Canad. Math. Bull.Vol. **36** (1), 1993 pp. 108–115

CLASSIFICATION AND ENUMERATION OF REAL QUADRATIC FIELDS HAVING EXACTLY ONE NON-INERT PRIME LESS THAN A MINKOWSKI BOUND

R. A. MOLLIN AND H. C. WILLIAMS

ABSTRACT. We will classify those real quadratic fields K having exactly one non-inert prime less than $\sqrt{\Delta}/2$ where Δ is the discriminant of K. Moreover, we will list all such K and prove that the list is complete with one possible exceptional value remaining (whose existence would be a counterexample to the Riemann hypothesis).

1. **Introduction.** Throughout $K = Q(\sqrt{d})$ where d is a positive square-free integer. Let $[\alpha, \beta]$ denote $\alpha\mathbb{Z}+\beta\mathbb{Z}$ where $\mathbb{Z}$ is the ring of rational integers. Thus the maximal order O_K in K is just $\mathbb{Z} + \omega\mathbb{Z}$ where

$$\omega = \frac{\sigma - 1 + \sqrt{d}}{\sigma} \text{ with } \sigma = \begin{cases} 1 & \text{if } d \equiv 2, 3 \pmod 4 \\ 2 & \text{if } d \equiv 1 \pmod 4 \end{cases}.$$

The discriminant of K is then $\Delta = (\frac{2}{\sigma})^2 d$.

An ideal I of O_K is *primitive* if the only rational integer divisors of I are units. If I is primitive then I is called *reduced* if there does not exist a non-zero $\alpha \in I$ such that both $|\alpha| < N(I)$ and $|\alpha'| < N(I)$ where α' is the algebraic conjugate of α and $N(I)$ is the *norm* of I; *i.e.*, $N(I) = |O_K/I|$. Furthermore, if I is a primitive ideal then $I = N(I)\mathbb{Z}+\beta\mathbb{Z}$ where $\beta = b + \omega$ with $b \in \mathbb{Z}$. If I is a primitive ideal and $N(I) < \sqrt{\Delta}/2$, then I is reduced. The conjugate ideal of I is denoted by I'.

Let C_K denote the *class group* of K, and $h(d)$ its order; *i.e.*, the *class number* of K. Equivalence of ideals I and J in C_K will be denoted $I \sim J$. We will be concerned primarily with the principal class in C_K. We now give an elucidation of the relationship between continued fractions and reduced ideals for this class.

The *continued fraction expansion* of ω is denoted by $\langle a_0, \overline{a_1, a_2, \ldots, a_\pi}\rangle$ of period length π. Here $a_0 = a = \lfloor \omega \rfloor$ where $\lfloor\ \rfloor$ denotes the greatest integer function. Also, $a_i = \lfloor (P_i + \sqrt{d})/Q_i \rfloor$ for $i \geq 1$ where, $(P_0, Q_0) = (\sigma - 1, \sigma)$, $P_i = a_{i-1}Q_{i-1} - P_{i-1}$, and $d = P_i^2 + Q_iQ_{i-1}$. Also note that $0 < P_i < \sqrt{d}$ and $0 < Q_i < 2\sqrt{d}$.

Now let $I_i = [Q_i/\sigma, (P_i + \sqrt{d})/\sigma]$ for $i = 0, 1, 2, \ldots$. By the continued fraction algorithm (see [10]), $I = I_0 \sim I_1 \sim I_2 \sim \cdots \sim I_{\pi-1}$ and $I_\pi = I$. Moreover the I_i's

The first author's research is supported by NSERC Canada grant #A8484 and that of the second author by grant #A7649.

Received by the editors September 25, 1991; revised May 5, 1992 .

AMS subject classification: Primary: 11R11, 11R29; secondary: 11Y40.

for $0 < i < \pi$ represent *all* of the distinct reduced principal ideals. Furthermore in this *cycle* (of period length π) of reduced principal ideals the Q_i/σ's represent the *norms* of *all* principal reduced ideals. Therefore given the following well-known result we will be able to cite a continued fraction class number one criterion.

