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On the Density of Cyclic Quartic Fields

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Abstract. An asymptotic formula is obtained for the number of cyclic quartic fields over Q with discriminant $\leq x$.

1 Introduction

Let $h(n)$ denote the number of cyclic quartic fields over the rational number field Q with discriminant n. We consider

$$N(x) = \sum_{n \leq x} h(n).$$

In [1, Theorem 9] Baily proved

$$N(x) \sim \frac{3}{\pi^2} \left\{ \frac{25}{24} \prod_{p \equiv 1 \ (\mathrm{mod}\ 4)} \left(1 + \frac{2}{(p+1)\sqrt{p}} \right) - 1 \right\} x^{1/2}, \tag{1.1}$$

where p runs through primes $p \equiv 1 \pmod 4$. Unfortunately Baily's generating function $f(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ is given incorrectly, and so the constant in (1.1) is wrong. In giving the Euler product for $f(s)$, Baily [1, p. 209] overlooks that the discriminant is $\frac{1}{2} f_4^3 f_2^2$ in one case rather than $f_4^3 f_2^2$ and so his term $4 \cdot 16^{-3s} = 4 \cdot 2^{-12s}$ should be replaced by $4 \cdot 2^{-11s}$.

In this paper, using the representation of a cyclic quartic field given by Hardy, Hudson, Richman, Williams and Holtz [2], see also [3], and an elementary method, we correct Baily's result and at the same time give an estimate for the error term. We prove

Theorem

$$N(x) = \frac{3}{\pi^2} \left\{ \frac{24 + \sqrt{2}}{24} \prod_{p \equiv 1 \ (\mathrm{mod}\ 4)} \left(1 + \frac{2}{(p+1)\sqrt{p}} \right) - 1 \right\} x^{1/2} + O(x^{1/3} \log^3 x). \tag{1.2}$$

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2 Representation of a Cyclic Quartic Field

In [2] the authors show that a cyclic quartic extension K of the rational number field Q can be expressed uniquely in the form

$$K = Q\left(\sqrt{A(D + B\sqrt{D})}\right), \tag{2.1}$$

where A, B, D are integers such that

$$\begin{cases} A \text{ is squarefree and odd,} \\ B \geq 1, D \geq 2, \\ D \text{ is squarefree and } D - B^2 \text{ is a square,} \\ (A, D) = 1, \end{cases} \tag{2.2}$$

where (A, D) denotes the gcd of A and D. The discriminant $d(K)$ of K is given by

$$d(K) = \begin{cases} 2^8 A^2 D^3, & \text{if } D \equiv 0 \pmod 2, \\ 2^6 A^2 D^3, & \text{if } D \equiv 1 \pmod 2,\ B \equiv 1 \pmod 2, \\ 2^4 A^2 D^3, & \text{if } D \equiv 1 \pmod 2,\ B \equiv 0 \pmod 2,\ A + B \equiv 3 \pmod 4, \\ A^2 D^3, & \text{if } D \equiv 1 \pmod 2,\ B \equiv 0 \pmod 2,\ A + B \equiv 1 \pmod 4. \end{cases} \tag{2.3}$$

3 Proof of the Theorem

Let K be a cyclic quartic extension of Q. From (2.1)–(2.3) we see that the discriminant $d(K)$ of K is of the form

$$d(K) = 2^\alpha (p_1 \cdots p_m)^2 (q_1 \cdots q_r)^3, \tag{3.1}$$

where $\alpha = 0, 4, 6$ or 11 and $p_1, \ldots, p_m, q_1, \ldots, q_r$ are distinct odd primes with $m \geq 0, r \geq 1$ if $\alpha = 0, 4, 6$, $r \geq 0$ if $\alpha = 11$, and $q_j \equiv 1 \pmod 4$, $j = 1, \ldots, r$. We set

$$A = p_1 \cdots p_m, \quad D = q_1 \cdots q_r. \tag{3.2}$$

We note that A and D defined in (3.2) are slightly different from the A and D in Section 2.

