4

Instantons in quantum mechanics

Instantons are solutions of the classical equations of motion with a finite Euclidean action. Such field configurations are not taken into account in perturbation theory. Instantons are characterized by a topological charge which may result in a conserved quantum number and never show up in perturbation theory. In Minkowski space, instantons are associated with tunneling processes between vacua labeled by a distinct topological charge.

Instantons first appear in Yang–Mills theory [BPS75], although this kind of classical solution was known long before in statistical physics [Lan67].

In this chapter we consider instantons in quantum mechanics as an illustration of path-integral calculations. We follow the original paper by Polyakov [Pol77] except for technical details.

4.1 Double-well potential

Let us consider a one-dimensional quantum-mechanical system with the double-well potential

$$V(x) \;=\; \frac{\lambda}{4}\left(x^2-\frac{\mu^2}{\lambda}\right)^2 \;=\; -\frac{1}{2}\mu^2x^2+\frac{1}{4}\lambda x^4+\frac{\mu^4}{4\lambda}. \tag{4.1}$$

This is nothing but an anharmonic oscillator with the opposite sign for the coefficient of the quadratic term,* which usually appears with a positive

* It is often called the mass term. This terminology comes from quantum field theory, where the potential (4.1) is considered in the context of a spontaneous breaking of the reflection symmetry $x \to -x$. In our quantum-mechanical problem, defined by the Euclidean action (4.3), the mass of the nonrelativistic particle is absorbed in τ which has, therefore, the dimension of $[\text{length}]^2$. This has already been explained in Sect. 1.6.

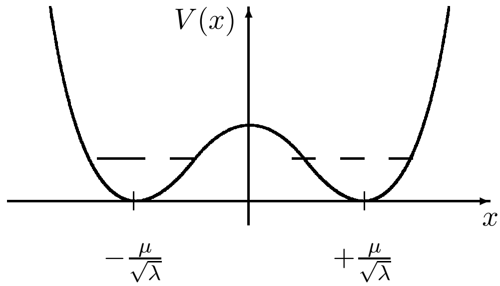

Fig. 4.1. The double-well potential (4.1). The short vertical lines represent the position of the minima (4.4). The dashed lines correspond to the energy E_0 of the lowest state in a single well, i.e. to that in the limit $\lambda \to 0$.

coefficient $\omega^2/2$. We have introduced

$$\mu^2 = -\omega^2 \tag{4.2}$$

in order to work with real numbered values. The constant term is added for later convenience. The potential (4.1) as a function x is depicted in Fig. 4.1.

The (Euclidean) action is defined by

$$S[x] = \int \mathrm{d}\tau \left[\frac{1}{2}\dot{x}^2(\tau) + V(x(\tau))\right] \tag{4.3}$$

with $V(x)$ given by Eq. (4.1). The plus sign between the kinetic and potential energies is because we are in Euclidean space.

It follows from Eqs. (4.1) and (4.3) that the parameter μ has the dimension of $[\text{length}]^{-2}$ or, in other words, the dimensions of x and τ are $[\mu]^{-1/2}$ and $[\mu]^{-1}$, respectively. Analogously, the dimension of the constant λ is $[\mu]^3$.

For $\lambda \ll \mu^3$, the potential (4.1) has superficially two degenerate vacua

$$x_0^{\pm} = \pm\frac{\mu}{\sqrt{\lambda}}, \tag{4.4}$$

the positions of which coincide with the minima of the potential in Fig. 4.1.

The degeneracy between the two minima is preserved at all orders of perturbation theory, where an expansion near one of the minima of the potential (either the left- or right-hand one) is carried out:

$$x(\tau) = \pm\frac{\mu}{\sqrt{\lambda}} + \chi(\tau) \tag{4.5}$$

with $\chi(\tau) \ll \mu/\sqrt{\lambda}$. The correlator at asymptotically large τ is

$$\langle\, x(0)\, x(\tau)\rangle \to \frac{\mu^2}{\lambda} + \cdots . \tag{4.6}$$

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Its nonvanishing value means that a particle is localized at one of the two vacua.

The next terms of the perturbative expansion in λ do not spoil this result since the potential (4.1) becomes

$$V = \mu^2\chi^2 \mp \sqrt{\lambda}\mu\chi^3 + \frac{\lambda}{4}\chi^4 \tag{4.7}$$

after the shift (4.5), and has a positive sign for the quadratic term. Therefore, a perturbation theory constructed around the vacuum $x_0^{\pm}$ is a normal one, and the particle lives perturbatively in one of the two vacua.