THEOREM 1.1. *The group C_K is generated by the non-inert prime ideals $\mathcal{P}$ with $N(\mathcal{P}) \leq \sqrt{\Delta}/2$.*

Thus immediate from Theorem 1.1 is

THEOREM 1.2. *$h(d) = 1$ if and only if $p = Q_i/\sigma$ for some i with $0 < i < \pi$ whenever p is a non-inert prime less than $\sqrt{\Delta}/2$.*

REMARK 1.1. In [5] we classified all those real quadratic fields K for which there were *no* non-inert primes less than $\sqrt{\Delta}/2$, and these turned out to be of narrow Richaud-Degert type; *i.e.*, of the form $d = \ell^2 + r$ where $|r| \in \{1, 4\}$. Moreover, in [2] Louboutin used a suitable Riemann hypothesis to list those real quadratic fields for which there were no split primes less than $\sqrt{\Delta}/2$, and he partially characterized such fields. In [3] we completely characterized such fields. These d's turned out in [2], [3] and [5] to be of extended Richaud-Degert type (or simply of ERD-type); *i.e.*, $d = \ell^2 + r$ where $4\ell \equiv 0 \pmod{r}$. In this paper we investigate and completely determine all d's for which there is *exactly* one non-inert prime less than $\sqrt{\Delta}/2$ (and not all of these are of ERD-type). The motivation for doing this is elucidated in the following.

DEFINITION 1.1. Let t be the number of distinct non-inert primes less than $\sqrt{\Delta}/2$.

DEFINITION 1.2. Let $f_d(x) = -x^2 + 2x(\sigma - 1)/\sigma + (d - \sigma + 1)/\sigma^2$ and let s be the maximum number of distinct primes dividing $f_d(x)$ for any given integer x such that $1 \leq x < \sqrt{\Delta}/2$.

LEMMA 1.1. $s \leq t + 1$.

PROOF. Assume there is an x_0 with $1 \leq x_0 < \sqrt{\Delta}/2$ such that $f_d(x_0)$ is divisible by at least $t + 2$ distinct primes. Since $d \equiv (\sigma x_0 + \sigma - 1)^2 \pmod{p}$ for each such prime p then all such primes are non-inert. By Definition 1.1 at least two of these primes are bigger than $\sqrt{\Delta}/2$; whence, $f_d(x_0) \geq \Delta/4$. However $f_d(x) \leq (d-1)/\sigma^2$ for any x with $1 \leq x \leq \sqrt{\Delta}/2$; whence, $(d-1)/\sigma^2 \geq \Delta/4 = d/\sigma^2$, a contradiction. ∎

REMARK 1.2. In Lemma 1.1 we see that if $t = 0$ then $f_d(x)$ is prime for all x with $1 \leq x \leq \sqrt{\Delta}/2$ and this provided us in [5] with a real quadratic field analogue of the well-known Rabinowitsch result for complex quadratic fields. In fact we found that if $t = 0$ then either $d \leq 11$ or $d \equiv 1 \pmod{4}$ and $d = \ell^2 + 1$ or $d = \ell^2 \pm 4$.

Now we are interested in what happens when $t > 0$. Specifically if $t = 1$ what can be said? The question is completely answered in the next section. All of the above notation will be in force and we also will have use for the following technical results.

LEMMA 1.2. *If $Q_{i-1}/\sigma > \sqrt{\Delta}/2$ then $Q_i/\sigma < \sqrt{\Delta}/2$.*

PROOF. Since $Q_iQ_{i-1} < d$ then $Q_i/\sigma < d/(\sigma Q_{i-1}) < 2d/(\sigma^2\sqrt{\Delta}) = \sqrt{\Delta}/2$. ∎

LEMMA 1.3. *If $Q_i \geq \sqrt{d}$ then $a_i = 1$.*

PROOF. $2\sqrt{d} > P_i + \lfloor\sqrt{d}\rfloor \geq a_i Q_i \geq a_i\sqrt{d}$. ■

The remaining result is well-known.

LEMMA 1.4. $(I_i)' = I_{\pi-i}$.

2. $t = 1$. In this section we classify those square-free positive integers d such that $t = 1$. Then we explicitly compute all such d, with one possible exceptional value remaining.

REMARK 2.1. We note that much is already known about the case where the unique non-inert, prime less than $\sqrt{\Delta}/2$ is ramified (see Remark 1.1). However the characterizations for $t = 1$ are not explicit in [2]–[3] and are not easy to extract. Therefore in the interest of a self-contained, complete result for $t = 1$, and because the explicit characterization, and elementary proof which we give in the ramified case is rather neat we include it below.

