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ON THE DISTRIBUTION OF HARMONIC MEASURE ON SIMPLY CONNECTED PLANAR DOMAINS

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Abstract

For a simply connected planar domain D with $0 \in D$ and $\operatorname{dist}(0, \partial D) = 1$, let $h_D(r)$ be the harmonic measure of $\partial D \cap \{|z| \leq r\}$ evaluated at 0. The function $h_D(r)$ is the distribution of harmonic measure. It has been studied by B. L. Walden and L. A. Ward. We continue their study and answer some questions raised by them by constructing domains with pre-specified distribution.

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1. Introduction

If D is a domain in the extended complex plane $\mathbb{C}_\infty$ and E is a Borel set on the boundary ∂D of D, we will denote by $\omega(z, E, D)$ the harmonic measure of E with respect to D, evaluated at the point $z \in D$. More generally, if K is a Borel set in the closure of D, then $\omega(z, K, D)$ will denote the harmonic measure of $\partial G \cap \partial K$ with respect to the component G of $D \setminus K$ that contains z.

Let $D \subset \mathbb{C}_\infty$ be a domain with $0 \in D$. The *distribution of harmonic measure* at 0 is the function

$$h_D(r) = \omega(0, \partial D \cap \{z : |z| \leq r\}, D). \tag{1.1}$$

This function has been used for a long time because its behavior near ∞ indicates how large D is; see for example [1, 2, 3, (page 112)] and references therein.

Here we restrict our attention to simply connected domains and study $h_D(r)$ for small values of r. First we need a normalization: Let $\mathcal{D}$ denote the class of all simply

connected domains D such that $0 \in D$ and $\operatorname{dist}(0, \partial D) = 1$. We will study the behaviour of $h_D(r)$ near $r = 1$, for $D \in \mathscr{D}$. Such a study has been initiated by Walden and Ward [4]. They used the Beurling-Nevanlinna projection theorem to prove that

$$h_D(r) \geq \frac{4}{\pi} \tan^{-1} \frac{\sqrt{r}-1}{\sqrt{r}+1}, \quad D \in \mathscr{D} \tag{1.2}$$

and

$$\liminf_{r \to 1+} \frac{h_D(r)}{r-1} \geq \frac{1}{\pi}, \quad D \in \mathscr{D}. \tag{1.3}$$

They also gave examples of domains $D \in \mathscr{D}$ such that

$$h_D(r) \approx (r-1)^\beta, \quad \text{near } r = 1, \tag{1.4}$$

for $\beta = 1/4$ and for $1/2 \leq \beta < 1$. The notation '$\approx$' in (1.4) means that $C_1 \leq h_D(r)/(r-1)^\beta \leq C_2$, where C_1, C_2 are positive constants that depend only on D (and not on r).

Walden and Ward [4] posed several interesting problems for the distribution of harmonic measure. Here we state only some of them:

QUESTION 1 ([4, Question 12]). Is it true that for every $D \in \mathscr{D}$, $h_D(r) \approx (r-1)^\beta$ near $r = 1$, for some β with $0 \leq \beta \leq 1$?

QUESTION 2 (see [4, pages 296–297]). Is it true that for every β with $0 < \beta < 1$ there is a domain $D \in \mathscr{D}$ such that $h_D(r) \approx (r-1)^\beta$ near $r = 1$?

In fact, in [4] a stronger conjecture is stated:

QUESTION 3 ([4, Conjecture 11]). Is it true that for any $0 < \beta < 1$, there exists $D \in \mathscr{D}$ such that

$$\lim_{r \to 1+} \frac{h_D(r)}{(r-1)^\beta} = c, \tag{1.5}$$

where $0 < c < \infty$?

QUESTION 4 ([4, Question 14]). If D_n is a sequence in $\mathscr{D}$ and $D_n \to D$ in the sense of Carathéodory, does $h_{D_n}(r) \xrightarrow[n \to \infty]{} h_D(r)$ for almost all r?

