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ON SPACES OF COMPACT OPERATORS IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. Let K be a non-trivial complete non-Archimedean valued field and let E be an infinite-dimensional Banach space over K. Some of the main results are:
(1) K is spherically complete if and only if every weakly convergent sequence in l^∞ is norm-convergent.
(2) If the valuation of K is dense, then c_0 is complemented in E if and only if $C(E, c_0)$ is not complemented in $L(E, c_0)$, where $L(E, c_0)$ is the space of all continuous linear operators from E to c_0 and $C(E, c_0)$ is the subspace of $L(E, c_0)$ consisting of all compact linear operators.

1. Throughout this paper, E and F denote non-Archimedean Banach spaces over a non-trivial, complete, non-Archimedean valued field K. A subset A of E is said to be compactoid if for every $\epsilon > 0$, there exists a finite subset X of E such that $A \subset B_\epsilon(0) + C_0(X)$, where $B_\epsilon(0) = \{x \in E : \|x\| \leqq \epsilon\}$ and $C_0(X)$ is the absolutely convex hull of X, i.e.,

$$C_0(X) = \left\{ \sum_{x \in X} \lambda_x x : \lambda_x \in K, |\lambda_x| \leqq 1, x \in X \right\}.$$

A linear operator $T : E \longrightarrow F$ is compact if $T(B_1(0))$ is compactoid in F. Clearly, every compact operator is continuous. $L(E, F)$ denotes the space of all continuous linear operators from E to F and $C(E, F)$ the subspace of $L(E, F)$ consisting of all compact operators.

In Archimedean analysis, a linear operator T from E to F is called compact if the subset $T(B_1(0))$ of F is precompact. However, in non-Archimedean analysis, if we use this definition for a compact operator, then the existence of non-zero compact operator implies that K is locally compact ([10] p. 160). Amice [1] proved that if K is locally compact, then a subset of E is compactoid if and only if it is precompact. For the properties on compact operators, see [2], [3], [5], [10] and [12] etc.

From here, we assume that E is infinite-dimensional. Every infinite-dimensional Banach space contains a subspace which is isomorphic to c_0.

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The purpose of this paper is to consider the space $L(E,F)$ having the sequential convergence property, defined as below, and its subspace $C(E,F)$.

DEFINITION. The space $L(E,F)$ has *the sequential convergence property* (*s.c.p.*), if for every sequence $(T_n) \subset L(E,F)$ such that for each $x \in E$, $\lim_n T_n(x) = 0$ we have that $\lim_n \|T_n\| = 0$.

The s.c.p. of $E'(= L(E,K))$ means that every weak*-convergent sequence in E' is norm-convergent. In Archimedean analysis, it is known ([4], [9]) that the dual of every infinite-dimensional Banach space fails to have the s.c.p. In our case, we prove the following:

(1) *If the valuation of K is discrete, then $L(E,F)$ has the s.c.p. And if the valuation of K is dense, then the following conditions* (i)–(iii) *are equivalent:*

(i) *E' fails to have the s.c.p.*

(ii) *c_0 is complemented in E.*

(iii) *$C(E,c_0)$ is not completed in $L(E,c_0)$.*

If E has a base, then c_0 is complemented in E ([10] p. 74), and hence it follows from (1) that $C(E,c_0)$ is not complemented in $L(E,c_0)$. The problem whether $C(E,F)$ is complemented in $L(E,F)$ has been extensively studied by many authors in the Archimedean case (cf. [6], [13], [14]). Our result gives an answer to the problem in the non-Archimedean case. Further, we obtain the following result, which shows the converse of Monna [7, p. 70, Theorem 6] is also true:

(2) *The field K is spherically complete if and only if every weakly convergent sequence in l^∞ is norm-convergent.*

The following theorem ([10], p. 142) shows that $C(E,F)$ is closed in $L(E,F)$ and it will be used in the sequel.

THEOREM 1. *Let $T \in L(E,F)$. Then T is compact if and only if for every $\epsilon > 0$, there exists $S \in L(E,F)$ such that S is of finite rank and $\|T - S\| < \epsilon$.*

Let π denote an arbitrary fixed element of K with $0 < |\pi| < 1$. The letter N stands for the set of positive integers. Other terms and symbols will be used as in [10].