However, we know from quantum mechanics that (nonperturbatively)

$$\langle\, x(0)\, x(\tau)\rangle = \sum_n |x_{n0}|^2\, \mathrm{e}^{-(E_n-E_0)\tau} \tag{4.8}$$

at imaginary time $\tau = \mathrm{i}t$, where E_n is the energy of the nth eigenstate of the Hamiltonian and x_{n0} is the proper matrix element. Therefore,

$$\langle\, x(0)\, x(\tau)\rangle \sim \mathrm{e}^{-\Delta E\,\tau} \tag{4.9}$$

for large τ, where

$$\Delta E = \mu\sqrt{\frac{48}{\pi}}\sqrt{\frac{2\sqrt{2}\,\mu^3}{3\lambda}}\exp\left(-\frac{2\sqrt{2}\,\mu^3}{3\lambda}\right) \tag{4.10}$$

is the energy splitting between the two lowest states (symmetric and antisymmetric) for $\lambda \ll \mu^3$, which vanishes exponentially as $\lambda \to 0$.

The appearance of imaginary time in Eq. (4.8) is because under a barrier particles live in imaginary time. We may say that imaginary time is an appropriate language for describing tunneling through a barrier.

Since the RHS of Eq. (4.9) vanishes as $\tau \to \infty$, the reflection symmetry $x \to -x$, which is broken in perturbation theory, is restored nonperturbatively as $\tau \to \infty$.

Problem 4.1 Derive Eq. (4.10) modulo a constant factor within standard quantum mechanics.

Solution Let us use the semiclassical formula [LL74] (Problem 3 in §50)

$$\Delta E = \frac{\sqrt{2}\,\mu}{\pi}\,\mathrm{e}^{-\int_{-a}^{+a}\mathrm{d}x\sqrt{2[V(x)-E_0]}}\,, \tag{4.11}$$

where $\pm a$ are the classical turning points, which are determined by

$$V(\pm a) = E_0\,, \tag{4.12}$$

and

$$E_0 = \sqrt{2}\,\mu \tag{4.13}$$

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is the lowest energy for the oscillator potential (4.7) as $\lambda \to 0$. Denoting

$$h = \sqrt{\frac{\lambda}{\sqrt{2}\,\mu^3}}, \qquad z = \frac{\sqrt{\lambda}}{\mu}\,x\,, \tag{4.14}$$

the integral in the exponent on the RHS of Eq. (4.11) can be calculated using an expansion in h which gives

$$\frac{1}{2h^2}\int\limits_{-1+h}^{1-h} \mathrm{d}z\sqrt{(1-z^2)^2-4h^2} = \frac{2}{3h^2}+\ln h+\mathcal{O}(1)\,. \tag{4.15}$$

Substituting into Eq. (4.11), one recovers Eq. (4.10) modulo a constant factor.

4.2 The instanton solution

In the path-integral approach, the correlator (4.8) is given by

$$\langle\, x(0)\, x(\tau)\rangle = \frac{\displaystyle\int \mathcal{D}x\ \mathrm{e}^{-S[x]}\, x(0)\, x(\tau)}{\displaystyle\int \mathcal{D}x\ \mathrm{e}^{-S[x]}} \tag{4.16}$$

with no restrictions on the integration over x. This is a quantum-mechanical analog of the path integrals defined in Sect. 2.1.

At small λ, the path integral (4.16) can be evaluated using the saddle-point method. The reason for this is that for x given by Eq. (4.4) (i.e. the minima of the action (4.3)), the Gaussian fluctuations around (4.4) are not essential as $\lambda \to 0$. This is most easily seen by making the shift (4.5) and noting that $\chi(\tau)$ is $\mathcal{O}(1)$ at the saddle points according to Eq. (4.7), the RHS of which is quadratic in $\chi(\tau)$ as $\lambda \to 0$.

Performing the saddle-point evaluation of the path integral (4.16), one obtains

$$\langle\, x(0)\, x(\tau)\rangle = \frac{\mu^2}{\lambda}+\cdots\,. \tag{4.17}$$

Note that $x(0)$ and $x(\tau)$ in the integrand can be substituted using the saddle-point values after which the integral over Gaussian fluctuations cancels with the same integral in the denominator. In other words, we have reproduced the fact that each of the trivial minima (4.4) results in Eq. (4.6).

Minima of the action (4.3) can also be obtained from the classical equation of motion

$$-\ddot{x}-\mu^2 x+\lambda x^3 = 0\,. \tag{4.18}$$

The trivial minima (4.4) obviously satisfy this equation.

