A GENERALIZED AVERAGING OPERATOR

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1. Introduction. The averaging operator $\nabla f(z) = \frac{1}{2}[f(z + h) + f(z)]$ has an extensive literature, the most detailed account being that of Nörlund **(4)**. In discussing solutions of the functional relation

$$\nabla f(z) = \phi(z), \tag{1.1}$$

he defines a "principal solution" **(4**, p. 41**)** by means of a summability process, and later, working in terms of complex numbers, he obtains **(4**, p. 70**)** a principal solution of (1.1) by means of a contour integral. He distinguishes his principal solution from other solutions, by showing that it is continuous at $h = 0$. His work includes a detailed account of the polynomial solutions of

$$\nabla f(z) = z^k, \tag{1.2}$$

the Euler polynomials with assigned values at $z = \frac{1}{2}$. Milne-Thomson **(3**, pp. 519–521**)** gives an account of generalized Euler numbers arising from the operator ∇^N, (N a positive integer) and of the generalized Euler numbers.

In this paper the ideas of Milne-Thomson are taken a step further. The operator ∇^λ is defined for all real λ, and is shown to be applicable to a wide class of functions. Polynomials corresponding to the generalized Euler polynomials of Milne-Thomson and a sequence of numbers corresponding to Nörlund's C-numbers **(4**, p. 27**)** are defined and some of their more important properties established. The inverse operator $\nabla^{-\lambda}$ is defined, and is shown to invert the operation ∇^λ and to give a unique solution in terms of the functions to which ∇^λ is applicable.

2. Generalized power of the averaging operator. The averaging (or mean) operator is defined for span h by

$$\nabla f(z) = \tfrac{1}{2}[f(z + h) + f(z)], \tag{2.1}$$

and its positive integer powers by

$$\nabla^M f(z) = \nabla \nabla^{M-1} f(z) = \sum_0^M \binom{M}{p} f(z + hp)/2^M. \tag{2.2}$$

To define $\nabla^\lambda f(z)$, where λ is related to the positive integer N by

$$N - 1 < \lambda \leqslant N, \tag{2.3}$$

we use the formal relation

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$$\nabla f(z) = \tfrac{1}{2}(1 + \exp hD).f(z),$$

and write

$$\nabla^\lambda = \frac{(1 + \exp hD)^{N+1}}{2^\lambda(1 + \exp hD)^\mu}, \qquad \mu = N + 1 - \lambda.$$

The operation in the numerator can be expressed by means of (2.2); and to obtain a representation of the operation in the denominator, we use the fact that

$$\frac{1}{(1 + \exp t)^\alpha} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\exp(-tw)dw}{E(\alpha, w)},$$

where t is real, α is positive, $0 < c < \alpha$ and

(2.4) $$E(\alpha, w) = \Gamma(\alpha)/\Gamma(w)\ \Gamma(\alpha - w).$$

Using the abbreviation

$$\int_c \text{ for } \int_{c-i\infty}^{c+i\infty},$$

we then have formally

(2.5) $$\begin{aligned}\nabla^\lambda f(z) &= \frac{1}{2^\lambda}\sum_0^{N+1}\binom{N+1}{p} e^{phD}\cdot \int_c \frac{\exp(-hDw)dw}{2\pi i\, E(\mu, w)}\cdot f(z)\\ &= \sum_0^{N+1}\binom{N+1}{p}\int_c \frac{f(z + ph - hw)dw}{2\pi i\, E(\mu, w)\, 2^\lambda},\end{aligned}$$

on using the shift operation $\exp(kD).f(z) = f(z + k)$. We take (2.5) *as the definition* of $\nabla^\lambda f(z)$, *if* λ *satisfies* (2.3), *the span h is positive or negative and the integrals exist.*

Although less restrictive assumptions as to the nature of $f(z)$ would be sufficient to ensure the existence of the integrals in (2.5), we shall assume throughout that