2. We begin by proving the following proposition.

PROPOSITION 2. $T = (T_1, T_2, \ldots, T_n, \ldots) \in L(E, \bigoplus_{n\in N} F)$. *Then $T \in C(E, \bigoplus_{n\in N} F)$ if and only if for every $n \in N$, $T_n \in C(E,F)$ and* $\lim_n \|T_n\| = 0$.

PROOF. Let $T \in C(E, \bigoplus_{n\in N} F)$. Then $T_n = P_n \circ T$ with

$$P_n : \bigoplus_{n\in N} F \longrightarrow F : (x_n)_n \longrightarrow x_n.$$

Since T is compact and P_n is continuous, T_n is compact. If $\lim_n \|T_n\| \neq 0$, then we can assume that there exists $\epsilon > 0$ with $\|T_n\| > \epsilon$ for all $n \in N$. Hence, for all $n \in N$,

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there exists $x_n \in E$ with $\|x_n\| \leqq 1$ and $\|T_n(x_n)\| > \epsilon$. Since T is compact, there are $a_1, a_2, \ldots, a_k$ in $\bigoplus_{n\in N} F$ such that

$$T(B_1(0)) \subset \{y \in \bigoplus_{n\in N} F : \|y\| \leqq \epsilon\} + C_0\{a_1, a_2, \ldots, a_k\}.$$

For $j = 1, 2, \ldots, k$, put $a_j = (a_{jn})_n$. Since $\lim_n a_{jn} = 0$, there exists $n_0 \in N$ such that $\|a_{jm}\| < \epsilon$ for $m \geqq n_0$, $j = 1, 2, \ldots, k$. Since $x_m \in B_1(0)$, we have

$$T(x_m) = d_m + \sum_{j=1}^{k} \alpha_{jm} a_j,$$

where $d_m = (d_{mn})_n \in \bigoplus_{n\in N} F$, $\|d_m\| \leqq \epsilon$ and $\alpha_{jm} \in K$, $|\alpha_{jm}| \leqq 1$. Hence $T_m(x_m) - d_{mm} = \sum_{j=1}^{k} \alpha_{jm} a_{jm}$. But $\|\sum_{j=1}^{k} \alpha_{jm} a_{jm}\| < \epsilon$ while $\|T_m(x_m) - d_{mm}\| > \epsilon$. This is a contradiction.

Conversely, assume that $T_n \in C(E, F)$, $n \in N$, and $\lim_n \|T_n\| = 0$. Then by Theorem 1, for each $n \in N$ and for each $\epsilon > 0$, there exists $S_{\epsilon n} \in L(E, F)$ such that $S_{\epsilon n}(E)$ is finite-dimensional and $\|T_n - S_{\epsilon n}\| < \epsilon$. Put

$$S_\epsilon^{(n)} = (S_{\epsilon 1}, S_{\epsilon 2}, \ldots, S_{\epsilon n}, 0, 0, \ldots) :$$

Then $S_\epsilon^{(n)} \in L(E, \bigoplus_{n\in N} F)$ and $S_\epsilon^{(n)}(E)$ is finite-dimensional. For every $\epsilon > 0$, there exists $n_0 \in N$ such that $\|T_m\| < \epsilon$ for all $m \geqq n_0$. Hence

$$\|T - S_\epsilon^{(n_0)}\| = \max_{\substack{i=1,\ldots,n_0 \\ m\geqq n_0}} (\|T_i - S_{\epsilon i}\|, \|T_m\|) < \epsilon.$$

This means that $T \in C(E, \bigoplus_{n\in N} F)$. □

Corollary 3. *If $L(E, \bigoplus_{n\in N} F) = C(E, \bigoplus_{n\in N} F)$, then $L(E, F)$ has the s.c.p. Moreover, if $L(E, F) = C(E, F)$, then the converse is also true.*

Proof. Let (T_n) be a sequence in $L(E, F)$ with $\lim_n T_n(x) = 0$ for every $x \in E$. Then by the Banach-Steinhaus Theorem, $\sup_n \|T_n\| < \infty$. Put

$$T : E \longrightarrow \bigoplus_{n\in N} F : x \longrightarrow (T_n(x))_n.$$

Then T is continuous and therefore compact. By Proposition 2, $\lim_n \|T_n\| = 0$.

