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A CHARACTERIZATION OF EXPONENTIAL FUNCTIONS WITH NON-LINEAR EXPONENTS

BY
HALINA ŚWIATAK

It is well known (see e.g. [1]) that the Cauchy functional equation

$$f(x+y) = f(x)f(y)$$

characterizes the function $f:x\to e^{\alpha x}$.

It was mentioned in [2] that the functions $f:x\to A\exp(\alpha x^{2m})$, $g:x\to e^{\alpha x}/A$ can be characterized by the equation

$$f(x-y) = f(x)f(y)g((x-y)^{2m}-x^{2m}-y^{2m}) \tag{1}$$

but the proof was done only for $m=1$ which was considerably simple.

The purpose of this paper is to show that the functions $f:x\to A\exp(\alpha x^{2m})$, $g:x\to e^{\alpha x}/A$ are the only solutions of (1) in the class of functions

$Z = \{(f, g):f:R\to R, g:R\to R, f(x)\not\equiv 0, f$ or g is continuous at the point $x=0\}$.

The following lemmas will be appplied:

LEMMA 1. *If $g:R\to R$, $h:R\to R$, $h(R)=R$ and if h and $g\circ h:x\to g(h(x))$ are continuous functions, then g is also a continuous function.*

LEMMA 2. *If $h:(-\delta, \delta)\to(-\varepsilon, \varepsilon)$ is a continuous and strictly monotonic function for which $h(0)=0$, then every function $g:(-\varepsilon, \varepsilon)\to R$ satsifying the condition $\lim_{u\to 0} g(h(u))=g(0)$ is continuous at the point $y=0$.*

The proofs of these lemmas follow almost immediately from the definition of the limit and from the definition of a continuous function.

LEMMA 3. *If the function g is continuous at the point $x=0$ and satisfies equation (1), where $f(x)\not\equiv 0$, then f and g are continuous functions.*

Proof. Notice first that the assumption $f(x)\not\equiv 0$ guarantees that $f(x)g(x)\neq 0$ for $x\in R$. In fact; $f(y)=0$ for a certain y implies $f(x)\equiv 0$ and $g(\xi)=0$ implies $f(\eta-1)=0$, where η is a solution of the equation $(\eta-1)^{2m}-\eta^{2m}-1=\xi$.

Setting in (1) $y=0$ one obtains $f(x)=f(x)f(0)g(0)$ and hence

$$f(0)g(0) = 1. \tag{2}$$

Setting in (1) $x=0$ one obtains $f(-y)=f(0)g(0)f(y)$ i.e., by (2),

$$f(-y) = f(y). \tag{3}$$

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The continuity of the function g at the point $x=0$ implies immediately the continuity of the function $f^2: x \to f(x)^2$ since, by (1),

$$\lim_{x\to 0} f(x)^2 = \lim_{x\to 0} \frac{f(0)}{g(-2x^{2m})} = \frac{f(0)}{g(0)} = f(0)^2.$$

Setting in (1) $x=u/2$, $y=-u/2$ one obtains

$$f(u) = f\left(\frac{u}{2}\right) f\left(-\frac{u}{2}\right) g\left(\left(1-\frac{1}{2^{2m-1}}\right)u^{2m}\right).$$

Hence, by (3),

$$f(u) = f\left(\frac{u}{2}\right)^2 g\left(\left(1-\frac{1}{2^{2m-1}}\right)u^{2m}\right)$$

and, consequently,

$$\lim_{u\to 0} f(u) = \lim_{u\to 0} f\left(\frac{u}{2}\right)^2 g\left(\left(1-\frac{1}{2^{2m-1}}\right)u^{2m}\right) = f(0)^2 g(0) = f(0).$$

The continuity of the functions f and g at the point $x=0$ and equation (1) imply the continuity of the function f at an arbitrary point.

To prove that also the function g is continuous everywhere notice that the function $h: x \to (x-1)^{2m} - x^{2m} - 1$ is a continuous function that maps R onto R. Setting in (1) $y=1$ one obtains

$$g(h(x)) = \frac{f(x-1)}{f(1)f(x)}$$

and since f is a continuous function, the function $g \circ h: x \to g(h(x))$ is continuous everywhere. Now, Lemma 1 implies the continuity of the function g.