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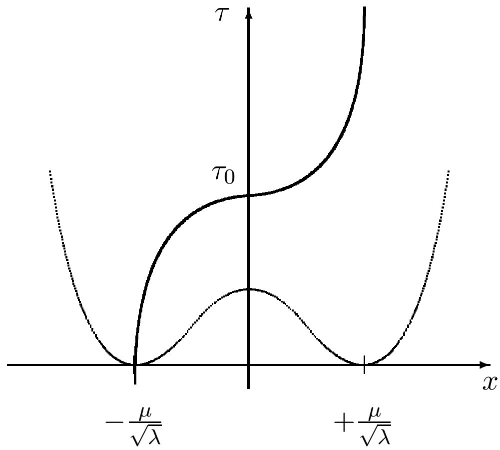

Fig. 4.2. Graphical representation of the one-kink solution (4.19) as a function of τ.

However, another solution of the classical equation of motion (4.18) exists:

$$x_{\rm inst}(\tau - \tau_0) = \frac{\mu}{\sqrt{\lambda}} \tanh \frac{\mu\,(\tau - \tau_0)}{\sqrt{2}}\,, \tag{4.19}$$

which is associated with another (local) minimum of the classical action. This solution is called an *instanton* or a pseudoparticle. The arbitrary constant τ_0 in Eq. (4.19) is the position of the center of the instanton.

The solution (4.19) is also known as a *kink* in this quantum-mechanical problem. It interpolates between the two minima (4.4) when τ changes from $-\infty$ to $+\infty$ as depicted in Fig. 4.2. Also shown in this figure is the double-well potential, $V(x)$, from Fig. 4.1.

An analogous solution which interpolates between $\mu/\sqrt{\lambda}$ at $\tau = -\infty$ and $-\mu/\sqrt{\lambda}$ at $\tau = +\infty$ is called an *anti-instanton*. It differs from Eq. (4.19) by an overall minus sign:

$$x_{\rm ainst}(\tau - \tau_0) = -\frac{\mu}{\sqrt{\lambda}} \tanh \frac{\mu\,(\tau - \tau_0)}{\sqrt{2}}\,, \tag{4.20}$$

and is obviously also a solution of the classical equation (4.18).

Problem 4.2 Find all solutions of Eq. (4.18) with the boundary conditions $x(-\infty) = -\mu/\sqrt{\lambda}$ and $x(+\infty) = \mu/\sqrt{\lambda}$.

Solution Equation (4.18) looks like Newton's equation for a classical particle, with unit mass, in the upside-down potential $-V(x)$ (its shape can be obtained from that depicted in Fig. 4.1 by reflecting with respect to the horizontal axis

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$V = 0$). The first integral of motion is the energy

$$\mathcal{E} \;=\; \frac{1}{2}\dot{x}^2 - \frac{\lambda}{4}\left(x^2 - \frac{\mu^2}{\lambda}\right)^2 \tag{4.21}$$

which is obviously conserved owing to Eq. (4.18).

Equation (4.21) can easily be solved for the velocity

$$\dot{x} \;=\; \sqrt{2[\mathcal{E} + V(x)]}, \tag{4.22}$$

where we have chosen the positive sign according to the boundary condition. It also says that $\mathcal{E} = 0$ in order for the particle to stay at $x = \mu/\sqrt{\lambda}$ for $\tau \to \infty$, since this point is associated with the maximum of $-V(x)$. Therefore, we find

$$\dot{x} \;=\; \sqrt{\frac{\lambda}{2}}\left(\frac{\mu^2}{\lambda} - x^2\right), \tag{4.23}$$

which after integration results in Eq. (4.19) with τ_0 being the integration constant. It is evident that the solution is unique.

For the instanton (or anti-instanton) minimum, one finds, substituting in Eq. (4.3),

$$S\left[x_{\text{inst}}\right] \;=\; \frac{2\sqrt{2}\,\mu^3}{3\lambda}, \tag{4.24}$$

which differs only by sign from the exponent in Eq. (4.10) for the energy splitting ΔE.

4.3 Instanton contribution to path integral

The contribution of the instanton configuration looks as if it is suppressed in the path integral by a factor of $\exp\left(-S\left[x_{\text{inst}}\right]\right)$, but, in fact, this exponential is multiplied by τ since the instanton has a zero mode. This factor of τ appears after an integration over the collective coordinate τ_0 – the instanton center. The explicit result for the one-kink contribution to the correlator (4.16) may be written as [Pol77]

$$\langle\, x(0)\, x(\tau)\rangle \;=\; \frac{\mu^2}{\lambda}\left[1 - C\tau\sqrt{\frac{2\sqrt{2}\,\mu^3}{3\lambda}}\exp\left(-\frac{2\sqrt{2}\,\mu^3}{3\lambda}\right)\right], \tag{4.25}$$

where C is a (dimensional) constant.

Problem 4.3 Derive Eq. (4.25) using the Faddeev–Popov method to deal with the collective coordinate τ_0.