Conversely, assume that $L(E, F) = C(E, F)$ and $L(E, F)$ has the s.c.p. Let $T = (T_1, T_2, \ldots, T_n, \ldots) \in L(E, \bigoplus_{n\in N} F)$. Then for every $n \in N$, $T_n \in L(E, F)$ and $\lim_n T_n(x) = 0$ for any $x \in E$. By assumption $\lim_n \|T_n\| = 0$. The conclusion $T \in C(E, \bigoplus_{n\in N} F)$ follows directly from Proposition 2. □

The following corollary is due to De Grande-de Kimpe [2].

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COROLLARY 4. *The dual E' has the s.c.p. if and only if $L(E, c_0) = C(E, c_0)$.*

A Banach space F is said to be weakly injective if for any Banach space X and for any subspace D of X, every $S \in L(D, F)$ has an extension $\bar{S} \in L(X, F)$, and F is said to be weakly projective if for every Banach space Y and for every continuous linear surjection $T : Y \longrightarrow F$, there exists $S \in L(F, Y)$ such that $TS = id_F$. It is known ([10, p. 177, 106]) that a Banach space is weakly projective if and only if it has a base and that if a Banach space is spherically complete, then it is weakly injective.

By Corollary 5.20 in [10] and Corollary 3 we obtain the following corollary.

COROLLARY 5. *Let the valuation of K be dense. Let E be weakly injective and F weakly projective. Then $L(E, F)$ has the s.c.p. if and only if $L(E, \bigoplus_{n \in N} F) = C(E, \bigoplus_{n \in N} F)$.*

THEOREM 6. *If E contains a complemented subspace of countable type, then for every F, $L(E, F)$ fails to have the s.c.p.*

PROOF. Let E_1 be a complemented subspace of countable type and E_0 a subspace of E such that $E = E_1 \oplus E_0$. Then there exist a number t with $0 < t \leqq 1$ and a t-orthogonal base (x_n) for E_1. We can assume that for all $n \in N$ $|\pi| \leqq \|x_n\| \leqq 1$. Every $x \in E$ can be written as

$$x = x_0 + \sum_{n \in N} \alpha_n x_n,$$

where $x_0 \in E_0$ and $\alpha_n \in K$, $n \in N$. Since there exists a number s $(0 < s \leqq 1)$ such that E_1 is s-orthogonal to E_0 ([10], p. 63), $\|x\| \geqq ts|\pi|\,|\alpha_n|$ for all $n \in N$. Let $a \in F$, $a \neq 0$, be fixed and we define a linear operator

$$T_n : E \longrightarrow F \text{ by } T_n(x) = \alpha_n a \quad (n \in N).$$

Then for every $x \in E$,

$$\|T_n(x)\| \leqq \frac{\|a\|}{|\pi| ts} \|x\| \quad (n \in N).$$

Hence $T_n \in L(E, F)$ and $\lim_n T_n(x) = 0$. While for all $n \in N$,

$$\|T_n\| \geqq \frac{\|T_n(x_n)\|}{\|x_n\|} = \frac{\|a\|}{\|x_n\|} \geqq \|a\|,$$

which completes the proof. □

If E has a base, then E satisfies the condition of Theorem 6 ([10], p. 74). Hence we have

COROLLARY 7. *If E has a base, then $L(E, F)$ fails to have the s.c.p.*

COROLLARY 8. *The valuation of K is discrete if and only if $(l^\infty)'$ fails to have the s.c.p.*

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PROOF. If the valuation of K is discrete, then l^∞ has a base. Hence by Corollary 7, $(l^\infty)'$ fails to have the s.c.p. If the valuation of K is dense, then $L(l^\infty, c_0) = C(l^\infty, c_0)$ ([10], p. 181). Hence $(l^\infty)'$ has the s.c.p. □

3. Next, we characterize the spherical completeness of K.

THEOREM 9. *If there exists $a \in F$, $a \neq 0$, such that the subspace $\{\lambda a : \lambda \in K\}$ of F is complemented, then the following statements are equivalent.*

(1) *K is not spherically complete.*

(2) $(\times_{n\in N} F / \bigoplus_{n\in N} F)' = \{0\}$.

