Some Remarks on de Longchamps Chain

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(*Received* 5*th February*, 1945. *Read* 2*nd March*, 1945).

The chain of circle theorems now associated with the name of de Longchamps[1] may be stated as follows.

(i) Any two lines $L(A)$, $L(B)$ meet in a point (AB).

(ii) Any three lines $L(A)$, $L(B)$, $L(C)$ determine three points (AB) (BC) (CA) which lie on a circle $S(ABC)$ with centre (ABC).

(iii) Any four lines $L(A)$, $L(B)$, $L(C)$, $L(D)$ determine four circles $S(ABC)$ $S(ABD)$ $S(ACD)$ $S(BCD)$ which meet in a point $P(ABCD)$ and four centres (ABC) (ABD) (ACD) (BCD) which lie on a circle $S(ABCD)$ with centre $(ABCD)$.

(iv) Any five lines $L(A)$, $L(B)$, $L(C)$, $L(D)$, $L(E)$ determine five points like $P(ABCD)$, which lie on a circle $SM(ABCDE)$ (Miquel's Theorem[2]), five circles like $S(ABCD)$, which meet in a point $P(ABCDE)$, and their five centres like $(ABCD)$, which lie on a circle $S(ABCDE)$ with centre $(ABCDE)$.

(v) Any six lines $L(A)$, ..., $L(F)$ determine six circles like $SM(ABCDE)$ which meet in a point $PM(ABCDEF)$, six circles like $S(ABCDE)$, which meet in a point $P(ABCDEF)$ and six centres like $(ABCDE)$ which lie on a circle $S(ABCDEF)$ with centre $(ABCDEF)$.

And so on indefinitely. The part of the chain concerning the points $P(ABCD)$, the circles $SM(ABCDE)$, the points $PM(ABCDEF)$, etc., is called Clifford's chain[3].

Two special relationships have been proved. (*a*) In the four line case, $P(ABCD)$ lies on $S(ABCD)$ (Steiner's theorem[4]). (*b*) In the five line case $P(ABCDE)$ lies on $SM(ABCDE)$ (Bath's theorem[5]).

In Richmond's extension[6] of the chain step (i) above is replaced by — (i). Any two points $P(A)$ $P(B)$, through which go lines $l(A)$ $l(B)$ respectively, determine a circle $S(AB)$, through $P(A)$ $P(B)$ and the intersection of $l(A)$ $l(B)$, whose centre is (AB); and appropriate modifications are made in the subsequent steps.

In section 1 of this note we define a construct of circles, the circular complete n-point, and apply this to deduce an analogue of the standard de Longchamps chain starting from circles instead of from lines. This is used in section 2 to show that ancillary to any case of the standard chain there exist infinitely many families of subsidiary points and circles. Section 3 then deduces corollaries of Steiner's and Bath's theorems, introducing further circles connected with the four and five line cases.

1. A circular analogue of Clifford's chain is readily obtained by inversion from the standard Clifford chain configurations. This case has been mentioned frequently; see for example Homersham Cox[7], Grace[8], or Godt[9]. We shall now find an analogue of the general de Longchamps chain; let us restrict ourselves initially to the 4-point case. Given four circles $S(A)$ $S(B)$ $S(C)$ $S(D)$ all passing through a point P, then any two, say $S(A)$ $S(B)$, cut again in a point (AB); the three points (AB) (AC) (AD) lie on the circle $S(A)$ and the three points (AB) (AC) (BC) determine a circle $S(ABC)$. Then, 1], $S(ABC)$ $S(ABD)$ $S(ACD)$ $S(BCD)$ all pass through a point $P(ABCD)$. We have in this manner eight points and eight circles, four points on each circle and four circles through each point—the configuration known as a complete 4-point.

Let A B .... (ABC) (ABD) .... be the centres of $S(A)$ $S(B)$ .... $S(ABC)$ $S(ABD)$ ..... Then in general (ABC) (ABD) (ACD) (BCD) do not lie on a circle. But now *let the four points A B C D lie on a circle $S(O)$*, and let us construct the complete 4-point from $S(A)$ $S(B)$ $S(C)$ $S(D)$: call it a *circular complete 4-point*.