Solution Let us approximate the path integrals in the numerator and denominator of Eq. (4.16) for small λ by the sum of the contributions from the trivial minima (4.4) and the one-kink minima (4.19) and (4.20). Since the one-kink

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contribution is suppressed by $\exp(-S[x_{\rm inst}])$, we can expand the denominator to give

$$\langle\, x(0)\, x(\tau)\rangle \;=\; \frac{\mu^2}{\lambda} + \mathrm{e}^{-S[x_{\rm inst}]}\, \frac{\displaystyle\int \mathcal{D}x(\tau)\left[x(0)x(\tau) - \frac{\mu^2}{\lambda}\right]\mathrm{e}^{-(S[x]-S[x_{\rm inst}])}}{\displaystyle\int \mathcal{D}\chi(\tau)\,\mathrm{e}^{-\int d\tau\left(\frac{1}{2}\dot\chi^2+\mu^2\chi^2\right)}}\,, \tag{4.26}$$

where the path integral in the numerator is over fluctuations around the instanton solution (4.19). The normalizing factor in the denominator is associated with averaging over the Gaussian fluctuations around the trivial minima (4.4), the potential energy of which is described by the quadratic term in Eq. (4.7). There are two such trivial minima (x_+ and x_-) and two one-kink minima (instanton and anti-instanton) so these factors of 2 cancel.

Keeping the quadratic term in the expansion around the instanton:

$$x(\tau) \;=\; x_{\rm inst}(\tau-\tau_0) + \chi(\tau-\tau_0)\,, \tag{4.27}$$

one obtains

$$S[x] - S[x_{\rm inst}] \;=\; \frac{1}{2}\int \mathrm{d}\tau\left(\dot\chi^2 - \mu^2\chi^2 + 3\lambda x_{\rm inst}^2\chi^2\right)\,. \tag{4.28}$$

The fluctuations around the instanton are Gaussian except for one mode, which is associated with a translation of the instanton center, τ_0. This zero mode is given by

$$\chi_0(\tau) \;\propto\; \dot x_{\rm inst}(\tau)\,. \tag{4.29}$$

This is obvious because

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} - \mu^2 + 3\lambda x_{\rm inst}^2\right)\dot x_{\rm inst} \;=\; 0 \tag{4.30}$$

as a result of differentiating Eq. (4.18) with respect to τ_0.

To deal with the zero mode, let us insert

$$1 \;=\; \int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\,\delta(u[x]-\tau) \tag{4.31}$$

into the path integral in the numerator on the RHS of Eq. (4.26). Here $u[x]$ is determined by the equation

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\, y(\tau - u[x])\, x(\tau) \;=\; 0 \tag{4.32}$$

with

$$y(\tau) \;=\; \frac{\dot x(\tau)}{\left[\displaystyle\int_{-\infty}^{+\infty} \mathrm{d}t\,\dot x^2(t)\right]^{1/2}} \tag{4.33}$$

which is the normalized derivative of $x(\tau)$.

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Under the translation,

$$\tau \quad \rightarrow \quad \tau' \;=\; \tau - \tau_0\,, \tag{4.34}$$

one obtains

$$x(\tau) \quad \rightarrow \quad x(\tau') \;=\; x(\tau - \tau_0)\,. \tag{4.35}$$

This leaves the measure and the action in the path integral (4.26) invariant, while

$$u[x] \quad \rightarrow \quad u[x] + \tau_0\,. \tag{4.36}$$

Therefore, the integration over the instanton center, τ_0, in the numerator of Eq. (4.26) factorizes and we find

$$\begin{aligned}\int \mathcal{D}x(\tau)\left(x(0)\,x(\tau) - \frac{\mu^2}{\lambda}\right)\,\mathrm{e}^{-(S[x]-S[x_{\text{inst}}])}&\\ = \int\limits_{-\infty}^{+\infty} \mathrm{d}\tau_0\left[x_{\text{inst}}(-\tau_0)\,x_{\text{inst}}(\tau-\tau_0) - \frac{\mu^2}{\lambda}\right]&\\ \times \int \mathcal{D}\chi(\tau)\,\delta(u[x_{\text{inst}}(\tau)+\chi(\tau)])\,\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}\tau\left(\dot{\chi}^2-\mu^2\chi^2+3\lambda x_{\text{inst}}^2\chi^2\right)}&.\end{aligned} \tag{4.37}$$

We have substituted the integration over the zero mode χ_0 by integration over the collective coordinate τ_0. The remaining path integral is finite since the integration runs over directions which are orthogonal to the zero mode.