(3) *For every $h \in (\times_{n\in N} F)'$, $\lim_n h(\mathbf{a}_n) = 0$, where $\mathbf{a}_1 = (a, 0, 0, \ldots)$, $\mathbf{a}_2 = (0, a, 0, 0, \ldots), \ldots$.*

PROOF. The implication (1) ⇒ (2) follows from Theorem 4.1 and Corollary 4.3 in [10]. We now show the implication (2) ⇒ (3). By hypothesis there exists a subspace F_1 of F such that $F = \{\lambda a : \lambda \in K\} \oplus F_1$. Let $x = (x_n)_n \in \times_{n\in N} F$ and let

$$x_n = \alpha_n a + y_n \quad (\alpha_n \in K, y_n \in F_1, n \in N).$$

There exists a number t $(0 < t \leqq 1)$ such that for every $n \in N$,

$$(*) \qquad t \max(\|\alpha_n a\|, \|y_n\|) \leqq \|x_n\|.$$

Suppose that (3) is not true. Then we can assume that there exist $\epsilon > 0$ and $h \in (\times_{n\in N} F)'$ such that $|h(\mathbf{a}_n)| \geqq \epsilon$ for each $n \in N$. Hence,

$$(**) \qquad \|\alpha_n (h(\mathbf{a}_n))^{-1} a\| \leqq \frac{\|x_n\|}{t\epsilon} \leqq \frac{\|x\|}{t\epsilon} \quad (n \in N)$$

and the linear operator

$$v_1 : \mathop{\times}_{n\in N} F \longrightarrow \mathop{\times}_{n\in N} F : x \longrightarrow (\alpha_n (h(\mathbf{a}_n))^{-1} a)_n$$

is continuous. If $x \in \bigoplus_{n\in N} F$, then by $(*)$ and $(**)$,

$$v_1(x) = \sum_{n\in N} \alpha_n (h(\mathbf{a}_n))^{-1} \mathbf{a}_n \in \bigoplus_{n\in N} F \text{ and } h \circ v_1(x) = \sum_{n\in N} \alpha_n.$$

Further, since for every $x \in \times_{n\in N} F$ and for every $n \in N$ $\|\alpha_n a\| \leqq \|x\|/t$, the linear operator $v_2 : \times_{n\in N} F \longrightarrow \times_{n\in N} F$ defined by

$$v_2(x) = (\alpha_2 a, \alpha_3 a, \ldots)$$

is continuous. Define $g \in (\times_{n\in N} F)'$ by

$$g(x) = \alpha_1 + (h \circ v_1 \circ v_2)(x) - (h \circ v_1)(x) \quad (x \in \mathop{\times}_{n\in N} F).$$

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If $x \in \bigoplus_{n\in N} F$, then $v_2(x) \in \bigoplus_{n\in N} F$ and $g(x) = 0$. Thus we can define $\hat{f} \in (\times_{n\in N} F/\bigoplus_{n\in N} F)'$ by $\hat{f}(\hat{x}) = g(x)$. Let $b = (a, a, \ldots, a, \ldots) \in \times_{n\in N} F$. Then $\hat{f}(\hat{b}) = g(b) = 1$ and $(\times_{n\in N} F/\bigoplus_{n\in N} F)' \neq \{0\}$. This contradicts (2). Finally, to prove (3) $\Rightarrow$ (1), let $x = (x_n)_n \in \bigoplus_{n\in N} F$, and let $x_n = \alpha_n a + y_n$ ($\alpha_n \in K, y_n \in F_1, n \in N$). Define a linear operator

$$f : \bigoplus_{n\in N} F \longrightarrow K \text{ by } f(x) = \sum_{n\in N} \alpha_n.$$