THEOREM 2.1. *(i) If $d \equiv 2, 3 \pmod 4$ then $t = 1$ if and only if $h(d) = 1$ and $d = m^2 \pm 2 > 3$.*

(ii) If $d \equiv 1 \pmod 4$ and $t = 1$ then one of the following must hold, where p denotes the unique prime with $p < \sqrt{d}/2$ and $(d/p) \neq -1$.

(a) $d = (pb)^2 + 4p \equiv 5 \pmod 8$, $h(d) = 1$, $b > 1$, and $pb^2 + 4$ is prime with $p \equiv pb^2 + 4 \equiv 3 \pmod 4$.

(b) $d = q^2 + 4qp^n \equiv 5 \pmod 8$ for some $n \geq 0$ with q and $q + 4p^n$ primes and $p^n < \sqrt{d}/2 < q$.

(c) $d = r^2 + 4p^{2m} \equiv 5 \pmod 8$ for some $m \geq 0$ where r is prime with $p^m < \sqrt{d}/2 < r < \sqrt{d}$ and d is a product of at most 2 primes, both larger than $\sqrt{d}/2$.

(d) $d = (2a-1)^2 + 2^\ell$ with $\ell \geq 3$ and d is a product of at most 2 primes both larger than $\sqrt{d}/2$.

PROOF.

CASE A: $d \equiv 2, 3 \pmod 4$. First assume that $d = m^2 \pm 2 > 3$ and $h(d) = 1$. If $t > 1$ then there exists a prime p with $2 < p < \sqrt{d}$ and $(d/p) \neq -1$. Therefore by Theorem 1.2, $p = Q_i$ for some i with $1 \leq i \leq \pi - 1$. However, if $d = m^2 + 2$ then $\pi = 2$ and $Q_i = 2 \neq p$, a contradiction. If $d = m^2 - 2$ then $\pi = 4$ with $Q_1 = Q_3 = 2m - 3$ and $Q_2 = 2$; whence, $2m - 3 = p$ is forced. However $2m - 3 \geq m > \sqrt{d}$, a contradiction. Thus $t \leq 1$. However by [5, Lemmas 2.2–2.3, pp. 146–148] $t \neq 0$.

Conversely, assume $t = 1$. We may always write $d = m^2 + r$ where either $m = \lfloor\sqrt{d}\rfloor$ and $0 < r \leq m$ or $m = \lfloor\sqrt{d}\rfloor + 1$ and $-m < r < 0$. If an odd prime q divides r then $(d/q) \neq -1$. However 2 ramifies and $t = 1$; whence, if $r > 0$ then $r \geq q > \sqrt{d} > m$, and if $r < 0$ then $-r \geq q > \sqrt{d} > m - 1$, both of which are contradictions. Thus $|r| = 2^\ell$ for $\ell \geq 0$. If $\ell > 1$ then $d \equiv 0, 1 \pmod 4$, a contradiction. If $\ell = 0$ then either

https://doi.org/10.4153/CMB-1993-016-7 Published online by Cambridge University Press

$d = m^2 + 1$ or $d = m^2 - 1$. If $d = m^2 + 1$ then m is odd and any prime q dividing m satisfies $(d/q) = 1$; whence, $m \geq q > \sqrt{d}$, a contradiction. If $d = m^2 - 1 > 3$ then there exists a prime q dividing d with $q < \sqrt{d}$, contradicting $t = 1$.

We note that $t = 0$ for $d = 3$. In total we have shown that $t = 1$ implies either $d = a^2 + 2 > 3$ or $d = (a+1)^2 - 2 > 2$ where $a = a_0$ is as in Section 1. In the former case $Q_2 = 2$. Thus by Theorem 1.1–1.2 we must have $h(d) = 1$.

