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AN EXTENSION OF A HARDY–LITTLEWOOD–PÓLYA INEQUALITY

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1.

The Hardy–Littlewood–Pólya inequality in question can be written in the form

$$\|x^{(1/p')-a}\int_0^\infty K(x/y)f(y)y^{-1}dy\|_p$$
$$\leqslant \|t^{-a}K(t)\|_1\|y^{(1/p')-a}f(y)\|_p. \tag{1.1}$$

Here and throughout, all functions are assumed to be locally integrable on $]0,\infty[$, $1\leqslant p\leqslant\infty$, $p^{-1}+(p')^{-1}=1$ (with similar conventions for q, r, s), $\|.\|_p$ is the usual norm on $L^p(0,\infty)$, and if the right hand side is finite, then (1.1) is understood to mean that

$$\tilde{K}f(x)=\int_0^\infty K(x/y)f(y)y^{-1}dy, \quad x>0 \tag{1.2}$$

defines a locally integrable function $\tilde{K}f$ for which (1.1) holds.

(1.1) is a paraphrase of Lemma 1 of (**3**). To see this, take

$$H(x,y)=x^{a-1-(1/p')}y^{(1/p')-a}K(y/x)$$

in that lemma. The lemma itself is an elaboration of Theorem 319 of (**2**) and was used by Kober for the investigation of the operators of fractional integration named after him. The form chosen here exhibits the relation of the inequality of Hardy, Littlewood, and Pólya to the well-known convolution inequality $\|f*g\|_p\leqslant\|f\|_1\|g\|_p$. This relationship becomes more conspicuous if one introduces the abbreviation

$$\|f\|_{a,p}=\|t^{(1/p')-a}f(t)\|_p, \tag{1.3}$$

when (1.1) becomes

$$\|\tilde{K}f\|_{a,p}\leqslant\|K\|_{a,1}\|f\|_{a,p}. \tag{1.4}$$

In several contexts, for instance in connection with the Stieltjes transformation, it is desirable to replace t^{-a} in (1.3) by a weighting factor which behaves at 0 like t^{-a} and at infinity like t^{-b}, say. The simplest such factor is

$$\zeta_{a,b}(t)=t^{-a}\quad\text{for}\quad 0<t\leqslant 1, =t^{-b}\quad\text{for}\quad t>1. \tag{1.5}$$

In analogy with (1.3) the notation

$$\|f\|_{a,b,p}=\|t^{1/p'}\zeta_{a,b}(t)f(t)\|_p \tag{1.6}$$

will be used. It is then natural to ask whether there is an inequality similar to (1.4) but

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involving (1.6). The result we give corresponds to $\|f*g\|_q \leq \|f\|_r\|g\|_p$ with $p^{-1}-q^{-1}+r^{-1}=1$(2, Theorem 280) (this includes $\|f*g\|_p \leq \|f\|_1\|g\|_p$), and the proof is modelled on that given in (2) for the corresponding inequality for sums.

Let

$$1 \leq p, q, r \leq \infty, p^{-1}-q^{-1}+r^{-1}=1 \tag{1.7}$$

$$a \geq \theta, a \geq A, b \leq \eta, b \leq B, A \leq \eta, B \geq \theta. \tag{1.8}$$

Then

$$\|\tilde{K}f\|_{A,B,q} \leq \|K\|_{\eta,\theta,r}\|f\|_{a,b,p}. \tag{1.9}$$

This inequality has applications to Mellin transforms, Stieltjes transforms, integrals of fractional order, and certain integral transforms with hypergeometric kernels (5).

2.

The key to the proof of (1.9) is the following

Lemma. *Under the conditions* (1.8),

$$\zeta_{A,B}(x) \leq \zeta_{a,b}(y)\zeta_{\eta,\theta}(x/y) \quad x>0, y>0.$$

The lines $x=1$, $y=1$, $x=y$ divide the positive quadrant of the x, y plane into six parts, and in each of the six parts the lemma can be verified by means of the explicit expressions for the ζ's. For instance, if $0<x\leq y\leq 1$, then

$$\zeta_{A,B}(x)/[\zeta_{a,b}(y)\zeta_{\eta,\theta}(x/y)] = (x/y)^{\eta-A}y^{a-A} \leq 1.$$

To prove (1.9), assume first $r>1$ so that $p<q$ and $p>1$ so that $r<q$. By the lemma we have

$$\begin{aligned} g(x) &= x^{1/q'}\zeta_{A,B}(x)|\tilde{K}f(x)| \\ &\leq x^{1/q'}\int_0^\infty \zeta_{\eta,\theta}(x/y)\zeta_{a,b}(y)|K(x/y)f(y)|y^{-1}dy \\ &= \int_0^\infty \phi\psi\chi\, dy, \end{aligned}$$

where

$$\phi = |y^{1/p'}\zeta_{a,b}(y)f(y)|^{1-(p/q)}$$
$$\psi = |xy^{-1-(1/r)}\zeta_{\eta,\theta}(x/y)K(x/y)|^{1-(r/q)}$$
$$\chi = |y^{1/p'}\zeta_{a,b}(y)f(y)|^{p/q}|x^{1/r'}y^{-1}\zeta_{\eta,\theta}(x/y)K(x/y)|^{r/q}.$$