If $\alpha = 0$ then $n = d(K) = A^2 D^3$ and $K = Q\left(\sqrt{\varepsilon A(D + B\sqrt{D})}\right)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$B \equiv 0 \pmod 2, \quad B \equiv 1 - \varepsilon p_1 \cdots p_m \pmod 4, \quad D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$\begin{aligned}
h(n) &= \sum_{\varepsilon=-1,+1} \sum_{\substack{B>0,2|B \\ B\equiv 1-\varepsilon p_1\cdots p_m \pmod 4 \\ D-B^2=\square}} 1 = \sum_{\substack{B>0,2|B \\ D-B^2=\square}} 1 \\
&= \sum_{\substack{C>0,2\nmid C \\ D-C^2=\square}} 1 = \frac{1}{2} \sum_{\substack{B>0 \\ D-B^2=\square}} 1 = \frac{1}{2} \sum_{\substack{B<0 \\ D-B^2=\square}} 1 \\
&= \frac{1}{4} \sum_{\substack{B\neq 0 \\ D-B^2=\square}} = \frac{1}{4} \sum_{\substack{B \\ D-B^2=\square}} 1 = \frac{1}{8} \sum_{\substack{B,C \\ D=B^2+C^2}} 1 = \frac{1}{8} r_2(D) \\
&= \frac{1}{8} 2^{r+2} = 2^{r-1} = \frac{1}{2} d(D),
\end{aligned}$$

where $r_2(k)$ denotes the number of representations of the positive integer k as the sum of two squares and $d(k)$ denotes the number of positive divisors of k.

If $\alpha = 4$ then $n = d(K) = 2^4 A^2 D^3$ and $K = Q\left(\sqrt{\varepsilon A(D + B\sqrt{D})}\right)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$B \equiv 0 \pmod 2, \; B \equiv 3 - \varepsilon p_1 \cdots p_m \pmod 4, D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$h(n) = \sum_{\varepsilon=-1,+1} \sum_{\substack{B>0,2|B \\ B\equiv 3-\varepsilon p_1\cdots p_m \pmod 4 \\ D-B^2=\square}} 1 = \sum_{\substack{B>0,2|B \\ D-B^2=\square}} 1 = \frac{1}{2} d(D).$$

If $\alpha = 6$ then $n = d(K) = 2^6 A^2 D^3$ and $K = Q\left(\sqrt{\varepsilon A(D + B\sqrt{D})}\right)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$B \equiv 1 \pmod 2, \quad D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$h(n) = 2 \sum_{\substack{B>0,2\nmid B \\ D-B^2=\square}} 1 = 2 \sum_{\substack{C>0,2|C \\ D-C^2=\square}} 1 = 2^r = d(D).$$

If $\alpha = 11$ then $n = d(K) = 2^{11} A^2 D^3$ and $K = Q\left(\sqrt{\varepsilon A(2D + B\sqrt{2D})}\right)$ for some $\varepsilon = \pm 1$ and some positive integer B such that

$$2D - B^2 = \text{square}.$$

Moreover distinct pairs (ε, B) give different fields K. Thus

$$h(n) = 2\sum_{\substack{B>0\\2D-B^2=\square}} 1 = 2\sum_{\substack{B<0\\2D-B^2=\square}} 1 = \sum_{\substack{B\neq 0\\2D-B^2=\square}} 1$$

$$= \sum_{\substack{B\\2D-B^2=\square}} 1 = \frac{1}{2}\sum_{\substack{B,C\\2D=B^2+C^2}} 1 = \frac{1}{2}r_2(2D) = \frac{1}{2}2^{r+2} = 2^{r+1} = 2d(D).$$