CASE B: $d \equiv 1 \pmod 4$. Let $p < \sqrt{d}/2$ be the non-inert prime. First assume that p ramifies. If $Q_1 \neq 2p$ then $Q_1 = 2q$ where $q > \sqrt{d}/2$ is prime. (Otherwise there is a prime $r \neq p$ with r dividing Q_1 and $r < \sqrt{d}/2$ where $(d/r) \neq -1$, contradicting that $t = 1$.) Therefore by Lemma 1.3 we have $a_1 = 1$. Moreover, if $\pi \neq 2$ then $Q_2 > 2$ and so $Q_2/2$ is divisible by a prime s with $s < \sqrt{d}/2$ (since $d = Q_1Q_2 + P_2^2$ and $Q_1 > \sqrt{d}$), but $(d/s) \neq -1$ contradicting that $t = 1$. Thus $\pi = 2$; whence, $d = q^2 + 4q$ since $d = P_2^2 + Q_1Q_2$. Now, since p ramifies, then p divides $q + 4$ which must be composite and square-free given that $p < \sqrt{d}/2 < q$. Thus $1 < (q+4)/p < \sqrt{d}/2$; whence, there is a prime $r \neq p$ dividing d with $r < \sqrt{d}/2$, contradicting that $t = 1$. We have shown that our initial assumption that $Q_1 \neq 2p$ is false. Thus, since $t = 1$ and the only possible ideal with norm less than $\sqrt{d}/2$ is principal, by Theorem 1.1 we get $h(d) = 1$. Since $d = P_1^2 + Q_0Q_1$ and p divides d we get p divides P_1 and $d = b^2p^2 + 4p$. Also, since $h(d) = 1$ then $pb^2 + 4$ is prime and $p \equiv pb^2 + 4 \equiv 3 \pmod 4$. (Observe that when $t = 1$ we cannot have $d = p(p+4)$ because, in this case, we get $\sqrt{d} - 2 < p$ but, since $p < \sqrt{d}/2$ we have a contradiction).

Now we assume that p splits; *i.e.*, $(p) = \mathcal{P}\mathcal{P}'$.

CASE (a): $d \equiv 5 \pmod 8$. If π is even then $\pi = 2i$ for $i \geq 1$, $P_{\pi/2+1} = P_{\pi/2}$ and by [1] (see also [3]), $Q_{\pi/2}/2$ divides d. Thus $Q_{\pi/2}/2$ must be prime since otherwise there is a ramified prime less than $\sqrt{d}/2$ dividing $Q_{\pi/2}$, contradicting that $t = 1$ and p splits. Let $q_1 = Q_{\pi/2}/2$; then we must have $q_1 > \sqrt{d}/2$; whence $a_{\pi/2} = 1$ by Lemma 1.3. Thus $a_{\pi/2}Q_{\pi/2} - P_{\pi/2} = 2q_1 - P_{\pi/2} = P_{\pi/2+1}$; whence, $P_{\pi/2} = q$. Therefore $d = P_{\pi/2}^2 + Q_{\pi/2}Q_{\pi/2-1} = q_1^2 + 2q_1Q_{\pi/2-1}$. By Lemma 2.1, $Q_{\pi/2-1} < \sqrt{d}/2$ so the only possible odd prime dividing $Q_{\pi/2-1}$ is p. Since $d \equiv 5 \pmod 8$ then $d = q_1^2 + 4q_1p^n$ for some $n \geq 0$. Moreover $d/q_1 = q_2$ is prime since there would be, otherwise, a ramified prime less than $\sqrt{d}/2$. Moreover we clearly have $\sqrt{d}/2 < q_1 < \sqrt{d}$ and $p^n < \sqrt{d}/2$.

If π is odd then $Q_{\frac{\pi-1}{2}} = Q_{\frac{\pi+1}{2}}$, and $d = P_{\frac{\pi+1}{2}}^2 + Q_{\frac{\pi+1}{2}}^2$. Since $d \equiv 5 \pmod 8$ and $t = 1$ then the only possible odd prime which can divide $Q_{\frac{\pi-1}{2}}$ is p. Thus $Q_{\frac{\pi-1}{2}} = 2p^m$ for $m \geq 0$ and $d = P_{\frac{\pi+1}{2}}^2 + 4p^{2m}$ with $p^m < \sqrt{d}/2$. If $P_{\frac{\pi+1}{2}}$ is not prime then there is a prime divisor q of it which splits and for which $q < \sqrt{d}/2$, a contradiction. Thus $P_{\frac{\pi+1}{2}} = r$, a prime and $d = r^2 + 4p^{2m}$ with $\sqrt{d} > r > \sqrt{d}/2$. Since $t = 1$, d is either prime or $d = q_1q_2$ for primes $q_1 > \sqrt{d}/2$ and $q_2 > \sqrt{d}/2$.