(2.6) *$f(z)$ is an entire function of exponential order κ, $\kappa h < \pi$.*

The following proposition is then an easy consequence of (2.6) and the fact that

$$|\Gamma(c + iv)\ \Gamma(\mu - c - iv)| \sim A \exp(-\pi|v|)\, .\, |v|^{N-\lambda}, \quad (|v| \to \infty):$$

if $\phi(z, h)$ is the function defined by (2.5) *and $f(z)$ satisfies* (2.6), *then $\phi(z, h)$ is an entire function of exponential order κ (in z) and*

(2.7) $$\lim_{h\to 0} \phi(z, h) = f(z).$$

Thus $\phi(z, h)$ has the property (2.7) which was noted by Nörlund (**4**, p. 46) as being characteristic of his principal solution of the functional equation $\nabla f(z) = \phi(z)$. It must be observed, however, that there do exist entire functions in z, for example, $\cos(\pi z/h)$ which satisfy neither (2.6) nor (2.7), but for which the operation ∇^λ is defined when λ is a positive integer but not otherwise.

In the particular case when $\lambda = N$, the definition (2.5) gives for $f(z)$ satisfying (2.6),

$$\begin{aligned}\nabla^N f(z) &= 2^{-N}\sum_0^N \binom{N}{p}\int_c \frac{f[z+h(p-w)]+f[z+h(p+1-w)]}{2\pi i\,E(1,w)}\,dw\\ &= 2^{-N}\sum_0^N\binom{N}{p}\operatorname{Res}\left\{\frac{\pi f[z+h(p-w)]}{\sin \pi w};0\right\}\\ &= 2^{-N}\sum_0^N\binom{N}{p} f(z+ph),\end{aligned}$$

which is the value given in (2.2).

We may confine ourselves to cases where $h \geqslant 0$ by reason of the following *extension property*: if $\phi(z, h) = \nabla^\lambda f(z)$, then

$$\phi(z+h\lambda, -h) = \phi(z,h). \tag{2.8}$$

For reversing the summation, and making the change of variable $w = \mu - \xi$, we have

$$\begin{aligned}\phi(z+h\lambda,-h) &= 2^{-\lambda}\sum_0^{N+1}\binom{N+1}{q}\int_{\mu-c}\frac{f(z+hq-h\xi)d\xi}{2\pi i\,E(\mu,\mu-\xi)}\\ &= 2^{-\lambda}\sum_0^{N+1}\binom{N+1}{q}\int_c\frac{f(z+hq-h\xi)d\xi}{2\pi i\,E(\mu,\xi)},\end{aligned}$$

by Cauchy's theorem, since $0 < c < \mu$, $0 < \mu - c < \mu$, and $E(\mu, \mu-\xi) = E(\mu,\xi)$.

3. The exponential property of ∇^λ.

We prove that

$$\nabla^\alpha\nabla^\beta f(z) = \nabla^{\alpha+\beta} f(z) \tag{3.1}$$

when α, β are positive. On account of (2.2) it is sufficient to give details for the cases

$$0 < \alpha+\beta \leqslant 1, \tag{3.2}$$

$$1 < \alpha+\beta < 2. \tag{3.3}$$

For the proof in the case (3.2) write $\alpha+\beta = \gamma$. Then

$$\nabla^\gamma f(z) = \sum_{n=0}^2\binom{2}{n}\int_c\frac{f(z+hn-hw)\,dw}{2^\gamma 2\pi i\,E(2-\gamma,w)} \qquad (0<c<2-\gamma); \tag{3.4}$$

and for $0 < a < 2-\alpha$, $0 < b < 2-\beta$,

$$\begin{aligned}\nabla^\alpha\nabla^\beta f(z) &= \sum_{p,q=0}^2\binom{2}{p}\binom{2}{q}\int_a\frac{ds}{2^\gamma 2\pi i\,E(2-\alpha,s)}\int_b\frac{f[z+h(p+q-s-w)]\,dw}{2\pi i\,E(2-\beta,w)}\\ &= \sum_{n=0}^2\binom{2}{n}\int_a\frac{ds}{2^\gamma 2\pi i\,E(2-\alpha,s)}\int_b\frac{F(s+w)\,dw}{2\pi i\,E(2-\beta,w)},\end{aligned} \tag{3.5}$$

