1}\right\}^{(2\lambda-1)/4}$ $\mathrm{Re}(\tau-\lambda) \geqq \frac{1}{2}$
$x\,\mathrm{th}\,\frac{\pi x}{2}B_{\frac{ix}{2}-\frac{1}{2}}(z)$	$\tau\,\mathrm{th}\,\frac{\pi\tau}{2}\left\{\frac{\hat{\beta}_2}{\beta_1}B_{\frac{i\tau}{2}-\frac{1}{2}}\left(2\frac{\hat{\beta}_2}{\beta_1}-1\right)\right\}\hat{\beta}_2 = \beta_2(\frac{1}{2}+\frac{1}{2}z)$
$\mathrm{ch}\,\eta x\,K_{ix}(\beta_2 r_0)$	$\frac{1}{\pi}\tau\,\mathrm{sh}\,\pi\tau\int_0^\infty K_{i\tau}(\beta_1 r)K_0\{\beta_0^2\surd(r^2+r_0^2+2rr_0\cos\eta)\}\frac{dr}{r}$

https://doi.org/10.1017/S0017089500002603 Published online by Cambridge University Press

is again valid only for $\eta \leqq \pi/2$, or otherwise the integral over m will not converge. In general, the validity of (4.18) and (5.12) will depend on the behaviour of $g(m)$ in the upper complex m-plane. In addition to its being even and entire it must have the properties: (1) $g(m) = 0[\exp(\pi m/2)]$ since ${}_2F_1(A, B; 2; \varepsilon)$ behaves like $\exp(\pi m/2)$ for large m; (2) it is such that the function $\left\{\frac{{}_2F_1(A, B; 2; \varepsilon)}{\operatorname{ch}\pi m - \operatorname{ch}\pi\tau} g(m)\right\}$ vanishes on the infinite upper semi-circle. Thus $g(m) = \cos\eta m$ violates (2) and the Q-transform for this case can only be evaluated by integration (Table 1).

Finally, for $g(m)$ with period $2in$,

$$\sum_{n=1}^{\infty} \frac{z^n}{n} P_{n-1}^{(1,i\tau)}(1-2\varepsilon)g(\tau-2in) = g(\tau)\frac{1-e^{-i\omega\tau}}{i\tau\varepsilon}.$$

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https://doi.org/10.1017/S0017089500002603 Published online by Cambridge University Press