BULL. AUSTRAL. MATH. SOC. 47H10
VOL. 45 (1992) [171–176]

# THE WEAK* KARLOVITZ LEMMA FOR DUAL LATTICES

BRAILEY SIMS

**We establish the Karlovitz lemma for a nonexpansive self mapping of a nonempty weak* compact convex set in a weak* orthogonal dual Banach lattice.**

We say a Banach space has the weak fixed point property (w–fpp) if every nonexpansive self mapping of a nonempty weak compact convex subset has a fixed point. In the case of a dual space we say it has the w*-fpp if every nonexpansive self mapping of a nonempty weak*compact convex subset has a fixed point.

Let $C$ be a nonempty weak (weak*) compact convex set and let $T : C \longrightarrow C$ be a nonexpansive mapping. The weak (weak*) compactness and Zorn's lemma ensure the existence of *minimal* nonempty weak (weak*) compact convex subsets of $C$ which are invariant under $T$. For brevity we will refer to such a set as a *weak (weak*) compact minimal invariant set for* $T$. It is readily verified that a space (dual space) has the w–fpp (w*-fpp) if and only if every such weak (weak*) compact minimal invariant set has precisely one element.

Fundamental for establishing the w–fpp for certain spaces has been the result of Brodskii and Mil'man [2], Garkarvi [3] and Kirk [7] that any such weak (weak*) compact minimal invariant set $D$ is *diametral* in the sense that, for all $x \in D$

$$\sup_{y \in D} \|x - y\| = \operatorname{diam} D := \sup_{x_1, x_2 \in D} \|x_1 - x_2\| .$$

Another useful observation has been the existence in any nonempty closed convex subset of $C$ which is invariant under $T$ of an *approximate fixed point sequence* for $T$, that is a sequence $(a_n) \subset C$ for which

$$\|a_n - Ta_n\| \longrightarrow 0.$$

(Such a sequence may be constructed by choosing $x_0$ in the set and taking $a_n$ to be the unique fixed point of the strict contraction $V_n x := (1 - 1/n)Tx + (1/n)x_0$, whose existence is ensured by the Banach contraction mapping theorem.)

Received 4th March 1991

In the weak case deeper more recent results (for example, Maurey [9]; Borwein and Sims [1]; Lin [8]) have relied on the *Karlovitz' lemma*:

(1) If $C$ is a nonempty weak compact convex set, $D \subseteq C$ is a minimal invariant set for the nonexpansive map $T : C \longrightarrow C$, and $(a_n)$ is an approximate fixed point sequence for $T$ in $D$, then

$$\lim_n \|x - a_n\| = \operatorname{diam} D, \quad \text{for all } x \in D.$$

Proofs of this result (Karlovitz [5] and Goebel [4]) have involved an appeal to Mazur's theorem; that the weak and norm closures of a convex set coincide, and so left open the question of whether a similar result holds in the weak* case. This impediment to progress in the weak* case was attacked by Khamsi [6], who established a weak* Karlovitz lemma for stable duals and dual spaces with a shrinking strongly monotone Schauder basis.

The purpose of this note is to extend these results to a weak* Karlovitz lemma for weak* orthogonal dual Banach lattices.

By analogy with Borwein and Sims [1] we say that a dual lattice $X$ is *weak* orthogonal* if whenever $(x_n)$ converges *weak** to 0 we have

$$\lim_n \| \, |x_n| \wedge |x| \, \| = 0, \text{ for all } x \in X.$$

In general it may be convenient to interpret $(x_n)$ as a net. However in smoothable dual spaces, in particular separable dual spaces, sequences suffice.

Proofs of the Brodskii–Mil'man result and the Karlovitz lemma have directly, or indirectly, relied on an idea captured in the following lemma which was first made explicit in the weak case by Maurey [9] while proving the w–fpp for $c_0$ and reflexive subspaces of $\mathcal{L}_1[0,1]$.