In Section 3 we present an example that shows that the answer to Question 2 is positive. Given $0 < \beta < 1$, we construct a domain $\Omega \in \mathscr{D}$ with $h_\Omega(r) \approx (r-1)^\beta$. However, we do not know if the limit $\lim_{r \to 1+} h_\Omega(r)/(r-1)^\beta$ exists. So our example does not give an answer to Question 3. In Section 4, we use a similar construction which shows that the answer to Question 1 is negative. For the domain G that we

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construct, h_G has logarithmic growth. Finally, we note that the answer to Question 4 is known to be positive. This follows at once from [1, Theorem 7.7] which is due to Baernstein. In Section 2 we state and prove some simple lemmas which we will need later.

2. Some lemmas

First a piece of notation: If $z_1, z_2 \in \mathbb{C}$, $[z_1, z_2]$ will denote the closed line segment joining z_1 with z_2. Open intervals (z_1, z_2) will be considered too.

The first lemma is a slight extension of the Carleman principle (domain monotonicity of harmonic measure) [3].

LEMMA 2.1. *Let $D_1 \subset D_2$ be two domains in $\mathbb{C}_\infty$ and let A be an open Jordan arc in $\partial D_1 \cap \partial D_2$. Let K be a Jordan arc in D_1 joining the endpoints of A. If z_o is a point in the component G of $D_1 \setminus K$ with $\partial G \cap A = \emptyset$, then*

$$\omega(z_o, A, D_1) \leq \omega(z_o, K, D_2). \tag{2.1}$$

PROOF. We apply the maximum principle to the domain G and (2.1) follows at once. □

LEMMA 2.2. *Let $\mathbb{H}$ be the upper half-plane. There exist absolute positive constants c_1, c_2 such that if $0 < a < 0.1$, then*

$$c_1 \leq \frac{\omega(i, [-a, a], \mathbb{H})}{a} \leq c_2. \tag{2.2}$$

PROOF. The estimates (2.2) follow from a straightforward computation of $\omega(i, [-a, a], \mathbb{H})$. □

LEMMA 2.3. *Let $y_o > 0$, $S = \{z : \Re z \geq 1$ and $|\Im y| \leq y_o\}$, and $\Sigma = \mathbb{C} \setminus S$. Then for $0 < a < 0.1$,*

$$c_3 \leq \frac{\omega(0, [1 - ia, 1 + ia], \Sigma)}{a} \leq c_4, \tag{2.3}$$

where the positive constants c_3, c_4 do not depend on a and y_o.

PROOF. We use (as comparison domains with Σ) the domains $\Sigma_1 = \{z : \Re z < 1\}$ and $\Sigma_2 = \mathbb{C}_\infty \setminus [1 - iy_o, 1 + iy_o]$. For the domains Σ_1, Σ_2, harmonic measures can be computed via conformal maps. The estimates (2.3) follow then from explicit computations. □

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LEMMA 2.4. *For fixed β with $0 < \beta < 1/2$, define the function h_β with*

$$h_\beta(\varphi) = (1 + \varphi^{1/\beta}) \cos \varphi, \quad \varphi > 0.$$

There exists $\varphi_\beta > 0$ such that h_β is strictly decreasing in the interval $(0, \varphi_\beta)$.

PROOF. We differentiate the function h_β and see that $h'_\beta(\varphi) < 0$ if

$$g_\beta(\varphi) := \frac{1}{\beta} \frac{\varphi^{1/\beta - 1}}{1 + \varphi^{1/\beta}} - \tan \varphi < 0. \tag{2.4}$$

We differentiate the function g_β:

$$g'_\beta(\varphi) = \frac{(1/\beta)(1/\beta - 1 - \varphi^{1/\beta})\varphi^{1/\beta - 2}}{(1 + \varphi^{1/\beta})^2} - \frac{1}{\cos^2 \varphi}. \tag{2.5}$$