Summarizing we have

$$(3.3)\qquad h(n) = \begin{cases} 2d(D), & \text{if } n = 2^{11}A^2D^3,\\ d(D), & \text{if } n = 2^6A^2D^3,\\ \frac{1}{2}d(D), & \text{if } n = 2^4A^2D^3 \text{ or } A^2D^3.\end{cases}$$

Recalling that $D = 1$ can only occur when $n = 2^{11}A^2D^3$, we have

$$\sum_{n\le x} h(n) = 2\sum_{2^{11}A^2\le x} 1 + 2\sum_{2^{11}A^2D^3\le x} d(D) + \sum_{2^6A^2D^3\le x} d(D)$$
$$+\frac{1}{2}\sum_{2^4A^2D^3\le x} d(D) + \frac{1}{2}\sum_{A^2D^3\le x} d(D),$$

so that

$$(3.4)\qquad \sum_{n\le x} h(n) = 2\sum_{\substack{A\le (x/2^{11})^{1/2}\\ A \text{ squarefree}\\ A \text{ odd}}} 1 + 2S(2^{-11}x) + S(2^{-6}x) + \frac{1}{2}S(2^{-4}x) + \frac{1}{2}S(x),$$

where

$$(3.5)\qquad S(x) = \sum_{A^2D^3\le x} d(D)$$

and the sum is over all positive integers A and D such that

$$(3.6)\qquad A = p_1\cdots p_m \quad (m\ge 0),\ D = q_1\cdots q_r \quad (r\ge 1),$$

where $p_1,\dots,p_m,q_1,\dots,q_r$ are distinct odd primes with $q_j \equiv 1 \pmod 4$ $(j = 1,\dots,r)$. We set

$$(3.7)\quad \mathcal{P} = \{D \mid D = q_1\cdots q_r (r\ge 1), q_1,\dots,q_r \text{ are distinct primes } \equiv 1 \pmod 4\},$$

so that

$$(3.8)\qquad S(x) = \sum_{\substack{D\le x^{1/3}\\ D\in\mathcal{P}}} d(D) \sum_{\substack{1\le A\le\sqrt{xD^{-3}}\\ A \text{ squarefree}\\ (A,2D)=1}} 1.$$

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Note that $1 \notin \mathcal{P}$.

We first estimate $\sum_{\substack{A \le y \\ A \text{ squarefree} \\ A \text{ odd}}} 1$, where $y = (x/2^{11})^{1/2}$. We have

$$\begin{aligned}
\sum_{\substack{A \le y \\ A \text{ squarefree} \\ A \text{ odd}}} 1 &= \sum_{\substack{A \le y \\ A \text{ odd}}} \sum_{d^2 \mid A} \mu(d) \\
&= \sum_{\substack{d \le y^{1/2} \\ d \text{ odd}}} \mu(d) \sum_{\substack{a \le y/d^2 \\ a \text{ odd}}} 1 \\
&= \sum_{\substack{d \le y^{1/2} \\ d \text{ odd}}} \mu(d) \Big(\frac{y}{2d^2} + O(1) \Big) \\
&= \frac{y}{2} \sum_{\substack{d \le y^{1/2} \\ d \text{ odd}}} \frac{\mu(d)}{d^2} + O(y^{1/2}) \\
&= \frac{y}{2} \sum_{\substack{d=1 \\ d \text{ odd}}}^{\infty} \frac{\mu(d)}{d^2} + O(y^{1/2}) \\
&= \frac{y}{2} \prod_{p \ne 2} \Big(1 - \frac{1}{p^2} \Big) + O(y^{1/2}) \\
&= \frac{y}{2} \Big(1 - \frac{1}{2^2} \Big)^{-1} \prod_{p} \Big(1 - \frac{1}{p^2} \Big) + O(y^{1/2}) \\
&= \frac{4}{\pi^2} y + O(y^{1/2}) \\
&= \frac{x^{1/2}}{2^{7/2} \pi^2} + O(x^{1/4}).
\end{aligned}$$