Since

$$|f(x)| \leqq \frac{\|x\|}{t\|a\|},$$

$f \in (\bigoplus_{n\in N} F)'$. If K is spherically complete, then f has an extension $g \in (\times_{n\in N} F)'$. For all $n \in N$, $g(\mathbf{a}_n) = f(\mathbf{a}_n) = 1$. This contradicts (3). □

In the following corollary, the implication (1) $\Rightarrow$ (2) is well known ([7], p. 70), (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious, and (4) $\Rightarrow$ (1) is a special case of Theorem 9. It is also well known that (5) is equivalent to (1) (cf. [10]); we include (5) here for the sake of completeness:

COROLLARY 10. *The following statements are equivalent.*

(1) *K is spherically complete.*

(2) *In every Banach space E over K, every weakly convergent sequence is norm-convergent.*

(3) *In l^∞, every weakly convergent sequence is norm-convergent*

(4) *There exists an element $x' \in (l^\infty)'$ such that* $\lim_n x'(e_n) \neq 0$, *where* $e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots), \ldots$.

(5) *l^∞ is spherically complete.*

COROLLARY 11. *The following statements are equivalent.*

(1) $L((l^\infty)', c_0) = C((l^\infty)', c_0)$.

(2) *K is spherically complete and* $L(l^\infty, c_0) = C(l^\infty, c_0)$.

PROOF. (1) $\Rightarrow$ (2). If K is not spherically complete, then $(l^\infty)'$ is linearly homeomorphic to c_0. This contradicts (1). Hence K is spherically complete. If the valuation of K is discrete, then $(l^\infty)'$ has a base. By Corollaries 4 and 7, we have $L((l^\infty)', c_0) \neq C((l^\infty)', c_0)$. Hence the valuation of K is dense. Combining Corollaries 4 and 8, we conclude that $L(l^\infty, c_0) = C(l^\infty, c_0)$.

(2) $\Rightarrow$ (1). Suppose that $L(l^\infty, c_0) = C(l^\infty, c_0)$, then the valuation of K is dense. Since K is spherically complete, $(l^\infty)'$ is spherically complete ([10], p. 101). By Corollary 5.20 in [10], $L((l^\infty)', c_0) = c((l^\infty)', c_0)$. □

4. Finally, we investigate spaces $L(E, c_0)$ in which $C(E, c_0)$ is complemented. If E' fails to have the s.c.p., then there exists a sequence $(x'_n) \subset E'$ such that for every $x \in E$ $\lim_n x'_n(x) = 0$ and $1 \leqq \|x'_n\| < 1/|\pi|$ for each $n \in N$. For $\lambda = (\lambda_n)_n \in l^\infty$, define $H_\lambda \in L(E, c_0)$ by $H_\lambda(x) = (\lambda_n x'_n(x))_n$.

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LEMMA 12. *Suppose that E' fails to have the s.c.p. Then the linear operator*

$$\Phi : l^\infty \longrightarrow L(E, c_0) : \lambda \longrightarrow H_\lambda$$

is injective and continuous. Moreover, if $\lambda \in c_0$, then $\Phi(\lambda) \in C(E, c_0)$.

REMARK. It is known for long (and in fact it follows easily from Proposition 2) that $T \in L(E, c_0)$ is compact iff $T(x) = (f_n(x))_n$ with $(f_n) \subset E'$, $\lim_n \|f_n\| = 0$.