CASE (b): $d \equiv 1 \pmod 8$. Thus $p = 2$.

If an odd prime q divides Q_1, then $q > \sqrt{d}/2$; whence, $d = (2a-1)^2 + 2^\ell q$ for $\ell \geq 3$. Thus $d > (2a-1)^2 + 4\sqrt{d} > (\sqrt{d} - 2)^2 + 4\sqrt{d} > d$, a contradiction. Therefore

$d = (2a-1)^2 + 2^\ell$ for $\ell \geq 3$. Moreover d must be the product of at most 2 primes both of which are greater than $\sqrt{d}/2$. ∎

REMARK 2.2. If all Q_i/Q_0's are powers of a single prime p then it follows that $h(d) = 1$ if and only if $t \leq 1$, (see [4]). Moreover as shown in [5], if $t = 0$ and $d > 11$ then $h(d) = 1$ if and only if $d \equiv 1 \pmod 4$ and $d = \ell^2 + 1$ or $d = \ell^2 \pm 4$.

Now we proceed to list all those d with $t = 1$, and verify that the list is complete (with one possible exception).

REMARK 2.3. As noted earlier, for the case where the unique non-inert prime p less than $\sqrt{\Delta}/2$ is ramified, Louboutin was able to use a suitable Riemann hypothesis in [2] to list all such fields. Theorem 2.1 shows that in this case $h(d) = 1$ and d is of ERD-type. In [6] we showed how to list all such fields (with one possible exceptional value). We include these fields in our list below and show how to complete the list for $t = 1$ when p splits. We enumerate all these fields and show that the list is complete with one possible exceptional value whose existence would be a counterexample to the Riemann hypothesis. This finite list, then, is not conditional upon the assumption of the Riemann hypothesis, but in order to eliminate the exceptional value arising from our use of Tatuzawa's result [9], we would have to invoke the Riemann hypothesis.

THEOREM 2.2. *$t = 1$ for $Q(\sqrt{d})$ if and only if d is one of the following 63 values (with one possible exceptional value remaining, the existence of which would be a counterexample to the Riemann hypothesis).*

(i) If $d \equiv 2, 3 \pmod 4$ then $d \in \{6, 7, 11, 14, 23, 38, 47, 62, 83, 167, 227, 398\}$.

(ii) If $d \equiv 1 \pmod 4$ then $d \in \{17, 33, 37, 41, 61, 65, 69, 85, 89, 93, 101, 113, 133, 137, 149, 157, 197, 213, 237, 257, 269, 317, 341, 353, 377, 397, 413, 453, 461, 557, 593, 629, 677, 717, 733, 773, 853, 941, 1077, 1097, 1133, 1217, 1253, 1333, 1553, 1877, 2273, 2917, 3053, 5297, 7213\}$.

PROOF. As seen in Theorem 2.1, if the non-inert prime p ramifies, then d is of ERD-type and $h(d) = 1$. In [6] we enumerated the list of all d's of ERD-type with $h(d) = 1$.

Now we consider the case where p splits. Suppose that $p^h < \sqrt{\Delta}$. Since $t = 1$ then C_K is generated by a single prime so the norms of the reduced principal ideals, the Q_i/Q_0's for $i = 1, 2, \ldots, \pi - 1$, can only be of the form $p^{jh(d)}$ or a prime q with $q > \sqrt{d}/2$. Observe that if the $p^{jh(d)}$ case holds then $j < \frac{1}{h(d)} \log_p \sqrt{\Delta}$. Furthermore, we note that we cannot have two primes $q_1 = Q_i/Q_0 > \sqrt{\Delta}/2$ and $q_2 = Q_{i+1}/Q_0 > \sqrt{\Delta}/2$.