The integral over τ_0 is equal to

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau_0\left[x_{\text{inst}}(-\tau_0)\,x_{\text{inst}}(\tau-\tau_0) - \frac{\mu^2}{\lambda}\right] \;=\; -\frac{2\mu^2}{\lambda}\tau \tag{4.38}$$

as $\lambda \rightarrow 0$. This is because

$$x_{\text{inst}}(\tau - \tau_0) \;=\; \frac{\mu}{\sqrt{\lambda}}\,\text{sign}\,(\tau-\tau_0) \tag{4.39}$$

as $\lambda \rightarrow 0$.

Expanding the delta-function in χ:

$$\delta(u[x]) \;=\; \left|\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\,\dot{y}_{\text{inst}}(\tau)\,x_{\text{inst}}(\tau)\right|\,\delta\!\left(\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\,y_{\text{inst}}(\tau)\chi(\tau)\right), \tag{4.40}$$

and noting that

$$\int\limits_{-\infty}^{+\infty} \mathrm{d}\tau\,\dot{x}_{\text{inst}}^2(\tau) \;=\; \frac{2\sqrt{2}\,\mu^3}{3\lambda}\,, \tag{4.41}$$

we obtain

$$\int \mathcal{D}\chi(\tau)\,\delta(u[x_{\rm inst}(\tau)+\chi(\tau)])\,{\rm e}^{-\frac{1}{2}\int {\rm d}\tau\left(\dot\chi^2-\mu^2\chi^2+3\lambda x_{\rm inst}^2\chi^2\right)}$$
$$= \sqrt{\frac{2\sqrt{2}\,\mu^3}{3\lambda}}\int \mathcal{D}\chi(\tau)\,\delta\left(\int\limits_{-\infty}^{+\infty} {\rm d}\tau\, y_{\rm inst}(\tau)\chi(\tau)\right)\,{\rm e}^{-\frac{1}{2}\int {\rm d}\tau\left(\dot\chi^2-\mu^2\chi^2+3\lambda x_{\rm inst}^2\chi^2\right)}. \tag{4.42}$$

Note the appearance of the factor of $\sqrt{S[x_{\rm inst}]}$.
We have thus obtained Eq. (4.25) with

$$C = 2\,\frac{\displaystyle\int \mathcal{D}\chi(\tau)\,\delta\left(\int_{-\infty}^{+\infty} {\rm d}\tau\, y_{\rm inst}(\tau)\chi(\tau)\right)\,{\rm e}^{-\frac{1}{2}\int {\rm d}\tau\left(\dot\chi^2-\mu^2\chi^2+3\lambda x_{\rm inst}^2\chi^2\right)}}{\displaystyle\int \mathcal{D}\chi(\tau)\,{\rm e}^{-\int {\rm d}\tau\left(\frac{1}{2}\dot\chi^2+\mu^2\chi^2\right)}}\,. \tag{4.43}$$

Problem 4.4 Calculate the ratio of determinants in Eq. (4.43).

Solution Let us introduce the notation

$$z = \frac{\mu\tau}{\sqrt{2}}\,, \qquad D = \frac{d}{dz}\,. \tag{4.44}$$

Noting that

$$\lambda x_{\rm inst}^2(\tau) = \mu^2\left(1-\frac{1}{\cosh^2 z}\right), \tag{4.45}$$

we can rewrite the ratio of determinants as

$$B^{-2} = \frac{4\pi}{\mu^2}\frac{\det'\left[-D^2+4-6/\cosh^2 z\right]}{\det\left[-D^2+4\right]}\,. \tag{4.46}$$

The notation $\det'$ means that the zero eigenvalue is excluded. An extra factor of 2π comes from the normalization of the Gaussian integral in the denominator which involves one further integral.

The RHS of Eq. (4.46) can be calculated using the limiting procedure

$$\frac{\det'\left[-D^2+4-6/\cosh^2 z\right]}{\det\left[-D^2+4\right]} = \lim_{\omega\to 2}\frac{\det\left[-D^2+\omega^2-6/\cosh^2 z\right]}{(\omega^2-4)\det\left[-D^2+\omega^2\right]}\,. \tag{4.47}$$

To compute the ratio of the Fredholm determinants

$$\mathcal{R}_\omega[v] \equiv \frac{\det\left[-D^2+\omega^2+v(z)\right]}{\det\left[-D^2+\omega^2\right]} \tag{4.48}$$

for the potential

$$v(z) = -\frac{6}{\cosh^2 z}\,, \tag{4.49}$$

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let us note that

$$\begin{aligned}\frac{\partial}{\partial\omega^2}\ln\mathcal{R}_\omega[v] &= \mathrm{Tr}\left[\frac{1}{-D^2+\omega^2+v(z)}\right]-\mathrm{Tr}\left[\frac{1}{-D^2+\omega^2}\right]\\ &= \int_{-\infty}^{+\infty}\mathrm{d}z\left[R_\omega(z,z;v)-\frac{1}{2\omega}\right],\end{aligned}\tag{4.50}$$

where the diagonal resolvent $R_\omega(z,z;v)$ is defined by Eq. (1.123) with $G=1$ and $V\equiv v$. The term $1/2\omega$ on the RHS, which equals the diagonal resolvent in the free case when $v=0$ (see Eq. (1.38)), comes from the free determinant in the denominator on the RHS of Eq. (4.48).