Since

$$\left(\frac{pq}{q-p}\right)^{-1}+\left(\frac{qr}{q-r}\right)^{-1}+q^{-1}=1$$

from (1.7), we may apply the extended Hölder inequality

$$g(x) \leqslant \|\phi\|_{pq/(q-p)} \|\psi\|_{qr/(q-r)} \|\chi\|_q.$$

Then

$$|g(x)|^q \leqslant \|f\|_{a,b,p}^{q-p} \|K\|_{\eta,\theta,r}^{q-r} \int_0^\infty \chi^q dy,$$

and by integration

$$\|g\|_q^q \leqslant \|f\|_{a,b,p}^{q-p} \|K\|_{\eta,\theta,r}^{q-r} \|K\|_{\eta,\theta,r}^r \|f\|_{a,b,p}^p,$$

which is (1.9).

If $r = 1$ and $p = q$, then $\phi = 1$,

$$(g(x))^p \leqslant \|K\|_{\eta,\theta,1}^{p-1} \int_0^\infty \chi^q dy,$$

and (1.9) follows as before; and if $p = 1$ so that $q = r$, then $\psi = 1$ and the proof is similar.

3.

Under suitable conditions operators of the form (1.2) can be composed, and it is of interest to know whether the composition is commutative. For instance, the commutative property of Kober's operators of fractional integration (**3**, Theorem 4) gives the so-called second index law (**4**) of fractional calculus.

Let

$$\begin{gathered} p^{-1} - q^{-1} + r^{-1} + s^{-1} = 2, \quad r^{-1} + s^{-1} \geqslant 1, \\ a \geqslant \max(\theta, \tau), \quad b \leqslant \min(\eta, \sigma), \quad \eta \geqslant \tau, \quad \theta \leqslant \sigma \\ A \leqslant \min(a, \eta, \sigma), \quad B \geqslant \max(b, \theta, \tau). \end{gathered} \tag{3.1}$$

If $\|f\|_{a,b,p}$, $\|H\|_{\sigma,\tau,s}$, $\|K\|_{\eta,\theta,r}$ are finite, then (1.9) shows that $\tilde{H}(\tilde{K}f)$ can be represented by a repeated integral in which the order of integrations may be interchanged by Fubini's theorem

$$\tilde{H}(\tilde{K}f) = \tilde{K}(\tilde{H}f) = \tilde{G}f, \tag{3.2}$$

where

$$G(t) = \int_0^\infty H(t/u)K(u)u^{-1}du. \tag{3.3}$$

Furthermore, if we set $m^{-1} = r^{-1} + s^{-1} - 1$, then

$$\|G\|_{\min(\eta,\sigma),\, \max(\theta,\tau),\, m} \leqslant \|H\|_{\sigma,\tau,s} \|K\|_{\eta,\theta,r} \tag{3.4}$$

by (1.9), and a second application of (1.9) results in

$$\|\tilde{G}f\|_{A,B,q} \leqslant \|H\|_{\sigma,\tau,s} \|K\|_{\eta,\theta,r} \|f\|_{a,b,p}. \tag{3.5}$$

Other deductions from (1.9) include results on the integral operator $\hat{K}$ defined by

$$\hat{K}f(x) = \int_0^\infty K(xt)f(t)dt. \tag{3.6}$$

If (1.7) holds and

$$a \geqslant 1-\eta, \quad b \leqslant 1-\theta, \quad A \leqslant \min(1-b, \eta), \quad B \geqslant \max(1-a, \theta),$$

then

$$\|\hat{K}f\|_{A,B,q} \leqslant \|K\|_{\eta,\theta,r}\|f\|_{a,b,p}. \tag{3.7}$$

4.

We shall illustrate the application of (1.9) to integral transforms by considering fractional integrals. The definition of these has been extended in (1) to functions which fail to satisfy the integrability conditions at the fixed limit of integration, and this extension will be used here.