Now if $Q_i/Q_0 = p^{jh(d)}$ and $Q_m/Q_0 = p^{jh(d)}$ then $I_{i+1} = (I_{m+1})' = I_{\pi-m-1}$. Hence at most two of the principal reduced ideals can have the same norm $p^{jh(d)}$. It follows that there are no more than $\frac{2}{h(d)} \log_p \sqrt{\Delta}$ reduced ideals in the principal class with norms of the form $p^{jh(d)}$. Also by Lemma 1.2 there is at most one ideal between any two of these with norm *not* of the form $p^{jh(d)}$. Hence there can be at most

$$\frac{2}{h(d)} \log_p \sqrt{\Delta} + \frac{2}{h(d)} \log_p\left(\sqrt{\Delta}\right) + 2 = \frac{4}{h(d)} \log_p\left(\sqrt{\Delta}\right) + 2$$

reduced ideals in the principal class. Thus,

$$\pi < \frac{4}{h(d)} \log_p \sqrt{\Delta} + 3.$$

Now since $R = \log \epsilon_d < \pi \log \sqrt{\Delta}$ where ϵ_d is the fundamental unit of $Q(\sqrt{d})$, (see [8]), then $h(d)R < 4 \log \sqrt{\Delta} \log_p \sqrt{\Delta} + 3h(d) \log \sqrt{\Delta}$. Since $p^{h(d)} < \sqrt{\Delta}$ we get

$$h(d)R < 7 \log \sqrt{\Delta} \log_p \sqrt{\Delta}; \tag{2.1}$$

whence, by the analytic class number formula

$$7 \log \sqrt{\Delta} \log_p \sqrt{\Delta} > \frac{\sqrt{\Delta}}{2} L(1, \chi). \tag{2.2}$$

(Observe that (2.1) is Corollary 2.1 below.)

Now consider the case where $p^{h(d)} > \sqrt{\Delta}$, then $Q_1/2$ can only be a prime $q > \sqrt{\Delta}/2$ and $Q_2/2$ can only be 1; whence, $d = q^2 + 4q$ with q and $q + 4$ both primes. Since $p < \sqrt{\Delta}/2$ then $h(d) \geq 2$. Put $m = \lfloor h(d)/2 \rfloor$, then if $p^m > \sqrt{\Delta}/2$, by [7, Theorem 2.1 and Lemma 2.1, p. 483], there is a reduced ideal I with $N(I) < \sqrt{\Delta}/2$ and $I \sim \mathcal{P}^m$ where $(p) = \mathcal{P}\mathcal{P}'$. If $N(I) = p^n$, then $n < m$. Also $I = \mathcal{P}^n$ or $(\mathcal{P}')^n$ in this case. Since $\mathcal{P}^m$ is not equivalent to $\mathcal{P}^n$ then $I = (\mathcal{P}')^n$ and $\mathcal{P}^m$ is equivalent to $(\mathcal{P}')^n$; whence, $\mathcal{P}^{m+n} \sim 1$. Therefore $m + n \equiv 0\big(\mathrm{mod}\, h(d)\big)$, a contradiction. We have shown that $h(d) \geq 2$ implies $\lfloor h(d)/2 \rfloor < \log_p(\sqrt{\Delta}/2)$, whence, $h(d) < 2\log_p(\sqrt{\Delta}/2)+2$. Since $\epsilon_d = (q+2+\sqrt{d})/2 = (\sqrt{d+4} + \sqrt{d})/2 < (2\sqrt{d} + 2)/2$, then $\epsilon_d < \sqrt{d} + 1$ which implies $R < \log(\sqrt{d} + 1)$. Hence,

$$h(d)R < \big(2 + 2\log_p(\sqrt{\Delta}/2)\big)(\log \sqrt{\Delta} + 1). \tag{2.3}$$

Thus,

$$\big(2 + 2\log_p(\sqrt{\Delta}/2)\big)\big(\log(\sqrt{\Delta} + 1)\big) > \frac{\sqrt{\Delta}}{2} L(1, \chi). \tag{2.4}$$

By Tatuzawa [9] we know that with one possible exception, we must have,

$$L(1, \chi) > .655\eta\Delta^{-\eta}$$

for $0 < \eta < 1/2$ and $\Delta > \max\{e^{1/\eta}, e^{11.2} < 73131\}$. Taking $\eta > 1/\log\Delta$ which decreases toward $1/\log\Delta$ we get $L(1, \chi) \geq 0.655/(e \log \Delta) > .24/\log\Delta$, and so

$$\sqrt{\Delta}L(1, \chi)/2 > .12\Delta/\log\Delta, \quad (\Delta \geq 73131).$$

Hence (2.4) cannot hold for $\Delta > 10^{10}$ and (2.2) cannot hold for $\Delta > 2 \cdot 10^{11}$. A computer check for those remaining Δ yielded only those on the list. Using the results of [5] we could show that the list is complete under the assumption of the generalized Riemann hypothesis. Thus the exceptional value, if it exists, would be a counterexample to that Riemann hypothesis. ∎