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where $F(\xi) = f[z + h(n - \xi)] + 2f[z + h(n + 1 - \xi)] + f[z + h(n + 2-\xi)]$. By Cauchy's theorem we may take $0 < a < b$; then

$$\int_b \frac{F(s + w)\, dw}{2\pi i\, E(2 - \beta, w)} = \int_b \frac{F(\xi)\, d\xi}{2\pi i\, E(2 - \beta, \xi - s)}.$$

Hypothesis (2.6) guarantees the absolute convergence of the integrals in (3.5), so that

$$\nabla^\alpha \nabla^\beta f(z) = \sum_0^2 \binom{2}{n} \int_b \frac{F(\xi)\, d\xi}{2\pi i} \int_a \frac{\Gamma(s)\, \Gamma(2 - \beta - \xi + s)\, \Gamma(2 - \alpha - s)\, \Gamma(\xi - s)\, ds}{2^\gamma 2\pi i\, \Gamma(2 - \alpha)\, \Gamma(2 - \beta)}$$
$$= \sum_0^2 \binom{2}{n} \int_b \frac{\Gamma(\xi)\, \Gamma(4 - \gamma - \xi)\, F(\xi)\, d\xi}{2^\gamma 2\pi i\, \Gamma(4 - \gamma)},$$

by Barnes's Lemma (**1**, p. 155). Abbreviating this expression as

$$2^{-\gamma} \sum_0^2 \binom{2}{n} [I_1 + 2I_2 + I_3]$$

we let the lines of integration in I_2 and I_3 be changed to $b + 1$ and $b + 2$ respectively; and since the only positive poles of the integrand are at $\xi = 4 - \gamma, 5 - \gamma, \ldots$ and since $4 - \gamma > 3$, no poles lie in the strip $b < R(\xi) < b + 2$. Cauchy's theorem may then be applied to give

$$I_1 + 2I_2 + I_3 = \int_b \frac{[\Gamma(\xi)\Gamma(4-\gamma-\xi)+2\Gamma(\xi+1)\Gamma(3-\gamma-\xi)+\Gamma(\xi+2)\Gamma(2-\gamma-\xi)]f[z+h(n-\xi)]\, d\xi}{2\pi i\, \Gamma(4 - \gamma)}$$
$$= \int_b \frac{\Gamma(\xi)\, \Gamma(2 - \gamma - \xi)\, f[z + h(n - \xi)]\, d\xi}{2\pi i\, \Gamma(2 - \gamma)}.$$

Thus we have from (3.4)

$$\nabla^\alpha \nabla^\beta f(z) = \nabla^{\alpha+\beta} f(z).$$

In the case (3.3)

$$\nabla^\gamma f(z) = \sum_0^3 \binom{3}{n} \int_c \frac{f[z + h(n - w)]\, dw}{2^\gamma 2\pi i\, E(3 - \gamma, w)},$$
$$\nabla^\alpha \nabla^\beta f(z) = \sum_{p,q=0}^2 \binom{2}{p}\binom{2}{q} \int_a \frac{ds}{E(2 - \alpha, s)} \int_b \frac{f[z + h(p + q - s - w)]\, dw}{2^\gamma (2\pi i)^2\, E(2 - \beta, w)}$$
$$= \sum_0^3 \binom{3}{n} \int_a \frac{ds}{2\pi i\, E(2 - \alpha, s)} \int_b \frac{\{f[z + h(n - s - w)] + f[z + h(n + 1 - s - w)]\}dw}{2\pi i\, E(2 - \beta, w)},$$

and the previous argument may then be used to establish the result.

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4. The numbers g_k^λ and the polynomials $g_k^\lambda(z)$. We digress here to define certain fundamental numbers and polynomials associated with ∇^λ. Let

$$\frac{2^\lambda}{(1+\exp t)^\lambda} = \sum_0^\infty \frac{g_k^\lambda t^k}{k!} \qquad (|t| < \pi), \tag{4.1}$$

$$g_k^\lambda(z) = \sum_0^k \binom{k}{m} z^{k-m} g_m^\lambda. \tag{4.2}$$

On writing $G(t) = 2^\lambda(1 + \exp t)^{-\lambda}$, we obtain

$$[1+\exp(-t)]\,G'(t) + \lambda G(t) = 0,$$

$$\sum_{k=0}^n \binom{n}{k}[1+\exp(-t)]^{(k)} G^{(n+1-k)}(t) + \lambda G^{(n)}(t) = 0,$$

