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DISPROOF OF THE CONJECTURED SUBEXPONENTIALITY OF CERTAIN FUNCTIONS IN PERCOLATION THEORY

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Abstract

Consider bond-percolation on a graph G with sites $S(G)$. We disprove the conjecture of Hammersley (1957) that the function $n \rightarrow \sup_{s \in S(G)} E$ [the number of sites s' at distance n from s which can be reached from s by an open path which, except for s', only passes through sites at distance smaller than n from s] is always subexponential.

1. Introduction

Percolation theory has been introduced by Broadbent and Hammersley (1957). For a recent introduction to the subject see Kesten (1982), Chapter 1.

Let G be a locally finite graph (i.e. the number of bonds incident to any site is finite) and denote the set of sites of G by $S(G)$. Let the bonds of G, independent of each other, be open with probability p and closed with probability $1-p$. The length of a path is the number of bonds it contains. The distance between two sites is the length of the shortest path which connects them. Define, for $s \in S(G)$:

$N^n(s)$ is the set of sites at distance $\leqq n$ from s.
$B^n(s)$ is the set of sites at distance n from s.
$E_n(s)$ is the expected number of sites $s' \in B^n(s)$ for which there exists an open path from s to s' which, except for s', only passes through sites in $N^{n-1}(s)$.
Finally, define $F_n = \sup_{s \in S(G)} E_n(s)$.
Though E_n and F_n also depend on p, we omit this parameter.

Hammersley (1957) conjectured that $F_{n+m} \leqq F_n F_m$ always. In the next section we show that there exists a case for which $F_2 > F_1^2$ so that the conjecture is false.

2. The counterexample

Consider, for a positive integer r, the graph with $1+r+r^2$ sites denoted by c; s_i, $1 \leqq i \leqq r$, and $s_{i,j}$, $1 \leqq i, j \leqq r$; and with bonds (c, s_i), $1 \leqq i \leqq r$; (s_i, s_j), $1 \leqq i, j \leqq r$, $i \neq j$; and $(s_i, s_{i,j})$, $1 \leqq i, j \leqq r$. This graph can be imagined as a central site c, surrounded by and

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connected with a complete graph on r sites, each of which having a bond to r other sites which have no further connections.

Now consider bond-percolation on this graph with p the probability of a bond to be open. It is clear that, for each site s, $E_1(s)$ equals p times the number of bonds incident to s and this is maximal if s is one of the s_i's, in which case it equals $2rp$. So

$$F_1 = 2rp. \tag{2.1}$$

Further, F_2 is at least $E_2(c)$ which, by symmetry, equals the number of sites at distance 2 from c multiplied by the probability of the event that at least one of them, say s_{11}, can be reached from c by an open path. (By the structure of the graph the condition of containing no sites, except s_{11}, outside $N^1(c)$ is automatically fulfilled.) Note that this event occurs if and only if the bond (s_1, s_{11}) is open (which happens with probability p) and there exists, inside the complete graph on the set $\{c, s_1, s_2, \cdots, s_r\}$ an open path from c to s_1. Denote the probability of the latter event by $P(p, r)$. Using independence we get

$$F_2 \geqq r^2 pP(p, r). \tag{2.2}$$

Hence, by (2.1) and (2.2)

$$\frac{F_2}{F_1^2} \geqq \frac{P(p, r)}{4p}. \tag{2.3}$$

It is easily seen that for fixed p

$$\lim_{r\to\infty} P(p, r) = 1, \qquad 0 < p \leqq 1. \tag{2.4}$$

Now fix p between 0 and $\frac{1}{4}$. Then, for r sufficiently large, the right-hand side of (2.3) is larger than 1, in contradiction to the conjecture.

Remarks.

(i) With the help of the finite graphs above it is easy to obtain a counterexample concerning an infinite connected graph. For example, connecting the site c with an infinite chain does not increase the value F_1.

(ii) One might think that the conjecture is true if, in the definition of $E^n(s)$, all open paths of which all sites are in $N^n(s)$ are allowed. However, consider the tree consisting of a site c which is connected with six sites $s_1, s_2, \cdots, s_6$, each s_i in its turn being connected with six sites $s_{i,1}, s_{i,2}, \cdots, s_{i,6}$. Add to this tree, for each $j \leqq 6$, the bonds $(s_{1,j}, s_{2,j})$, $(s_{3,j}, s_{4,j})$ and $(s_{5,j}, s_{6,j})$. For the graph thus obtained it is easily verified that $F_1 = 7p$ and according to the new definition of E_n, $F_2 \geqq E_2(c) = 36P$ [there exists an open path from c to $s_{1,1}] \geqq 36(p^2 + p^3 - p^5)$ which, if $p = \frac{1}{2}$, appears to be larger than $49p^2$.

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