BULL. AUSTRAL. MATH. SOC.
VOL. 13 (1975), 337-342.

05B10, 20K99
(20D99)

Maximal sum-free sets in finite abelian groups, V

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Let $\lambda(G)$ be the cardinality of a maximal sum-free set in a group G. Diananda and Yap conjectured that if G is abelian and if every prime divisor of $|G|$ is congruent to 1 modulo 3, then $\lambda(G) = |G|(n-1)/3n$ where n is the exponent of G. This conjecture has been proved to be true for elementary abelian p-groups by Rhemtulla and Street and for groups $G = Z_{p^2} \oplus Z_p$ by Yap. We now prove this conjecture for groups $G = Z_{pq} \oplus Z_p$ where p and q are distinct primes.

Let G be an additive group with non-empty subsets S and T. Let $S \pm T = \{s \pm t;\ s \in S,\ t \in T\}$ and let $|S|$ be the cardinality of S. We say that S is sum-free in G if $(S+S) \cap S = \emptyset$ and that S is a maximal sum-free set in G if $|S| \geq |T|$ for every T sum-free in G. We denote by $\lambda(G)$ the cardinality of a maximal sum-free set in G.

Exact values $\lambda(G)$ for all finite abelian groups G, except when every prime divisor of $|G|$ is congruent to 1 modulo 3, were determined by Diananda and Yap [1]. In this exceptional case,

(1) $$|G|(n-1)/3n \leq \lambda(G) \leq (|G|-1)/3$$

where n is the exponent of G. It is conjectured that in this exceptional case, $\lambda(G)$ equals its lower bound [1]. Rhemtulla and Street [3] prove this conjecture for elementary abelian p-groups. Yap [4] proves this conjecture for groups $G = Z_{p^2} \oplus Z_p$.

Received 1 August 1975.

https://doi.org/10.1017/S0004972700024588 Published online by Cambridge University Press

We now prove this conjecture for groups $G = Z_{pq} \oplus Z_p$ where p and q are distinct primes. We shall in fact prove the following

THEOREM 1. *Let* $G = Z_{pq} \oplus Z_p$ *where* $p = 3k + 1$ *and* $q = 3l + 1$ *are distinct primes. Then* $\lambda(G) = p(kq+l)$.

Proof. We shall adopt the definitions and notations given in [4] and [5]. In particular, we let

$$H_0 = [(p, 0)] \oplus [(0, 1)] , \quad H_i = [(1, i)] , \quad i = 1, 2, \ldots, p ,$$

$$L = [(q, 0)] \oplus [(0, 1)] , \text{ and } K = [(p, 0)] .$$

From the above definitions, we observe that each H_i is a cyclic group of order pq , $H_i \cap H_j = K$ for each $i \neq j$, L is an elementary p-group such that $L_i = L \cap H_i$ is a group of order p for every $i = 0, 1, \ldots, p$, and $G = \bigcup_{i=0}^{p} H_i$.

From (1) we have

$$(2) \qquad p(kq+l) = p(lp+k) \leq \lambda(G) \leq (qp^2-1)/3 .$$

Now let S be a maximal sum-free set in G . Let $|S \cap H_i| = lp + \lambda_i = kq + \mu_i$, $i = 0, 1, \ldots, p$, $\lambda = \max\{\lambda_i ; i = 0, 1, \ldots, p\}$, $\mu = \max\{\mu_i ; i = 0, 1, \ldots, p\}$, and let $H \in \{H_0, H_1, \ldots, H_p\}$ be such that

$$(3) \qquad |S \cap H| = lp + \lambda = kq + \mu , \quad \lambda \leq k , \quad \mu \leq l .$$

(The fact that $\lambda \leq k$ and $\mu \leq l$ follows from (1).)

By definition we have, for each $i = 0, 1, \ldots, p$,

$$(4) \qquad \lambda_i = k - l + \mu_i \text{ and, in particular, } \lambda = k - l + \mu .$$

From (1) we have

$$(5) \qquad |S \cap K| = m , \quad 0 \leq m \leq l .$$

By looking at the distribution of the elements of S in the subgroups H_i , we have

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$$|S| = \sum_{i=0}^{p} |S \cap H_i| - p|S \cap K|$$

$$= (p+1)lp + \sum_{i=0}^{p} \lambda_i - pm .$$

If $|S| > p(lp+k)$, then

(6) $$\sum_{i=0}^{p} \lambda_i > p(k-l+m) .$$

We also have

(7) $$|S| \le (p+1)(kq+\mu) - pm$$
$$= kpq + p(\mu-m) + lp + k - l + \mu .$$

Thus if $\mu < m$, then (7) contradicts (2). Hence

(8) $$\mu \ge m .$$

Next, let $K_0 = K$, $H = \bigcup_{i=0}^{p-1} K_i$ where $K_i = x_i + K$ are distinct cosets of K in H such that $x_1 + x_1 = x_2$, $x_1 + x_2 = x_3$, $\ldots$, $x_i \in H \cap L$. Let $x_i + S_i = S \cap K_i$. Then we have

(9) $$m(H) = \max\{|S_i|;\ i = 1, 2, \ldots, p-1\} \ge l + 1 ,$$

for otherwise $kq + \mu = |S \cap H| \le (p-1)l + m = kq - k + m$ which contradicts (8).