**Lemma 1.** *Let $T$ be a nonexpansive mapping of a nonempty weak (weak*) compact convex set and let $D$ denote a minimal invariant set for $T$. If $\psi : D \longrightarrow \mathbf{R}$ is a weak (weak*) lower semi-continuous convex mapping with $\psi(Tx) \leqslant \psi(x)$ for all $x \in D$, then $\psi$ is constant on $D$.*

**Proof:** Since $D$ is weak (weak*) compact and $\psi$ is weak (weak*) lower semi-contininuous, $\psi$ achieves its minimum on $D$. Let $x_0 \in D$ be such that $\psi(x_0) = \min \psi(D)$ and let $E = \{x \in D : \psi(x) = \psi(x_0)\}$; then $E$ is a nonempty weak (weak*) closed convex set which is invariant under $T$. Thus, by minimality $E = D$, establishing the lemma. $\square$

To illustrate how the lemma may be used we prove the result of Brodskii – Mil'man in the weak* case. A substantially simplified version of the same argument establishes the corresponding result for weak compact sets.

**THEOREM 2.** *If $D$ is a weak* compact minimal invariant set for a nonexpansive mapping $T$ then $D$ is diametral.*

PROOF: It suffices to verify that $\psi$ defined by

$$\psi(x) := \sup\{\|x - y\| : y \in D\}$$

satisfies the hypotheses for Lemma 1, as then $\psi$ is a constant on $D$ with value equal to

$$\sup_{x\in D} \psi(x) = \sup_{x\in D}\sup_{y\in D} \|x - y\| = \operatorname{diam}(D).$$

To complete the proof we first note that, since $\|\cdot\|$ is a dual norm, $\psi$ is the supremum of weak* lower semi-continuous functions and so is itself weak* lower semi-continuous. Next, observe that

$$\psi(x) = \sup_{y\in \operatorname{co} T(D)} \|x - y\|.$$

This follows, since by the minimality of $D$, we have $D = \overline{\operatorname{co}}^{w^*} T(D)$, so given $\varepsilon > 0$ there exists a $y_\varepsilon \in D$ with $\psi(x) - \varepsilon \leqslant \|x - y_\varepsilon\|$ and a net $y_\alpha \rightharpoonup^{w^*} y_\varepsilon$ with $y_\alpha \in \operatorname{co} T(D)$. Thus,

$$\psi(x) - \varepsilon \leqslant \|x - y_\varepsilon\| \leqslant \liminf_\alpha \|x - y_\alpha\|$$

and so there exists a $y \in \operatorname{co} T(D)$ with $\psi(x) - 2\varepsilon \leqslant \|x - y\|$ establishing the claim.

It now follows by standard convexity arguments that

$$\psi(x) = \sup_{y\in T(D)} \|x - y\|,$$

from which it is readily seen that $\psi(Tx) \leqslant \psi(x)$, completing the proof. ∎

The Karlovitz' lemma for a weak compact minimal invariant set $D$ follows from the weak lower semi-continuity of the function $\psi(x) := \limsup_n \|x - a_n\|$, where $(a_n)$ is an approximate fixed point sequence for $T$ in $D$, which in turn follows since the *epigraph* of $\psi$ is a norm closed convex set and hence also weak closed by Mazur's theorem.

As the following result shows, Karlovitz' lemma also holds for a weak* compact minimal invariant set $D$ whenever functions of the above form are weak* lower semi-continuous.

**LEMMA 3.** *Let $(a_n)$ be an approximate fixed point sequence for the nonexpansive mapping $T$ in the weak* compact minimal invariant set $D$. If for each subsequence $(y_k)$ of $(a_n)$ the function*

$$\psi(x) := \limsup_k \|x - y_k\|,$$

*is weak* lower semi-continuous on $D$, then*

$$\lim_n \|x - a_n\| = \operatorname{diam}(D), \quad \textit{for all} \quad x \in D.$$