We now turn to the estimation of $S(x)$. The inner sum in (3.8) is

$$\begin{aligned}
\sum_{\substack{A \le \sqrt{xD^{-3}} \\ (A,2D)=1}} \sum_{d^2 \mid A} \mu(d) &= \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \mu(d) \sum_{\substack{a \le d^{-2}\sqrt{xD^{-3}} \\ (a,2D)=1}} 1 \\
&= \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \mu(d) \sum_{e \mid 2D} \mu(e) \sum_{b \le e^{-1} d^{-2} \sqrt{xD^{-3}}} 1 \\
&= \sum_{e \mid 2D} \mu(e) \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \mu(d) \Big[\frac{\sqrt{xD^{-3}}}{d^2 e} \Big]
\end{aligned}$$

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$$= \sqrt{xD^{-3}} \sum_{e|2D} \frac{\mu(e)}{e} \sum_{\substack{d \le (xD^{-3})^{1/4} \\ (d,2D)=1}} \frac{\mu(d)}{d^2} + O\left(d(2D) \left(\frac{x}{D^3} \right)^{1/4} \right)$$

$$= \sqrt{xD^{-3}} \sum_{e|2D} \frac{\mu(e)}{e} \sum_{(d,2D)=1} \frac{\mu(d)}{d^2} + O\left(d(D) \left(\frac{x}{D^3} \right)^{1/4} \right)$$

$$+ O\left(\left(\frac{x}{D^3} \right)^{1/2} \sum_{e|2D} \frac{1}{e} \left(\frac{x}{D^3} \right)^{-1/4} \right)$$

$$= \sqrt{xD^{-3}} \frac{\varphi(2D)}{2D} \frac{6}{\pi^2} \left(\prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1} \right) + O\left(d(D) \left(\frac{x}{D^3} \right)^{1/4} \right),$$

since

$$\sum_{\substack{d=1 \\ (d,2D)=1}}^{\infty} \frac{\mu(d)}{d^2} = \prod_{(p,2D)=1} \left(1 - \frac{1}{p^2} \right) = \prod_{p} \left(1 - \frac{1}{p^2} \right) \Big/ \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)$$

$$= \frac{1}{\zeta(2)} \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1} = \frac{6}{\pi^2} \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1}$$

and Euler's phi function $\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$. Thus

$$S(x) = \frac{3}{\pi^2} x^{1/2} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d(D) \varphi(D) D^{-5/2} \prod_{p|2D} \left(1 - \frac{1}{p^2} \right)^{-1}$$

$$+ O\left(x^{1/4} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) D^{-3/4} \right)$$

$$= \frac{4}{\pi^2} x^{1/2} \sum_{\substack{D=1 \\ D \in \mathcal{P}}}^{\infty} d(D) \varphi(D) D^{-5/2} \prod_{p|D} \left(1 - \frac{1}{p^2} \right)^{-1}$$

$$+ O\left(x^{1/2} \sum_{\substack{D > x^{1/3} \\ D \in \mathcal{P}}} d(D) \varphi(D) D^{-5/2} \right) + O\left(x^{1/4} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) D^{-3/4} \right),$$

as

$$\prod_{p|D} \left(1 - \frac{1}{p^2} \right)^{-1} = \frac{\pi^2}{6} \prod_{p \nmid D} \left(1 - \frac{1}{p^2} \right) < \frac{\pi^2}{6}.$$

Clearly

$$\sum_{\substack{D=1 \\ D \in \mathcal{P}}}^{\infty} d(D) \varphi(D) D^{-5/2} \prod_{p|D} \left(1 - \frac{1}{p^2} \right)^{-1} = \prod_{p \equiv 1 \pmod 4} \left(1 + \frac{2}{(p+1)\sqrt{p}} \right) - 1.$$