COROLLARY 2.1. *If $t = 1$ for $Q(\sqrt{d})$ and p is the unique split prime less than $\sqrt{\Delta}/2$ then we have*

$$h(d)R \leq \frac{7}{4}\log(\Delta)\log_p \Delta.$$

PROOF. We actually verified this in the proof of Theorem 2.2 ∎

REMARK 2.3. We observe in the proof of Theorem 2.2 that if we could show that there are no d's of the form $d = q^2 + 4q$ with $q > \sqrt{\Delta}/2$ and $q + 4$ both primes when $t = 1$ and p splits, then we could avoid the use of the Tatuzawa result in that case. It is possible that this could be proved without making use of the Tatuzawa result.

REMARK 2.4. In [5] we saw that $t = 0$ was tantamount to d being of narrow RD-type, $h(d) = 1$ and $d \equiv 1 \pmod 4$, if $d > 11$. Moreover from Lemma 1.1, $f_d(x)$ is prime for all x with $1 \leq x \leq \sqrt{d}/2$, this being a real quadratic field analogue of the Rabinowitsch condition for complex quadratic fields.

Our classification of the $t = 1$ case herein yields more than the ERD-types and possible $h(d) > 1$. Moreover by Lemma 1.1, $f_d(x)$ is a product of at most 2 distinct primes. However as $h(5297) = 3$ it is not the case that $h(d) \leq 2$; *i.e.*, we cannot generalize the $t = 0$ case to say that if $f_d(x)$ is a product of at most s distinct primes then $h(d) \leq s$. What is the general relationship between s and $h(d)$? We can say something about the relationship between t and $h(d)$; viz., if the exponent of C_K is e that $h(d) \leq e^t$. We are currently examining the case where $e = 2$, and the results will be published at a later date.

ACKNOWLEGEMENTS. The first author wishes to thank the mathematics department of the University of Bordeaux 1 for providing an excellent working environment while he was visiting there in April–May, 1991, and during which this research was done. In particular thanks must go to Jacques Martinet as the host, and to Henri Cohen, Francisco Diaz y Diaz, Anne-Marie Berge and Michel Olivier for their impeccable hospitality.

REFERENCES

1. S. Louboutin, *Continued fractions and real quadratic fields*, J. Number Theory **30**(1988), 167–176.

2. ———, *Groupes des classes d'idéaux triviaux*, Acta Arith. **54**(1989), 61–74.

3. S. Louboutin, R. A. Mollin and H. C. Williams, *Class numbers of real quadratic fields, continued fractions, reduced ideals, prime-producing quadratic polynomials, and quadratic residue covers*, Canadian J. Math. **44**(1992), 1–19.

4. R. A. Mollin, *Powers in continued fractions and class numbers of real quadratic fields*, Utilitas Math., to appear.

5. R. A. Mollin and H. C. Williams, *On prime valued polynomials and class numbers of real quadratic fields*, Nagoya Math. J. **112**(1988), 143–151.

6. ———, *Solution of the class number one problem for real quadratic fields of Extended Richaud-Degert type (with one possible exception)*. In: Number Theory, Walter de Gruyter and Co., Berlin, (1990), (ed. R. A. Mollin), 417–425.

7. ———, *Class number one for real quadratic fields, continued fractions and reduced ideals*. In: Number Theory and Applications, NATO ASI series **C265**, (ed. R. A. Mollin), (1989), 481–496.

8. ———, *Computation of the Class numbers of a real quadratic field*, Utilitas Math. **41**(1992), 259–308.
9. T. Tatuzawa, *On a theorem of Siegel*, Japan J. Math. **21**(1951), 163–178.
10. H. C. Williams and M. C. Wunderlich, *On the parallel generation of the residues for the continued fraction factoring algorithm*, Math. Comp. **177**(1987), 405–423.

Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
T2N 1N4
e-mail: ramollin@acs.ucalgary.ca

Computer Science Department
University of Manitoba
Winnipeg, Manitoba
R3T 2N2
e-mail: Hugh_Williams@csmail.cs.umanitoba.ca