LEMMA 4. *If the function f ($f(x) \not\equiv 0$) is continuous at the point $x=0$ and satisfies equation* (1), *the functions f and g are continuous everywhere.*

Proof. Setting in (1) $x=2u$, $y=u$ one obtains

$$(4') \qquad g(h_1(u)) = \frac{1}{f(2u)} \quad \text{with} \quad h_1(u) = -2^{2m}u^{2m}.$$

Setting in (1) $x=-u$, $y=u$ one obtains

$$(4'') \qquad g(h_2(u)) = \frac{f(-2u)}{f(-u)f(u)} \quad \text{with} \quad h_2(u) = (2^{2m}-2)u^{2m}.$$

Let

$$h(u) = \begin{cases} h_1(u) & \text{for } u \le 0 \\ h_2(u) & \text{for } u > 0 \end{cases}.$$

The function h defined above is a continuous and strictly increasing function satisfying the condition $h(0)=0$. Since the function f is continuous at the point $u=0$,

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equalities (4′), (4″) and (2) imply

$$\lim_{u\to 0} g(h(u)) = \frac{1}{f(0)} = g(0).$$

Applying Lemma 2 one concludes that the function g is continuous at the point $y=0$ and, by Lemma 3, the functions f and g are continuous everywhere.

THEOREM. *If the functions f ($f(x)\not\equiv 0$) and g satisfy equation* (1) *and at least one of them is continuous at the point $x=0$, then*

$$f(x) = A\exp(\alpha x^{2m}), \qquad g(x) = e^{\alpha x}/A \tag{5}$$

where A ($A\neq 0$) and α are constants.

Proof. Similarly as in the proof of Lemma 3 one can show that $f(x)g(x)\neq 0$. Lemmas 3 and 4 imply the continuity of the functions f and g and it follows from (1) that $\operatorname{sgn} f(x) = \operatorname{sgn} g(x)$ for all $x\in R$. Therefore it suffices to find only the positive continuous solutions f, g of (1) considering the equation

$$F(x-y) = F(x)+F(y)+G((x-y)^{2m}-x^{2m}-y^{2m}), \tag{6}$$

where

$$F(x) = \ln f(x), \qquad G(x) = \ln g(x). \tag{7}$$

The continuity of the functions F and G follows from the continuity of the functions f and g.

Multiplying (6) by $-2m[(x-y)^{2m-1}+y^{2m-1}]$ and integrating with respect to y from α to β one obtains

$$\mathscr{T}_1(x) = \mathscr{T}_2(x)F(x)+\mathscr{T}_3(x)+\mathscr{T}_4(x), \tag{8}$$

where

$$\mathscr{T}_1(x) = -2m\sum_{i=0}^{2m-2}\binom{2m-1}{i}(-1)^i x^{2m-1-i}\int_{x-\beta}^{x-\alpha} u^i F(u)\,du,$$

$$\mathscr{T}_2(x) = (x-\beta)^{2m}-(x-\alpha)^{2m}-\beta^{2m}+\alpha^{2m},$$

$$\mathscr{T}_3(x) = -2m\sum_{i=0}^{2m-2}\binom{2m-1}{i}(-1)^i x^{2m-1-i}\int_{x-\beta}^{x-\alpha} y^i F(y)\,dy,$$

$$\mathscr{T}_4(x) = \int_{(x-\alpha)^{2m}-x^{2m}-\alpha^{2m}}^{(x-\beta)^{2m}-x^{2m}-\beta^{2m}} G(u)\,du.$$

The continuity of the functions F and G guarantees that the functions $\mathscr{T}_1$, $\mathscr{T}_2$, $\mathscr{T}_3$, $\mathscr{T}_4$ have continuous first derivatives. Therefore (8) implies that the function F has the continuous first derivative at all the points x such that $\mathscr{T}_2(x)\neq 0$. Since $\mathscr{T}_2$ is a strictly monotonic function and $\mathscr{T}_2(0)=0$, the function F has the continuous first derivative in $R\backslash\{0\}$.