If $m(H) \ge l + 2$, then we can show that at least one of S_i is empty (see the proof of Theorem 4 [5]) and following the proof of Theorem 2 [4], we can show that $\lambda(G)$ equals its lower bound given in (2). Thus if

(10) $$|S| > p(kq+l) , \text{ then } m(H) = l + 1 .$$

Suppose $\mu > m$. Then

$$kq + \mu - m \ge kq + 1 = (k+1)(l+1) + (2k-1)l ,$$

and applying the theorem of Cauchy-Davenport (see [2]) to S_i , we get a contradiction (for details of the proof of this part, see the proof of Theorem 4 [5]). Hence if

(11) $$|S| > p(kq+l) , \text{ then } \mu = m .$$

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Substituting (11) into (6) we get

(12) $$\sum_{i=0}^{p} \lambda_i > p\lambda ,$$

from which it follows that if

(13) $|S| > p(kq+l)$, then $\lambda_i > 0$ for each $i = 0, 1, \ldots, p$.

Next, from (11), we have

$$kq + \mu = |S \cap H| = k(l+1) + 2kl + m ,$$

from which it follows that there are k elements in $\{S_1, S_2, \ldots, S_{p-1}\}$ each of which has cardinality $l + 1$. Suppose $|S_i| = l + 1$. Then applying the theorem of Cauchy-Davenport to $S_i + S_i$ and noting that $(S_i+S_i) \cap S_{2i} = \emptyset$, we know that $S_{2i} = K\backslash(S_i+S_i)$, the set complement of $S_i + S_i$ with respect to K . Thus, we have either $0 \in S_i$ or $0 \in S_{2i}$; that is, either $x_i \in S$ or $x_{2i} \in S$. Hence $|S \cap L \cap H| = k$.

Let $I = \{i_1, i_2, \ldots, i_{r+1}\}$ be such that for each $i \in I$, $\lambda_i < \lambda$, and for each $i \in \{0, 1, \ldots, p\}\backslash I$, $\lambda_i = \lambda$. Let $H_j = \bigcup_{i=0}^{p-1} K_{ij}$ where $K_{ij} = x_{ij} + K$ are distinct cosets of K in H_j such that $x_{1j} + x_{1j} = x_{2j}$, $x_{1j} + x_{2j} = x_{3j}, \ldots, x_{ij} \in L_j$. Let $x_{ij} + S_{ij} = S \cap K_{ij}$. Then for each $j \in I$, we have

$$\begin{aligned} kq + \mu_j = |S \cap H_j| &\le r_j(l+1) + (p-r_j-1)l + m \\ &= kq - k + r_j + m , \end{aligned}$$

from which it follows that $r_j \ge k + \mu_j - m$.

By the same reasoning as in showing that $|S \cap L \cap H| = k$, we can show that if $|S_{ij}| = l + 1$, and if $|S_{(2i)j}| \le l - 1$, then there are at least $r_j + 1$ elements in $\{S_{1j}, \ldots, S_{(p-1)j}\}$ each of which has cardinality $l + 1$. Thus, by the replacement process, we have $|S \cap L_j| \ge r_j$.

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Putting the above results together, we have

$$(14) \qquad |S \cap L| \ge (p-r)k + (r+1)(k-m) + \sum_{j \in I} \mu_j$$
$$= (p-r)k + (r+1)(k-m) + (r+1)(l-k) + \sum_{j \in I} \lambda_j$$
$$= kp + (r+1)(l-m-k) + k + r\lambda$$
$$= kp + (r+1)(l-m-k+\lambda) + k - \lambda$$
$$= kp + k - \lambda \ .$$

Finally, we decompose H_i as a union of q cosets L_{ij} , $j = 1, \ldots, q$ of L_i . If for some i ,

$$\max\{|S \cap L_{ij}|; \ j = 1, 2, \ldots, q\} \ge k + 2 ,$$

then following the proof of Theorem 2 [4], we can show that $\lambda(G)$ equals the lower bound given in (2). Otherwise, from

$$lp + \lambda_i = |S \cap H_i| = l(k+1) + 2lk + \lambda_i ,$$

it follows that $|S \cap L_i| \ge \lambda_i$ and thus $|S \cap L| \ge \sum_{i=0}^{p} \lambda_i > p\lambda$ which cannot be true if $\lambda = k$ (see Theorem 1 [3]). Consequently, if

$$(15) \qquad |S| > p(kq+l) \text{ , then } \lambda < k \ .$$

From (14) and (15), we have

$$|S \cap L| > kp ,$$

which again contradicts Theorem 1 [3].

Hence $\lambda(G) = p(kq+l)$.

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https://doi.org/10.1017/S0004972700024588 Published online by Cambridge University Press