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It remains to estimate $R_1 = \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) D^{-3/4}$ and $R_2 = \sum_{\substack{D > x^{1/3} \\ D \in \mathcal{P}}} d(D)\varphi(D) D^{-5/2}$. Firstly

$$\sum_{\substack{D \le x \\ D \in \mathcal{P}}} d^2(D) = \sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D) \sum_{a|D} 1 = \sum_{\substack{ab \le x \\ a,b \in \mathcal{P} \\ (a,b)=1}} d(ab) + 2 \sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D)$$
$$\le \sum_{\substack{a \le x \\ a \in \mathcal{P}}} d(a) \sum_{\substack{b \le x/a \\ b \in \mathcal{P}}} d(b) + 2 \sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D)$$

$$\sum_{\substack{D \le x \\ D \in \mathcal{P}}} d(D) = \sum_{\substack{D \le x \\ D \in \mathcal{P}}} \sum_{a|D} 1 \le \sum_{\substack{a \le x \\ a \in \mathcal{P}}} \Big(2 + \sum_{\substack{b \le x/a \\ b \in \mathcal{P}}} 1 \Big) \ll x \log x,$$

so

$$\sum_{\substack{D \le x \\ D \in \mathcal{P}}} d^2(D) \ll x \log x + \sum_{\substack{a \le x \\ a \in \mathcal{P}}} d(a) \frac{x}{a} \log \frac{x}{a}$$
$$\ll x \log x + x \log x \sum_{\substack{a \le x \\ a \in \mathcal{P}}} \frac{d(a)}{a}$$
$$\ll x \log^3 x.$$

By partial summation we have

$$R_1 = x^{-\frac{1}{4}} \sum_{\substack{D \le x^{1/3} \\ D \in \mathcal{P}}} d^2(D) - \int_1^{x^{1/3}} \Big(\sum_{\substack{D \le y \\ D \in \mathcal{P}}} d^2(D) \Big) d(y^{-3/4})$$
$$= O(x^{1/3 - 1/4} \log^3 x) = O(x^{1/12} \log^3 x)$$

and

$$R_2 \le \sum_{\substack{D > x^{1/3} \\ D \in \mathcal{P}}} d(D) D^{-3/2} = -\int_{x^{1/3}}^{\infty} \Big(\sum_{\substack{D \le y \\ D \in \mathcal{P}}} d(D) \Big) d(y^{-3/2}) = O(x^{-1/6} \log x).$$

Therefore

$$S(x) = \frac{4c_0}{\pi^2} x^{1/2} + O(x^{1/3} \log^3 x),$$

where

$$c_0 = \prod_{p \equiv 1 \ (\mathrm{mod}\ 4)} \Big(1 + \frac{2}{(p+1)\sqrt{p}} \Big) - 1,$$

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and

$$\begin{aligned}\sum_{n\le x} h(n) &= \frac{x^{1/2}}{2^{5/2}\pi^2} + O(x^{1/4}) \\ &\quad + \frac{4c_0}{\pi^2}\left(2\cdot 2^{-11/2} + 2^{-3} + \frac{1}{2}2^{-2} + \frac{1}{2}\right)x^{1/2} + O(x^{1/3}\log^3 x) \\ &= \left(\frac{(24+\sqrt{2})}{8\pi^2}c_0 + \frac{\sqrt{2}}{8\pi^2}\right)x^{1/2} + O(x^{1/3}\log^3 x) \\ &= \left(\frac{24+\sqrt{2}}{8\pi^2}\prod_{p\equiv 1 \pmod 4}\left(1+\frac{2}{(p+1)\sqrt{p}}\right) - \frac{24}{8\pi^2}\right)x^{1/2} + O(x^{1/3}\log^3 x) \\ &= \frac{3}{\pi^2}\left\{\frac{24+\sqrt{2}}{24}\prod_{p\equiv 1 \pmod 4}\left(1+\frac{2}{(p+1)\sqrt{p}}\right) - 1\right\}x^{1/2} + O(x^{1/3}\log^3 x).\end{aligned}$$

This completes the proof of (1.2).

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