from which, on setting $t = 0$, and using the definition $g_k^\lambda = G^{(k)}(0)$, we have the recurrence relations

$$\sum_1^n \binom{n}{k}(-)^k g_{n+1-k}^\lambda + 2g_{n+1}^\lambda + \lambda g_n^\lambda = 0$$

or

$$(-)^n \sum_0^{n-1} \binom{n}{p}(-)^p g_{p+1}^\lambda + 2g_{n+1}^\lambda + \lambda g_n^\lambda = 0, \qquad n \geqslant 0, \tag{4.3}$$
$$g_0^\lambda = 1.$$

It is an easy calculation to establish for the polynomials $g_k^\lambda(z)$ the *generating relation*

$$\frac{2^\lambda \exp(zt)}{(1+\exp t)^\lambda} = \sum_0^\infty \frac{t^k g_k^\lambda(z)}{k!}. \tag{4.4}$$

The numbers g_k^λ have the following *explicit value* in terms of the Stirling numbers:

$$g_k^\lambda = (-)^k \sum_{p=1}^k \mathscr{S}_k^p\, \Gamma(\lambda+p)/\Gamma(\lambda)\, 2^p. \tag{4.5}$$

For we have

$$g_k^\lambda = \lim_{t\to 0} G^{(k)}(t) = \lim_{t\to 0} \frac{(-)^k 2^\lambda}{2\pi i}\int_c \frac{w^k \exp(-tw)dw}{E(\lambda, w)}$$
$$= \lim_{t\to 0}\, (-)^k 2^\lambda \sum_1^k \frac{\mathscr{S}_k^p \Gamma(\lambda+p)}{\Gamma(\lambda)\, 2\pi i}\int_c \frac{\exp(-tw)dw}{E(\lambda+p, \lambda-w)},$$

where

$$\mathscr{S}_k^p = \lim_{x\to 0}\frac{\Delta^p x^k}{p!}$$

are the Stirling numbers of the second kind (**2**, p. 134), and use is made of the identity

$$w^k = \sum_{p=1}^k \mathscr{S}_k^p\, \Gamma(w+p)/\Gamma(w).$$

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Thus, using the notation $(\lambda)_p = \Gamma(\lambda + p)/\Gamma(\lambda)$,

$$g_k^\lambda = \lim_{t\to 0} (-)^k 2^\lambda \sum_{p=1}^{k} \frac{\mathcal{S}_k^p(\lambda)_p}{(1+e^t)^{\lambda+p}} = (-)^k 2^{-p} \sum_{p=1}^{k} \mathcal{S}_k^p(\lambda)_p.$$

We prove next that

$$\nabla^\lambda h^k g_k^\lambda(z/h) = z^k. \tag{4.6}$$

Writing $\xi = z/h$, we have from (4.4),

$$\frac{z^\lambda e^{t(\xi+n-w)}}{(1+e^t)^{\lambda-1}} = \sum_0^\infty \frac{t^k}{k!}[g_k^\lambda(\xi+n-w) + g_{k+1}^\lambda(\xi+n+1-w)].$$

Multiplying throughout by

$$\binom{N}{n}/2\pi i\, 2^\lambda E(N+1-\lambda, w),$$

summing from $n = 0$ to N, and integrating with respect to w along the line $R(w) = c$ makes the right-hand side equal to

$$\sum_0^\infty \frac{t^k}{k!} \nabla^\lambda g_k^\lambda(\xi),$$

and the left-hand side equal to

$$\frac{e^{t\xi}}{(1+e^t)^{\lambda-1}} \sum_{n=0}^{N} \binom{N}{n} e^{nt} \int_c \frac{\exp(-tw)dw}{2\pi i\, E(N+1-\lambda, w)}$$
$$= \frac{e^{t\xi}(1+e^t)^N}{(1+e^t)^{\lambda-1}(1+e^t)^{N+1-\lambda}} = e^{t\xi}.$$

Thus

$$e^{t\xi} = \sum_0^\infty \frac{t^k}{k!} \nabla^\lambda g_k^\lambda(\xi),$$

and the result stated follows by comparing coefficients.