PROOF. Since $\|x_n'\| \geqq 1$, Φ is injective. For every $\lambda \in l^\infty$, $\lambda \neq 0$,

$$\frac{\|\Phi(\lambda)\|}{\|\lambda\|} \leqq \frac{\|H_\lambda\|}{\|\lambda\|} \leqq \frac{1}{|\pi|}.$$

Hence Φ is continuous. Let $\lambda = (\lambda_n)_n \in c_0$. The conclusion $\Phi(\lambda) \in C(E, c_0)$ then follows directly from the above remark. □

PROPOSITION 13. *If E' fails to have the s.c.p., then there exists a continuous projection P in $L(E, c_0)$ such that $P(L(E, c_0)) = \Phi(l^\infty)$ and $P(C(E, c_0)) = \Phi(c_0)$.*

PROOF. For a number ϵ_0 $(0 < \epsilon_0 < 1)$, there exists $y_n \in E$ $(n \in N)$ such that

$$1 - \epsilon_0 \leqq \|x_n'\| - \epsilon_0 \leqq \frac{|x_n'(y_n)|}{\|y_n\|}.$$

Put $x_n = (x_n'(y_n))^{-1} y_n$. Then $x_n'(x_n) = 1$ and $\|x_n\| \leqq 1/(1 - \epsilon_0)$. For $T \in L(E, c_0)$, let $T(x) = (f_n(x))_n$, and define $\lambda_T \in l^\infty$ by $\lambda_T = (f_n(x_n))_n$. Set $P(T) = \Phi(\lambda_T)$. Then P is the required operator as we now show the following: Since $\|P\| \leqq 1/(1 - \epsilon_0)|\pi|$, P is continuous. For every $x \in E$,

$$(\Phi(\lambda_{\Phi(\lambda_T)}))(x) = (f_n(x_n) x_n'(x))_n = (\Phi(\lambda_T))(x).$$

It follows that $(P \circ P)(T) = P(T)$. Hence the operator P is a projection in $L(E, c_0)$. Let $\lambda = (\lambda_n)_n \in l^\infty$, and define $U \in L(E, c_0)$ by $U(x) = (\lambda_n x_n'(x))_n$ $(x \in E)$. Then $P(U) = \Phi(\lambda)$. This means that $P(L(E, c_0)) = \Phi(l^\infty)$. Let $S \in C(E, c_0)$. By the remark in Lemma 12, $S(x) = (g_n(x))_n$ with $(g_n) \subset E'$, $\lim_n \|g_n\| = 0$. Since $\|g_n(x_n)\| \leqq \|g_n\|/(1 - \epsilon_0)$, $\lambda_S = (g_n(x_n))_n \in c_0$. This implies that $P(C(E, c_0)) \subset \Phi(c_0)$. Conversely, let $\mu = (\mu_n)_n \in c_0$, and define a linear operator $R : E \longrightarrow c_0$ by $R(x) = (\mu_n x_n'(x))_n$. Then by the same proof as that of Lemma 12, it can be proved that $R \in C(E, c_0)$. It follows immediately from the definition of P that $P(R) = \Phi(\mu)$. Thus the proof is complete. □

THEOREM 14. *Let the valuation of K be dense. If E' fails to have the s.c.p. then $C(E, c_0)$ is not complemented in $L(E, c_0)$.*

PROOF. Suppose that $C(E, c_0)$ is complemented in $L(E, c_0)$. Then there exists a subspace L_0 of $L(E, c_0)$ such that $L(E, c_0) = C(E, c_0) \oplus L_0$. For every $\lambda \in l^\infty$, there

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exists $T_1 \in C(E, c_0)$ and $T_2 \in L_0$ such that $\Phi(\lambda) = T_1 + T_2$. By Lemma 12 and Proposition 13 there is a unique element λ_{T_1} of c_0 such that $P(T_1) = \Phi(\lambda_{T_1})$. Consider a linear operator Q from l^∞ into c_0 defined by $Q(\lambda) = \lambda_{T_1}$. By Lemma 12, $\Phi(\lambda_{T_1}) \in C(E, c_0)$, and thus $Q \circ Q(\lambda) = Q(\lambda_{T_1}) = \lambda_{\Phi(\lambda_{T_1})}$. Let (x_n) be the sequence in E which is defined in the proof of Proposition 13, and let $T_1(x) = (h_n(x))_n$ $(x \in E)$. Then $\lambda_{T_1} = (h_n(x_n))_n$ and $(\Phi(\lambda_{T_1}))(x) = (h_n(x_n)x_n'(x))_n$ for every $x \in E$. Hence,

$$\lambda_{\Phi(\lambda_{T_1})} = (h_n(x_n)x_n'(x_n))_n = (h_n(x_n))_n = \lambda_{T_1}.$$