Observe that $g_\beta(0) = 0$ and $g'_\beta(0) = -1 < 0$. By continuity, $g'_\beta(\varphi) < 0$ in an interval $(0, \varphi_\beta)$, which implies that $g_\beta(\varphi) < 0$ and hence $h'_\beta(\varphi) < 0$ for $\varphi \in (0, \varphi_\beta)$. Thus the lemma is proved. □

3. Answer to Question 2

Let $0 < \beta < 1/2$ be given. We will construct a domain $\Omega \in \mathscr{D}$ with $h_\Omega(r) \approx (r-1)^\beta$ near $r = 1$. We denote by ρ, φ the polar coordinates in the plane and consider the curves $L_\beta = \{\rho e^{i\varphi} : \rho = 1 + \varphi^{1/\beta},\ \varphi \in [0, \pi/2]\}$, $\widetilde{L_\beta} = \{z : \bar{z} \in L_\beta\}$. Let Ω be the simply connected domain bounded by $L_\beta \cup \widetilde{L_\beta}$ and containing 0. It is clear that $\Omega \in \mathscr{D}$. We will prove that

$$h_\Omega(r) \approx (r-1)^\beta, \quad \text{for } r \text{ near } 1. \tag{3.1}$$

Let $1 < r < 1.1$. By the maximum principle,

$$h_\Omega(r) \geq \omega(0, \{re^{i\varphi} : |\varphi| \leq (r-1)^\beta\}, D_r) = \frac{1}{\pi}(r-1)^\beta, \tag{3.2}$$

where D_r is the disk with center 0 and radius r.

To establish an inequality in the opposite direction we first need to prove some geometric properties of the domain Ω. Let $r \in (1, r_o)$. The number r_o will be specified later. Let $\theta_r > 0$ be such that $r = 1 + \theta_r^{1/\beta}$. The circle $\{z : |z| = r\}$ intersects $\partial\Omega$ at two points, namely at $re^{i\theta_r}$ and $re^{-i\theta_r}$.

LEMMA 3.1. *Let K_r be the line segment $(re^{-i\theta_r}, re^{i\theta_r})$. Then $K_r \subset \Omega$.*

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PROOF. Let $k \in K_r$ and write $k = |k|e^{is}$, $s \in (0, \theta_r)$. We must prove that $|k| < 1 + s^{1/\beta}$. Since $k \in K_r$, we have $|k| \cos s = r \cos \theta_r$. Hence

$$|k| = r\frac{\cos \theta_r}{\cos s} = (1 + \theta_r^{1/\beta})\frac{\cos \theta_r}{\cos s}. \tag{3.3}$$

Therefore we must prove the inequality

$$(1 + \theta_r^{1/\beta}) \cos \theta_r < (1 + s^{1/\beta}) \cos s. \tag{3.4}$$

By Lemma 2.4, the function $h_\beta(\varphi) = (1 + \varphi^{1/\beta}) \cos \varphi$ is decreasing in an interval $(0, \varphi_\beta)$. Hence (3.4) is true provided that $r_o < 1 + \varphi_\beta^{1/\beta}$, and the lemma is proved. □

LEMMA 3.2. *Let* $M_r = \{z : \Re z = r \cos \theta_r,\ r \sin \theta_r < \Im z < r \cos \theta_r \tan \varphi_\beta\}$. *Then* $M_r \cap \Omega = \emptyset$.