We note here that the function $h^k g_k^\lambda(z/h)$ has the property (2.7).

5. The inverse operator. A definition for negative powers of ∇ is obtained from the observation that formally

$$\begin{aligned} \nabla^{-\lambda}\phi(z) &= \frac{2^\lambda}{(1+\exp hD)^\lambda}\phi(z) = \frac{2^\lambda}{2\pi i}\int_c \frac{\exp(-hDw)dw}{E(\lambda, w)} \cdot \phi(z) \\ &= \frac{2^\lambda}{2\pi i}\int_c \frac{\phi(z-hw)dw}{E(\lambda, w)}, \qquad 0 < c < \lambda. \end{aligned} \tag{5.1}$$

We take (5.1) *as the definition of* $\nabla^{-\lambda}\phi(z)$, and as before assume that $\phi(z)$ is of exponential order κ, $\kappa h < \pi$, as a sufficient condition for assuring the existence of the integral in (5.1). This definition is valid for any real h, but we are justified in confining ourselves to the case $h \geqslant 0$ by the following *extension property*

$$\text{(5.2)} \qquad \textit{if } f(z) = \nabla^{-\lambda}\phi(z), \textit{ then } f(z - h\lambda, -h) = f(z, h).$$

For on setting $w = \lambda - \xi$, and observing that

$$0 < \mathscr{R}(\xi) = \lambda - c < \lambda,$$

we may apply Cauchy's theorem to see that

$$\begin{aligned} 2^{-\lambda} f(z - h\lambda, -h) &= \int_c \frac{\phi(z - \lambda h + hw)dw}{2\pi i\, E(\lambda, w)} \qquad (0 < c < \lambda) \\ &= \int_{\lambda - c} \frac{\phi(z - h\xi)d\xi}{2\pi i\, E(\lambda, \lambda - \xi)} \\ &= \int_c \frac{\phi(z - h\xi)d\xi}{2\pi i\, E(\lambda, \xi)} = 2^{-\lambda} f(z, h). \end{aligned}$$

The definition (5.1) is easily applied in special cases. Since

$$\begin{aligned} \text{(5.3)} \quad \nabla^{-\lambda}\phi(z) &= \frac{2^\lambda}{2\pi i}\int_c \frac{dw}{E(\lambda, w)} \sum_{m=0}^{\infty} \frac{(-hw)^m}{m!}\, \phi^{(m)}(z) \\ &= \phi(z) + \frac{2^\lambda}{2\pi i}\int_c \frac{dw}{E(\lambda, w)} \sum_{m=1}^{\infty} \frac{(-h)^m}{m!}\, \phi^{(m)}(z) \sum_{p=1}^{m} \mathscr{S}_m^p \frac{\Gamma(w+p)}{\Gamma(w)} \\ &= \phi(z) + 2^\lambda \sum_{m=1}^{\infty} \frac{(-h)^m}{m!}\, \phi^{(m)}(z) \sum_{p=1}^{m} \mathscr{S}_m^p \frac{\Gamma(p+\lambda)}{\Gamma(\lambda)} \\ &\qquad\qquad \int_c \frac{\Gamma(w+p)\,\Gamma(\lambda - w)dw}{2\pi i\, \Gamma(p+\lambda)} \\ &= \phi(z) + \sum_{m=1}^{\infty} \frac{(-h)^m}{m!}\, \phi^{(m)}(z) \sum_{p=1}^{m} \mathscr{S}_m^p \frac{\Gamma(p+\lambda)}{\Gamma(\lambda)\, 2^p} \\ &= \phi(z) + \sum_{m=1}^{\infty} \frac{h^m}{m!}\, \phi^{(m)}(z)\, g_m^\lambda = \sum_{m=0}^{\infty} \frac{h^m\, \phi^{(m)}(z)}{m!}\, g_m^\lambda, \end{aligned}$$

by (4.5) when the series converge. Thus when $\phi(z) = z^k$,

$$\text{(5.4)} \qquad \nabla^{-\lambda}\phi(z) = z^k + \sum_1^k \binom{k}{m} h^m z^{k-m} g_m^\lambda \cdot = \sum_0^k \binom{k}{m} h^m z^{k-m} g_m^\lambda .$$