PROOF: Let $(y_k)$ be any subsequence of the approximate fixed point sequence $(a_n)$ then lemma 1 applies to show that $\psi(x) := \limsup_k \|x - y_k\|$ is constant on $D$ with value $c$ say. Now let $(y_{k_\alpha})$ be a subnet with $y_{k_\alpha} \xrightarrow{w^*} y_0$; then

$$c \geqslant \limsup_\alpha \|x - y_{k_\alpha}\| \geqslant \liminf_\alpha \|x - y_{k_\alpha}\| \geqslant \|x - y_o\|$$

and so

$$c \geqslant \sup_{x\in D} \|x - y_0\| = \operatorname{diam}(D), \quad \text{by Theorem 2.}$$

Thus for each subsequence $(y_k)$ of $(a_n)$ we have

$$\limsup_k \|x - y_k\| = \operatorname{diam}(D),$$

for all $x$ in $D$ and the result follows. □

Unfortunately in a dual space not all functions of the form $\psi(x) := \limsup_n \|x - y_n\|$, even when $(y_n)$ is a norm one weak* null sequence, need be weak* lower semi-continuous.

EXAMPLE 4. In $\ell_\infty$ define $\psi$ by

$$\psi(x) := \limsup_n \|x - y_n\|,$$

where

$$y_n(i) = \begin{cases} 0, & i = 1, 2, \cdots, n-1, \\ -1, & i = n, \cdots \end{cases}$$

Then for

$$x_n(i) := \begin{cases} 1, & i = 1, \cdots, n, \\ 0, & \text{otherwise}, \end{cases}$$

we have $x_n \xrightarrow{w^*} x_\infty := (1, 1, \cdots, 1, \cdots)$, while $\psi(x_n) = 1 \not\to \psi(x_\infty) = 2$, so $\psi$ is not weak* lower semi-continuous.

The next example, due to Simon Fitzpatrick (private communication), shows that even in separable dual spaces such a $\psi$ may not be weak* lower semi-continuous.

EXAMPLE 5. Equivalently renorm $c_0$ by

$$\|(x(i))\| = \sup\{|x(1) - x(i) + x(j)| : 1 \leqslant i \leqslant j\},$$

and let $X$ be its dual space $(\ell_\infty, \|\cdot\|^*)$.

https://doi.org/10.1017/S0004972700037126 Published online by Cambridge University Press

The natural basis vectors, $e_n := (\delta_{ni})_{i=1}^{\infty}$, $n = 1, 2, \cdots$, form a norm one weak* null sequence in $X$ and we define $\psi$ by

$$\psi(x) := \limsup_n \|x - e_n\|^* .$$

Then taking $x_n := e_n - e_1$ we have $x_n \rightharpoonup^{w^*} -e_1$, while

$$\begin{aligned} \psi(x_j) &= \limsup_n \|e_j - e_1 + e_n\|^* \\ &= 1 \\ &\not\to \psi(-e_1) = \limsup_n \|e_1 + e_n\|^* = 2. \end{aligned}$$

Thus $\psi$ is not weak* lower semi-continuous.

On the other hand, we now show that in a weak* orthogonal dual lattice such a function $\psi$ is always weak* lower semi-continuous.

LEMMA 6. *Let $X$ be a weak* orthogonal dual Banach lattice and let $y_n \rightharpoonup^{w^*} 0$ with $\|y_n\| \leqslant 1$. Then*

$$\psi(x) := \limsup_n \|x - y_n\|,$$

*is weak* lower semi-continuous.*

PROOF: It suffices to show that for each $\lambda$ the *sub-level* set

$$D_\lambda := \{x : \psi(x) \leqslant \lambda\}$$

is weak* closed. Thus, suppose $(x_\alpha) \subseteq D_\lambda$ with $x_\alpha \rightharpoonup^{w^*} x$, we must show that $x \in C_\lambda$. Now given $\varepsilon > 0$ we may by the weak* orthogonality choose $\alpha_0$ 'sufficiently large' so that $\| |x| \wedge |x_{\alpha_0} - x| \| < \varepsilon/3$. Then, for all sufficiently large $n$ we have $\|x_{\alpha_0} - y_n\| \leqslant \psi(x_{\alpha_0}) + \varepsilon/3$ and $\| |y_n| \wedge |x_{\alpha_0} - x| \| \leqslant \varepsilon/3$, and so, since