Notice now that the functions

$$h_i : x \to (x-i)^{2m} - x^{2m} - i^{2m} \qquad (i = 1, 2)$$

are strictly monotonic in R. (This follows from the continuity of h_i' and from the fact that $h_i'(x)=2m[(x-i)^{2m-1}-x^{2m-1}]\neq 0$ which guarantees sgn $h_i'(x)$=const for all $x \in R$). Therefore h_i are invertible functions. Moreover, h_i^{-1} have the continuous first derivatives in R. Setting in (6) $y=1$, $h_1(x)=(x-1)^{2m}-x^{2m}-1=u_1$, $x=h_1^{-1}(u_1)$ one proves the existence of the continuous first derivative of the function G in $R\setminus\{0, -2\}$. Similarly, setting in (6) $y=2$, $h_2(x)=(x-2)^{2m}-x^{2m}-2^{2m}=u_2$, $x=h_2^{-1}(u_2)$ one proves the existence of the continuous first derivative of the function G in $R\setminus\{0, -2^{2m+1}\}$. Thus the function G has the continuous first derivative in $R\setminus\{0\}$.

Setting in (6) $y=x-u$ one obtains

$$F(u) = F(x)+F(x-u)+G(k(x, u)), \tag{9}$$

where $k(x, u)=u^{2m}-x^{2m}-(x-u)^{2m}$.

If $x \in \langle 1, 2\rangle$ and $|u|\leq 1/4$, $k(x, u)\leq (1/2^{4m})-1-(1-1/4)^{2m}<-1$ and $x-u\geq 3/4$. Therefore the existence of the continuous first derivatives of the functions F and G in $R\setminus\{0\}$ implies that the right-hand side of (9) is differentiable with respect to $u \in \langle -1/4, 1/4\rangle$ for every fixed $x \in \langle 1, 2\rangle$. Moreover,

$$u \to -F'(x-u)+\frac{\partial}{\partial u} k(x, u)G'(k(x, u))$$

is a continuous function and (9) implies the existence of the continuous first derivative of the function F in $\langle -1/4, 1/4\rangle$. Consequently, (6) implies the existence of the continuous first derivative of the function G in a neighborhood of the point $x=0$. This completes the proof of the fact that F, $G \in C^1(R)$. Now, analogous considerations allow one to prove that F, $G \in C^2(R)$.

Differentiating (6) with respect to x and y one obtains

$$\begin{aligned} F''(x-y) = {} & 2m(2m-1)(x-y)^{2m-2} \\ & \times G'((x-y)^{2m}-x^{2m}-y^{2m})+4m^2[(x-y)^{2m-1}-x^{2m-1}] \\ & \times [(x-y)^{2m-1}+y^{2m-1}]G''((x-y)^{2m}-x^{2m}-y^{2m}). \end{aligned}$$

Setting in the last equation $y=0$ one obtains

$$F''(x) = 2m(2m-1)G'(0)x^{2m-2}$$

and hence

$$F(x) = \alpha x^{2m}+ax+b \quad \text{with} \quad \alpha = G'(0).$$

Since, by (6), $F(y)=F(-y)$,

$$F(x) = \alpha x^{2m}+b. \tag{10}$$

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Substituting this into (6) one obtains

$$\alpha(x-y)^{2m} = \alpha(x^{2m}+y^{2m})+b+G((x-y)^{2m}-x^{2m}-y^{2m}). \tag{11}$$

Setting in (11) $y=x$ yields $G(-2x^{2m})=-2\alpha x^{2m}-b$ and hence $G(u)=\alpha u-b$ for $u\leq 0$. Setting in (11) $y=-x$ yields $G((2^{2m}-2)x^{2m})=(2^{2m}-2)\alpha x^{2m}-b$ and hence $G(u)=\alpha u-b$ for $u\geq 0$ which, together with (10) and (7), implies that the positive continuous solutions f ($f(x)\not\equiv 0$), g of (1) have the form (5), where $A=e^b$ is an arbitrary positive constant. In view of previous remarks all the solutions f, g of (1) satisfying the assumptions of the theorem have the form (5), where A ($A\neq 0$) is an arbitrary constant.

References

1. J. Aczél, *Lectures on functional equations and their applications*, Academic Press, New York and London, 1966.

2. H. Świątak, *On a class of functional equations with several unknown functions*, Aequationes Math. (to appear).

Université de Montréal,
Montréal, Québec

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