Other simple cases would be

$$\text{(5.5)} \qquad \nabla^{-\lambda} e^z = 2^\lambda e^z / (1 + e^h)^\lambda$$

$$\text{(5.6)} \qquad \nabla^{-\lambda} \sin z = \sin\left(z - \frac{h}{2}\right) \Big/ \left(\cos \frac{h}{2}\right)^\lambda .$$

That the operation (5.1) does indeed invert $\nabla^\lambda f(z)$ is shown in the theorem:

THEOREM. *If* $\phi(z) = O(\exp \kappa |z|)$, $(|z| \to \infty)$, $\kappa h < \pi$, *and* $F(z)$ *is defined by* (5.1), *then* $F(z)$ *is of exponential order* κ, *and*

$$\text{(5.7)} \qquad \nabla^\lambda F(z) = \phi(z).$$

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That $F(z) = O(\exp \kappa|z|)$ may be proved in a manner similar to that by which (2.7) was established. To prove (5.7), let $0 < a < N + 1 - \lambda$, $0 < b < \lambda$, and consider

$$\begin{aligned}\nabla^\lambda F(z) &= 2^{-\lambda} \sum_{p=0}^{N+1} \binom{N+1}{p} \int_a \frac{F(z+ph-hs)ds}{2\pi i\, E(N+1-\lambda, s)} \\ &= \sum_0^{N+1} \binom{N+1}{p} \int_a \frac{ds}{2\pi i\, E(N+1-\lambda, s)} \\ &\qquad\qquad \int_b \frac{\phi[z+ph-h(s+w)]dw}{2\pi i\, E(\lambda, w)} \\ &= \sum_0^{N+1} \binom{N+1}{p} \int_a \frac{ds}{2\pi i\, E(N+1-\lambda, s)} \int_{a+b} \frac{\phi(z+ph-h\xi)d\xi}{2\pi i\, E(\lambda, \xi - s)}.\end{aligned}$$

Since $a + b < a + \lambda$, and the poles of the inner integrand lie on the lines

$$\mathscr{R}(\xi) = a, a-1, \ldots, \mathscr{R}(\xi) = a+\lambda, a+\lambda+1, \ldots.$$

Cauchy's theorem may be applied to give

$$\nabla^\lambda F(z) = \sum_0^{N+1} \binom{N+1}{p} \int_a \frac{ds}{2\pi i\, E(N+1-\lambda, s)} \int_b \frac{\phi(z+ph-h\xi)d\xi}{2\pi i\, E(\lambda, \xi - s)}.$$

The exponential order of $\phi(z)$ and the order properties on vertical lines of the Γ-function (**5**, p. 151), are sufficient to establish the absolute convergence of this iterated integral, and Fubini's theorem may be applied to give

$$\begin{aligned}\nabla^\lambda F(z) &= \sum_0^{N+1} \binom{N+1}{p} \int_b \frac{\phi(z+ph-h\xi)d\xi}{2\pi i} \\ &\qquad \int_a \frac{\Gamma(s)\,\Gamma(\lambda-\xi+s)\,\Gamma(N+1-\lambda-s)\,\Gamma(\xi-s)ds}{2\pi i\,\Gamma(\lambda)\,\Gamma(N+1-\lambda)} \\ &= \sum_0^{N+1} \binom{N+1}{p} \int_b \frac{\phi(z+ph-h\xi)d\xi}{2\pi i} \\ &\qquad \int_{L_a} \frac{\Gamma(s)\,\Gamma(\lambda-\xi+s)\,\Gamma(N+1-\lambda-s)\,\Gamma(\xi-s)ds}{2\pi i\,\Gamma(\lambda)\,\Gamma(N+1-\lambda)}\end{aligned}$$

by Cauchy's theorem, where the contour L_a is obtained by deforming $R(s) = a$ in such a way that the poles of $\Gamma(N+1-\lambda-s)\,\Gamma(\xi - s)$ lie to the right of L_a, while the poles of $\Gamma(s)\,\Gamma(\lambda - \xi + s)$ lie to the left. Then by Barnes's Lemma (**1**, p. 155),

$$\nabla^\lambda F(z) = \sum_0^{N+1} \binom{N+1}{p} \int_b \frac{\phi[z-h(\xi-p)]d\xi}{2\pi i\, E(N+1, \xi)} \equiv A + B.$$