A crucial observation is that the diagonal resolvent for the potential (4.49) is given by the simple formula

$$R_\omega(z,z;v) = \frac{1}{2\omega}-\frac{v(z)}{4\omega(\omega^2-1)}+\frac{v^2(z)}{8\omega(\omega^2-1)(\omega^2-4)},\tag{4.51}$$

which can easily be verified by substituting into the Gel'fand–Dikii equation (1.127) with $\mathcal{G}=1$. The reason for this is that the potential (4.49) is integrable and possesses two bound states (see, for example, §23 of [LL74]).

Calculating the integral over z on the RHS of Eq. (4.50), using the formulas

$$\int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{\cosh^2 z} = 2\,,\qquad \int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{\cosh^4 z} = \frac{4}{3}\,,\tag{4.52}$$

we obtain

$$\frac{\partial}{\partial\omega^2}\ln\mathcal{R}_\omega[v] = \frac{1}{\omega}\left(\frac{1}{\omega^2-1}+\frac{2}{\omega^2-4}\right),\tag{4.53}$$

which is easily integrated over ω to give

$$\frac{\det\left[-D^2+\omega^2-6/\cosh^2 z\right]}{\det\left[-D^2+\omega^2\right]} = \frac{(\omega-2)(\omega-1)}{(\omega+2)(\omega+1)}\,.\tag{4.54}$$

The integration constant has been determined by requiring that

$$\lim_{\omega\to\infty}\mathcal{R}_\omega[v] = 1\,.\tag{4.55}$$

Substituting into Eq. (4.47), we obtain

$$C = 2B = \sqrt{\frac{48}{\pi}}\mu\tag{4.56}$$

which coincides with the constant in Eq. (4.10).

For other methods of calculating the ratio of determinants in the one-instanton contribution, see the original papers [Lan67, Pol77], the reviews [Col77, VZN82] or Chapter 4 of the book [Pol87].

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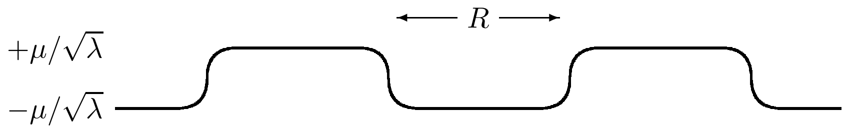

Fig. 4.3. The many-kink configuration $x_{M\text{-kink}}(\tau)$ which is combined from the solution (4.19).

4.4 Symmetry restoration by instantons

At $\tau \sim 1/\Delta E$, many kinks become essential. A many-kink "solution" can be approximately constructed from several single kinks and antikinks, which are separated along the τ-axis by the some distance $R \gg 1/\mu$, since the interaction between kinks would be $\sim \exp(-\mu R)$. Such a configuration is depicted in Fig. 4.3 for the case when the number of kinks is equal to the number of antikinks. An analogous configuration with the number of kinks being one more greater than the number of antikinks connects the $-\mu/\sqrt{\lambda}$ and $\mu/\sqrt{\lambda}$ vacua.

It is not an exact solution of Eq. (4.18) since the kink and the antikink attract and have a tendency to annihilate. However, it is an approximate solution as $\lambda \to 0$.

Analytically, the M-kink configuration can be represented as

$$x_{M\text{-kink}}(\tau) \quad = \quad \frac{\mu}{\sqrt{\lambda}} \prod_{i=1}^{M} \operatorname{sign}(\tau - \tau_i)\,, \tag{4.57}$$

where τ_i are the centers of the instantons (or anti-instantons), from which the M-kink configuration is built out, and

$$\tau_1 \leq \tau_2 \leq \cdots \leq \tau_M\,. \tag{4.58}$$

Equation (4.57) assumes that the kinks do not interact and are infinitely thin as $\lambda \to 0$. The action of the configuration (4.57) is therefore given by

$$S[x_{M\text{-kink}}] \quad = \quad \frac{2\sqrt{2}\mu^3}{3\lambda} M\,, \tag{4.59}$$

i.e. it equals M times the action for the one-kink case.

Summing over many-kink configurations, one finds [Pol77]

$$\langle\, x(0)\, x(\tau)\rangle \quad = \quad \frac{\mu^2}{\lambda}\, \mathrm{e}^{-\tau\Delta E}\,, \tag{4.60}$$

where ΔE is given by Eq. (4.10). The $x \to -x$ symmetry is now restored as $\tau \to \infty$. This restoration is produced by instantons = classical trajectories with a finite (Euclidean) action.