Let

$$K_0(t) = 0 \quad \text{for} \quad 0 < t \leqslant 1, \; = \frac{1}{\Gamma(\lambda)}\frac{1}{t}\left(1-\frac{1}{t}\right)^{\lambda-1} \quad \text{for} \quad t > 1,$$

$$K_h(t) = K_0(t) + \sum_{r=0}^{h-1}(-t)^{-r-1}/[r!\Gamma(\lambda-r)] \quad h = 1, 2, 3, \ldots.$$

Then $K_h(t) = O(t^{-h})$ if $h \geqslant 1$, and $t \to 0$, and $K_h(t) = O(t^{-h-1})$ for all h as $t \to \infty$. It follows that $\|K_0\|_{\eta,\theta,r} < +\infty$ for $\mathrm{Re}\lambda > 1/r'$, $\theta > 0$, and any η, while for $h \geqslant 1$ we have $\|K_h\|_{\eta,\theta,r} < +\infty$ for $\mathrm{Re}\lambda > 1/r'$, $\eta < 1-h$, $\theta > -h$. Also, $\|K_0\|_{\eta,\theta,\infty} < +\infty$ for $\mathrm{Re}\lambda \geqslant 1$, $\theta \geqslant 0$; and for $h \geqslant 1$, $\|K_h\|_{\eta,\theta,\infty} < +\infty$ for $\mathrm{Re}\lambda \geqslant 1$, $\eta \leqslant 1-h$, $\theta \geqslant -h$.

The extended definition (1, (1)) of I^λ is

$$I^\lambda f(x) = x^\lambda \tilde{K}_h f(x),$$

where h is a non-negative integer ($h \in Z_+$) and f is subject to certain integrability conditions depending on h. From (1.9) we now have the following results:–

Let $\mathrm{Re}\lambda > p^{-1} - q^{-1} \geqslant 0$, and let $h \in Z_+$. If $h = 0$, assume $a > 0$ and if $h \geqslant 1$ assume $0 < a + h < 1$, $b + h < 1$. Also let $A \leqslant a$, $B \geqslant b$, $B + h > 0$. Then

$$\|I^\lambda f\|_{A+\mathrm{Re}\lambda, B+\mathrm{Re}\lambda, q} \leqslant C\|f\|_{a,b,p}. \tag{4.1}$$

Let $\mathrm{Re}\lambda \geqslant 1$ and let $h \in Z_+$. If $h = 0$, assume $a \geqslant 0$, and if $h \geqslant 1$ assume $0 \leqslant a + h < 1$, $b + h \leqslant 1$. Also let $A \leqslant a$, $B \geqslant b$, $B + h \geqslant 0$. Then (4.1) holds with $p = 1$, $q = \infty$.

Again, let

$$H_0(t) = \frac{1}{\Gamma(\lambda)}\frac{1}{t}\left(\frac{1}{t}-1\right)^{\lambda-1} \quad \text{for} \quad 0 < t < 1, = 0 \quad \text{for} \quad t \geqslant 1$$

$$H_h(t) = H_0(t) + \sum_{r=0}^{h-1}(-1)^{r+1}t^{r-\lambda}/[r!\Gamma(\lambda-r)] \quad h = 1, 2, 3, \ldots,$$

and define J^λ by

$$J^\lambda f(x) = x^\lambda \tilde{H}_h f(x).$$

Then $\|H_0\|_{\eta,\theta,r} < +\infty$ for $\mathrm{Re}\lambda > 1/r'$, $\eta < 1 - \mathrm{Re}\lambda$, and any θ, while for $h \geqslant 1$, $\|H_h\|_{\eta,\theta,r} < +\infty$ for $\mathrm{Re}\lambda > 1/r'$, $\eta < h + 1 - \mathrm{Re}\lambda$, $\theta > h - \mathrm{Re}\lambda$. Also, $\|H_0\|_{\eta,\theta,\infty} < +\infty$ for $\mathrm{Re}\lambda \geqslant 1$, $\eta \leqslant 1 -$

Reλ, and for $h \geqslant 1$, $\|H_h\|_{\eta,\theta,\infty} < +\infty$ for Re$\lambda \geqslant 1$, $\eta \leqslant h + 1 - \text{Re}\lambda$, $\theta \geqslant h - \text{Re}\lambda$. From (1.9) we now have the following results:–

Let Re$\lambda > p^{-1} - q^{-1} \geqslant 0$, and let $h \in Z_+$. If $h = 0$, assume $b < 1$, and if $h \geqslant 1$ assume $a > h$, $h < b < h + 1$. Also let $A \leqslant a$, $A < h + 1$, $B \geqslant b$. Then

$$\|J^\lambda f\|_{A,B,q} \leqslant C\|f\|_{a-\text{Re}\lambda, b-\text{Re}\lambda, p}. \tag{4.2}$$

Let Re$\lambda \geqslant 1$ and let $h \in Z_+$. If $h = 0$, assume $b \leqslant 1$, and if $h \geqslant 1$ assume $a \geqslant h$, $h < b \leqslant h + 1$. Also let $A \leqslant a$, $A \leqslant h + 1$, $B \geqslant b$. Then (4.2) holds with $p = 1$, $q = \infty$.

In (4.1) and (4.2) C is independent of f.

The inequality (3.5) has applications to composition of integral transformations, leading to results corresponding to e.g., (**1**, (14), (15)) or (**5**, (7), (11)).

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