To evaluate

$$A = \sum_0^{N} \binom{N+1}{p} \int_b \frac{\phi[z-h(\xi-p)]d\xi}{2\pi i\, E(N+1, \xi)},$$

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Cauchy's theorem may be applied, since $0 \leqslant p \leqslant N$, to give

$$A = \sum_0^N \binom{N+1}{p} \int_{b+p} \frac{\phi[z - h(\xi - p)]d\xi}{2\pi i\, E(N+1, \xi)}$$
$$= \sum_0^N \binom{N+1}{p} \int_b \frac{\phi(z - h\xi)d\xi}{2\pi i\, E(N+1, p+\xi)}$$
$$= \int_b \frac{\phi(z-h\xi)}{2\pi i} \sum_0^N \binom{N+1}{p} \frac{\Gamma(p+\xi)\,\Gamma(N+1-p-\xi)}{\Gamma(N+1)} d\xi$$
$$= \int_b \frac{\phi(z-h\xi)}{2\pi i\, E(N+1,\xi)} \sum_0^N \frac{(-N-1)_p(\xi)_p}{p!\,(\xi-N)_p} d\xi$$

where

$$\sum_0^N \frac{(-N-1)_p(\xi)_p}{p!\,(\xi-N)_p} = {}_2F_1\left[\begin{matrix}-N-1, \xi\\ \xi - N\end{matrix}; 1\right] - \frac{(-)^{N+1}(\xi)_{N+1}}{(\xi-N)_{N+1}}$$
$$= \frac{(-N)_{N+1}}{(\xi-N)_{N+1}} - \frac{\Gamma(N+1+\xi)\,\Gamma(-\xi)}{\Gamma(\xi)\,\Gamma(N+1-\xi)}$$
$$= -\frac{\Gamma(N+1+\xi)(-\xi)}{\Gamma(\xi)\,\Gamma(N+1-\xi)}.$$

Thus $A + B$

$$= \int_b \frac{\Gamma(\xi)\,\Gamma(N+1-\xi)\,\phi[z-h(\xi-N-1)] - \Gamma(N+1+\xi)\,\Gamma(-\xi)\,\Gamma(z-h\xi)}{2\pi i\,\Gamma(N+1)} d\xi$$
$$= \left\{\int_{b-N-1} - \int_b\right\} \frac{\Gamma(N+1+w)\,\Gamma(-w)\,\phi(z-hw)dw}{2\pi i\,\Gamma(N+1)}$$
$$= -\operatorname{Res}\left\{\frac{\Gamma(N+1+w)\,\Gamma(-w)\,\phi(z-hw)}{\Gamma(N+1)}; 0\right\} = \phi(z),$$

which completes the proof.

6. Remarks. It is well known that the functional equation

(6.1) $$\nabla^N f(z) = \phi(z), \qquad (N = 1, 2, \ldots)$$

has solutions other than that given by (5.1). For example, if $p(z)$ has the property

(6.2) $$p(z+h) + p(z) = 0,$$

it is a solution of the homogeneous equation

$$\nabla^N f(z) = 0;$$

and if it is added to the solution of (6.1) given by (5.1), the resulting function is still a solution of (6.1). It does not, however, have the property (2.7), since for example $p(z)$ could be $\sin(\pi z/h)$ or $\cos(\pi z/h)$. Moreover it need not satisfy requirement (2.6), since

$$\cos(\pi z/h) = O[\exp(\pi|y|/h)], \qquad (|y| \to \infty),$$

and ∇^λ need not then be defined except when $\lambda = 1, 2, \ldots$. These facts suggest the possible existence of a set of eigenvalues $\lambda = 1, 2, \ldots$, with a family of eigenfunctions corresponding to each eigenvalue for the operator ∇^λ.

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