PROOF. Let $\zeta_1 = r \cos \theta_r + iy_1$ be a point of M_r. We write $r \cos \theta_r + iy_1 = \rho_1 e^{i\varphi_1}$. With this notation we have

$$y_1 = \rho_1 \sin \varphi_1, \tag{3.5}$$

$$\rho_1 \cos \varphi_1 = r \cos \theta_r, \tag{3.6}$$

and we must prove

$$\rho_1 > 1 + \varphi_1^{1/\beta}, \tag{3.7}$$

or, equivalently (by (3.6)),

$$(1 + \theta_r^{1/\beta}) \cos \theta_r > (1 + \varphi_1^{1/\beta}) \cos \varphi_1. \tag{3.8}$$

Because of Lemma 2.4, in order to prove (3.8), it suffices to prove that

$$\theta_r < \varphi_1 < \varphi_\beta. \tag{3.9}$$

Since $\zeta_1 \in M_r$, we have

$$r \sin \theta_r < y_1 < r \cos \theta_r \tan \varphi_\beta. \tag{3.10}$$

Now using (3.5) and (3.6), it is easy to verify that (3.9) is equivalent to (3.10) and the lemma is proved. □

For r near 1, the number $r \cos \theta_r \tan \varphi_\beta$ is close to $\tan \varphi_\beta$ which is a fixed positive number. By Lemma 3.2, we can fix a number y_o close to $\tan \varphi_\beta$ such that $r \cos \theta_r + iy_o \in M_r$ for all $r \in (1, r_1)$, where r_1 is a constant greater than 1. For such a y_o consider the domain $\Omega_r = \mathbb{C} \setminus \{z : \Re z \geq r \cos \theta_r, |\Im z| \leq y_o\}$. Note that by Lemma 3.1, K_r is a crosscut of Ω. Also, it follows easily from Lemma 3.2 that Ω_r contains the component Ω' of $\Omega \setminus K_r$ with $0 \in \Omega'$. So we may apply Lemma 2.1 to obtain

$$h_\Omega(r) = \omega(0, \{z : |z| \leq r\} \cap \partial\Omega, \Omega) \leq \omega(0, K_r, \Omega_r).$$

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Therefore it remains to prove that $\omega(0, K_r, \Omega_r) \le C(r-1)^\beta$ for some positive constant C. By Lemma 2.3, $\omega(0, K_r, \Omega_r) \le c_4 r \sin\theta_r \le C\theta_r = C(r-1)^\beta$ and so we have proved that $h_\Omega(r) \le C(r-1)^\beta$. This together with (3.2) imply that $h_\Omega(r) \approx (r-1)^\beta$ near $r = 1$.

4. Answer to Question 1

We will construct a domain $G \in \mathscr{D}$ such that near $r = 1$ we have $h_G(r) \not\approx (r-1)^\beta$ for any $0 \le \beta \le 1$. This shows that the answer to Question 1 is negative. The construction of G is similar to the construction of Ω in Section 3.

Let

$$L_1 = \left\{ \rho e^{i\varphi} : \rho = 1 + \frac{1}{e^{1/\varphi} - 1}, \varphi \in (0, \pi] \right\} \cup \{1\} \quad \text{and} \quad \widetilde{L_1} = \{z : \bar{z} \in L_1\}.$$

Let G be the domain bounded by $L_1 \cup \widetilde{L_1}$ and containing 0. It is clear that $G \in \mathscr{D}$. By the maximum principle, for $r > 1$,

$$h_G(r) \ge \omega\left(0, \left\{ re^{i\varphi} : |\varphi| \le \frac{1}{\log(1 + 1/(r-1))} \right\}, D_r\right) = \frac{1}{\pi \log(1 + 1/(r-1))},$$

where D_r is the disk with center 0 and radius r. Since

$$\lim_{r \to 1+} (r-1)^\beta \log\left(1 + \frac{1}{r-1}\right) = 0, \quad \text{for any } 0 < \beta \le 1,$$

we have $h_G(r) \not\approx (r-1)^\beta$.

REMARK. One can actually prove that

$$h_G(r) \approx \log \frac{1}{1 + 1/(r-1)}, \quad \text{near } r = 1.$$

NOTE ADDED ON JANUARY 17, 2003: The paper [5] by B. L. Walden and L. A. Ward which appeared recently is very much related with the present paper and contains a more complete treatment of Questions 1–3. Some of our results are proved in [5] but the proofs are different.

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