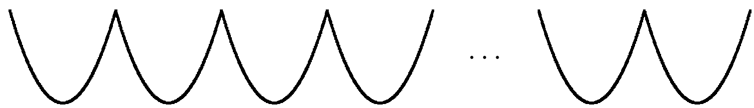

Fig. 4.4. Graphical representation of a periodic potential.

Problem 4.5 Obtain the exponentiation of the one-kink contribution (4.25) after summing over the M-kink configurations (4.57) in the dilute gas approximation when the interaction between kinks is disregarded.

Solution The calculation of the contribution of the M-kink configuration (4.57) to the path integral is quite analogous to that for the one-kink case which is described in Problem 4.3. One finds

$$\langle\, x(0)\, x(\tau)\,\rangle \;=\; \frac{\mu^2}{\lambda}\sum_{M=0}^{\infty}(-\Delta E)^M\int_0^{\tau}\mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\cdots\int_0^{\tau_{M-1}}\mathrm{d}\tau_M\,, \tag{4.61}$$

which reproduces Eq. (4.60) by noting that the ordered integral is equal to

$$\int_0^{\tau}\mathrm{d}\tau_1\int_0^{\tau_1}\mathrm{d}\tau_2\cdots\int_0^{\tau_{M-1}}\mathrm{d}\tau_M \;=\; \frac{\tau^M}{M!}\,. \tag{4.62}$$

This calculation is very similar to that in statistical mechanics for the exponentiation of a single-particle contribution to the partition function in the case of an ideal gas.

4.5 Topological charge and θ-vacua

Let us consider a periodic potential whose period equals 1, which is depicted in Fig. 4.4. It can be viewed as being defined on a circle S^1 of unit length. The boundary conditions are

$$\left.\begin{array}{lcll} x(1) & = & x(0) & \boxed{\text{in perturbation theory}}\,, \\ x(1) & = & x(0)+n & \boxed{\text{for } n\text{-instanton solution}}\,. \end{array}\right\} \tag{4.63}$$

The multi-instanton solution always exists because of the topological formula*

$$\pi_1(S^1) \;=\; \mathbb{Z}\,, \tag{4.64}$$

where $\pi_k(M)$ is the kth homotopy group with elements that are classes of continuous maps of the k-sphere S^k onto M. Equation (4.64) describes the fact that an (integer) winding number $n \in \mathbb{Z}$ is associated with the mapping $S^1 \to S^1$, which counts how many times the target is covered.

* See, for example, the book [DNF86] (§17.5 of Part II).

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We see the difference between the M-kink configuration for the double-well potential and the multi-instanton solution for the periodic potential. The former was not an exact solution of the classical field equation (4.18). Only a single instanton or anti-instanton was a solution that connects the two vacua. This is why we need a periodic potential for the multi-instanton solution to exist owing to the topological argument.

The value of n in the boundary condition (4.63) is called the *topological charge* of the instantons, while $n < 0$ is associated with anti-instantons. The vacuum states are labeled by n: $|n\rangle$. The n-instanton configuration connects the $|m\rangle$ and $\langle m+n|$ states. Therefore, instantons are associated in Minkowski space with the process of tunneling between topologically distinct vacua* rather than with real particles. For this reason, they are sometimes called pseudoparticles in Euclidean space.

It is convenient to consider another representation of vacuum states

$$|\theta\rangle \;=\; \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\theta n}\,|n\rangle\,, \tag{4.65}$$

which are called the θ-vacua. The θ-vacua are orthogonal

$$\langle\theta \mid \theta'\rangle \;=\; \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(\theta n-\theta' m)}\,\langle m \mid n\rangle \;=\; \delta_{2\pi}\left(\theta-\theta'\right), \tag{4.66}$$

where $\delta_{2\pi}$ is a periodic delta-function with period 2π. Here we have used the orthogonality of the n-states:

$$\langle m \mid n\rangle \;=\; \delta_{mn}\,. \tag{4.67}$$

The θ-vacuum partition function is given by

$$Z(\theta) \;=\; \int \mathcal{D}x\,\mathrm{e}^{-S[x]+\mathrm{i}\theta\int_0^1 \mathrm{d}\tau\,\dot{x}(\tau)}. \tag{4.68}$$

Here in the exponent θ is multiplied by the topological charge

$$\int_0^1 \mathrm{d}\tau\,\dot{x}(\tau) \;=\; x(1)-x(0) \tag{4.69}$$

* The Minkowski-space interpretation of instantons is attributed to V.N. Gribov (unpublished). It is based on the fact that when the particle is localized in one of the two wells its momentum is indefinite and can sometimes be very large so that the proper energy is above the barrier between the two wells. Such a particle jumps from the given well to the other one. The characteristic time of this process is small in the typical units given by μ. In other words, this process is instantaneous, which explains the term “instanton” as introduced by ’t Hooft. The exponential suppression with λ of the one-instanton contribution (4.25) represents quantitatively the fact that the probability of having large momentum is small.