$$\begin{aligned} |x - y_n| &\leqslant |(x - y_n) + (x_{\alpha_0} - x)| + |x - y_n| \wedge |x_{\alpha_0} - x| \\ &= |x_{\alpha_0} - y_n| + |x - y_n| \wedge |x_{\alpha_0} - x| \\ &\leqslant |x_{\alpha_0} - y_n| + |x| \wedge |x_{\alpha_0} - x| + |y_n| \wedge |x_{\alpha_0} - x|, \end{aligned}$$

we have

$$\begin{aligned} \|x - y_n\| &\leqslant (\psi(x_{\alpha_0}) + \varepsilon/3) + \varepsilon/3 + \varepsilon/3 \\ &\leqslant \lambda + \varepsilon. \end{aligned}$$

It follows that $\psi(x) = \limsup_n \|x - y_n\| \leqslant \lambda$, as required. ∎

We now obtain our main result as a corollary to Lemma 6 and Lemma 3, where by a suitable dilation and translation we may assume without loss of generality that $(a_n)$ is weak* null with $\|a_n\| \leqslant 1$.

https://doi.org/10.1017/S0004972700037126 Published online by Cambridge University Press

**THEOREM 7.** *Let $X$ be a weak* orthogonal dual Banach lattice and let $(a_n)$ be an approximate fixed point sequence for the nonexpansive mapping $T$ in the weak* compact minimal invariant set $D$, then*

$$\lim_n \|x - a_n\| = \operatorname{diam}(D), \quad \textit{for all} \quad x \in D.$$

Since the condition of Opial is a geometric analogue of weak orthogonality, Sims [**10**], we are led to ask: is a weak* Karlovitz' lemma true for dual spaces satisfying the weak* Opial condition?

We conclude by observing that this result combined with analogous arguments in the weak* case to those in Sims [**10**] establish the weak*- fpp for weak* orthogonal dual lattices, a result which in part subsumes the conclusions of Soardi [**11**], and Khamsi [**6**].

## REFERENCES


[**1**] J. Borwein and B. Sims, 'Non-expansive mappings on Banach lattices and related topics', *Houston J. Math.* **10** (1984), 339–355.

[**2**] M.S. Brodskii and D.P. Mil'man, 'On the center of a convex set', *Dokl. Akad. Nauk. SSSR* **59** (1948), 837–840.

[**3**] A.L. Garkavi, 'The best possible net and the best possible cross-section of a set in a normed linear space', *Amer. Math. Soc. Trans. Ser. 2* **39** (1964), 111-131. .

[**4**] K. Goebel, 'On the structure of minimal invariant sets for nonexpansive mapings', *Ann. Univ. Mariae Curie-Sktodowska Sect A (Lublin)* **9** (1975), 73–77.

[**5**] L.A. Karlovitz, 'Existence of fixed points of nonexpansive mappings in a space without normal structure', *Pacific J. Math.* **66** (1976), 153–159.

[**6**] M.A. Khamsi, 'On the weak*–fixed point property', *Contemp. Math.* **85** (1989), 325–337.

[**7**] W.A. Kirk, 'A fixed point theorem for mappings which do not increase distances', *Amer. Math. Monthly* **72** (1965), 1004–1006.

[**8**] P-K. Lin, 'Unconditional bases and fixed points of nonexpansive mappings', *Pacific J. Math.* **116** (1985), 69–76.

[**9**] B. Maurey, 'Seminaire d'Analyse Fonctionnelle', Exposé No. VIII (1980).

[**10**] B. Sims, 'Orthogonality and fixed points of nonexpansive maps', *Proc. Centre Math. Anal. Aust. Nat. Uni.* **20** (1988), 178–186.

[**11**] P. Soardi, 'Existence of fixed points of nonexpansive mappings in certain Banach lattices', *Proc. Amer. Math. Soc.* **73** (1979), 25–29.



Deaprtment of Mathematics
The University of Newcastle
New South Wales 2308
Australia