This means that $Q \circ Q(\lambda) = Q(\lambda)$. We next show that Q is continuous. For every $\lambda \in l^\infty$, $\lambda \neq 0$,

$$\frac{\|Q(\lambda)\|}{\|\lambda\|} = \frac{\max\{|h_n(x_n)| : n \in N\}}{\|\lambda\|} \leqq \frac{\|T_1\|}{\|\lambda\|\,|\pi|}.$$

Since there exists t $(0 < t \leqq 1)$ such that $t\|T_1\| \leqq t\max(\|T_1\|, \|T_2\|) \leqq \|\Phi(\lambda)\|$ ([10], p. 63),

$$\frac{\|Q(\lambda)\|}{\|\lambda\|} \leqq \frac{\|\Phi(\lambda)\|}{t\|\lambda\|\,|\pi|} \leqq \frac{1}{t|\pi|^2}.$$

This implies that Q is continuous. Finally, for every $\mu \in c_0$, $Q(\mu) = \mu$. Thus Q is a continuous projection from l^∞ onto c_0. However, since the valuation of K is dense, this contradicts Corollary 5.19 in [10]. □

COROLLARY 15. *Let the valuation of K be dense. Then the following are equivalent.*

(1) *c_0 is not complemented in E. (Recall that E contains a subspace which is isomorphic to c_0).*

(2) *$L(E, c_0) = C(E, c_0)$.*

(3) *$C(E, c_0)$ is complemented in $L(E, c_0)$.*

PROOF. The equivalence (1) ⇔ (2) is due to De Grande-de Kimpe [2]. (Throughout her paper the spherical completeness of K is assumed, however this part holds without this assumption.) The implication (2) ⇒ (3) is obvious. We shall show the implication (3) ⇒ (2). If $C(E, c_0)$ is complemented in $L(E, c_0)$, then by Theorem 14 E' has the s.c.p. Hence $L(E, c_0) = C(E, c_0)$, which completes the proof. □

If the valuation of K is discrete, then by Corollary 6, $L(E, c_0) \neq C(E, c_0)$ and $C(E, c_0)$ is complemented in $L(E, c_0)$ ([10], p. 108). Combining these with Corollaries 7 and 15, we have the following:

COROLLARY 16. *Let the valuation of K be dense. If E has a base, then $C(E, c_0)$ is not complemented in $L(E, c_0)$.*

There is a variety of Banach spaces with basis; many of them appear in [10] and [11]. Further, we obtain the following result.

COROLLARY 17. *If K is not spherically complete and E has a base, then $C(l^\infty, E')$ is not complemented in $L(l^\infty, E')$. In particular, $C(l^\infty, l^\infty)$ is not complemented in $L(l^\infty, l^\infty)$.*

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To show this we need the following lemma:

LEMMA 18. *Let F be an infinite-dimensional Banach space such that F is complemented in F''. Suppose that $C(E,F)$ is not complemented in $L(E,F)$, then $C(F',E')$ is not complemented in $L(F',E')$.*

PROOF. In the Archimedean case this lemma was shown by T. H. Kuo [6]. In our case we can also prove similarly to his proof: Let $T \in L(E,F)$ and let $T' \in L(F',E')$ be the dual operator of T, then it is well known that $\|T\| = \|T'\|$, $T''|_E = T$ and that if $T \in C(E,F)$, then $T' \in C(F',E')$. Let Q be a continuous projection of F'' onto F. Suppose that R is a continuous projection of $L(F',E')$ onto $C(F',E')$. Define an operator P of $L(E,F)$ to $C(E,F)$ by $(PT)(x) = Q((RT')'(x))$. Then P is a continuous projection of $L(E,F)$ onto $C(E,F)$. This is a contradiction.

PROOF OF COROLLARY 17. By hypothesis c_0 is reflexive. Hence c_0 is complemented in c_0'' and $(c_0)' = l^\infty$. Since E has a base, c_0 is complemented in E. Using Corollary 16 and Lemma 18, we complete the proof. □

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