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which never appears in perturbation theory. Therefore, the partition function (4.68) can be alternatively represented as

$$Z(\theta) = \sum_n \int_{x(1)=x(0)+n} \mathcal{D}x\, \mathrm{e}^{-S[x]+\mathrm{i}\theta n}. \tag{4.70}$$

The second term in the exponent in Eq. (4.68) is known as the θ-*term*. The parameter θ plays the role of a new fundamental constant that does not show up in perturbation theory. The amplitude of physical processes generated by instantons may depend on θ.

Remark on description of instantons

A description of instantons in the first-quantized language can only be given in quantum mechanics (where the first and second quantizations do not differ essentially). The path-integral representation (4.16) is more in the spirit of second quantization, which is discussed in Chapter 2, where $x(\tau)$ plays the role of a field that depends on the one-dimensional coordinate τ.

Remark on instantons in Yang–Mills theory

In the Yang–Mills theory, instantons are conveniently described by a (Euclidean) path integral over fields. The saddle-point equation, which describes instantons in the $SU(2)$ Yang–Mills theory, is given by [BPS75]

$$F^a_{\mu\nu}(x) = \widetilde{F}^a_{\mu\nu}(x)\,, \tag{4.71}$$

for which nontrivial solutions exist owing to the fact that the mapping of the asymptotic boundary S^3 of four-dimensional Euclidean space onto $SU(2)$ is nontrivial:

$$\pi_3\,(SU(2)) = \mathbb{Z}\,. \tag{4.72}$$

Correspondingly, the topological charge is given by*

$$n = \frac{g^2}{16\pi^2}\int \mathrm{d}^4x \sum_{a=1}^{3} F^a_{\mu\nu}(x)\widetilde{F}^a_{\mu\nu}(x)\,, \tag{4.73}$$

which equals one-half of the nonconservation of the axial charge given by the Minkowski-space integral of the chiral anomaly (3.63). This expression is also known in topology as the Pontryagin index or the second Chern class. See, for example, the lectures/reviews [Col77, VZN82, SS98] and the book [Shi94] for an introduction to instantons in Yang–Mills theory.

* Concerning the coefficient, see the footnote on p. 59.

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Bibliography to Part 1

Reference guide

The operator formalism in quantum field theory is described in the canonical books [AB69, BS76, BD65] which were written in the 1950s or at the beginning of the 1960s. Modern textbooks on this subject include those by Brown [Bro92] and Weinberg [Wei98].

Feynman disentangling is contained in the original paper [Fey51], the appendices of which are especially relevant. A classic book on path integrals in quantum mechanics is that by Feynman and Hibbs [FH65]. The path-integral approach to the very closely related problem of Brownian motion is discussed in the books [Kac59, Sch81, Wie86, Roe94]. Many information on path integrals can be found in the book by Kleinert [Kle95].

An introduction to path integrals in quantum mechanics and quantum field theory can be found in many books. I shall list some of those that I have on my bookshelf: [Ber86, Pop91, FS80, IZ80, Ram89, Sak85, Riv88]. The ordering is according to the appearance of the first edition. The book by Berezin [Ber86], which is mathematically more rigorous, contains an excellent description of operations with Grassmann variables.

An introduction to path integrals in statistical mechanics can be found in the books [Kac59, Fey72, Pop91, Wie86, ID91, Roe94]. The well-written book by Parisi [Par88] describes a modern view of the relation between statistical mechanics and quantum field theory. A very good, while slightly more advanced, book where contemporary problems of quantum field theory and statistical mechanics are discussed using the unified language of Euclidean path integrals is that by Polyakov [Pol87].

The derivation of quantum anomalies from the noninvariance of the measure in the path integral is contained in the original papers [Ver78, Fuj79, Fuj80] (see also the review [Mor86]). It can also be found in Chapter 22 of the book by Weinberg [Wei98].

Instantons in the Yang–Mills theory were discovered by Belavin, Polyakov, Schwartz and Tyupkin [BPS75]. The role of instantons in quantum mechanics is clarified in the original paper by Polyakov [Pol77]. Their description is given in the books by Sakita [Sak85] and Polyakov [Pol87]. The review articles [Col77, VZN82, SS98] are also useful for an introduction to the subject. The original papers on instantons in quantum field theory are collected in the book edited by Shifman [Shi94].

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