2] (ABC) (ABD) (ACD) (BCD) lie on a circle $S(ABCD)$. For angle $(BCD)(ABD)(ABC) + (ABC)(ACD)(BCD) = (BP).P(ABCD).(AB) + (AC).P(ABCD).(CD) = (BD)(AD)(AB) + (AC)(AD)(CD) = (BD)(AD)(AC) + (AC)(AD)(AB) + (AC)(AD)(BD) + (BD)(AD)(CD) = (AC)P(AB) + (BD)P(CD) = CAB + BDC = 0$. (The angles are directed and the equations are congruences modulo π.)

3] A B $(ABC)(ABD)$ lie on a circle $S(A_1B_1)$. For angle $(ABD)\,B\,(ABC) = (DB)(BA)(BC) = (BD)\,P\,(BC) = DBC$ and similarly angle $(ABD)\,A\,(ABC) = DAC$. There are six such circles.

The eight points A B C D $(ABC)(ABD)(ACD)(BCD)$ and the eight circles $S(O)$ $S(ABCD)$ $S(A_1B_1)$ .... $S(C_1D_1)$ clearly form a complete 4-point. Let O .... (C_1D_1) be the centres of $S(O)$ .... $S(C_1D_1)$. Then angle $(A_1D_1)(A_1C_1)\,O = (ACD)\,AC = (AD)(AC)\,P$ and similarly angle $(A_1D_1)(A_1B_1)\,O = (AD)(AB)\,P$, and since (AB) (AC) (AD) P are concyclic so are (A_1B_1) (A_1C_1) (A_1D_1) O.

4] *The centres of the circles of a circular complete 4-point are the*

https://doi.org/10.1017/S0013091500024433 Published online by Cambridge University Press

points of a second circular complete 4-*point.* The centres of the circles of this second set are the points of a third set and so on, the process continuing indefinitely.

Now take a fifth circle $S(E)$ through P whose centre E lies on $S(O)$. We have then ten points (AB) .... (DE), ten circles $S(ABC)$ .... $S(CDE)$, and five points $P(ABCD)$ .... $P(BCDE)$ as defined above. By the inversion of the 5-line Clifford's chain configuration, these five points $P(ABCD)$ .... $P(BCDE)$ lie on a circle $SM(ABCDE)$. This closes a complete 5-point of 16 points and 16 circles, five points on each circle and five circles through each point, and we will call it circular because $A\ B\ C\ D\ E$ lie on $S(O)$. Then, as in 2], (ABC) (ABD) (ACD) (BCD) lie in a circle $S(ABCD)$ of centre $(ABCD)$; there are five such circles. Again, as in 3], $A\ B\ (ABC)\ (ABD)$ lie on a circle, and so do $A\ B\ (ABC)\ (ABE)$, so that $A\ B\ (ABC)\ (ABD)\ (ABE)$ lie on a circle $S(A_1B_1)$ of centre (A_1B_1); there are ten such circles.

6] $S(ABCD)\ S(ABCE)\ S(ABDE)\ S(ACDE)\ S(BCDE)$ all pass through $M(ABCDE)$, the centre of $SM(ABCDE)$. For $S(A)$ $S(ABE)\ S(ACE)\ S(ADE)$ all meet in (AE) and have their centres on $S(A_1E_1)$; hence on forming their complete 4-point it is circular. Therefore these four circles with $S(ABC)\ S(ABD)\ S(ACD)$ $SM(ABCDE)$ are the circles of a circular complete 4-point; so their centres are the points of a circular complete 4-point and $S(ABCD)$ $S(ABCE)\ S(ABDE)\ S(ACDE)$ go through $M(ABCDE)$. By a similar argument $S(BCDE)$ goes through $M(ABCDE)$.

The 16 centres thus lie on 16 circles and form a complete 5-point. The proof of 6] above, with a further obvious application of 4], shows that it is circular. So that

7] *the centres of the circles of a circular complete* 5-*point are the points of a second circular complete* 5-*point.* As happened in 4] we obtain in general from the second set a third and then a fourth and so on as far as we wish.

It is clear that both the conceptions and the proofs used above may be extended to a 6-point, to a 7-point and so generally.

We have proved *en passant* a circle analogue of de Longchamp's chain.

8] *Let* $A\ B\ C\ D\ E$ .... *be any concyclic points and* P *any other point taken generally. Then any point* A *say determines a circle* $S(A)$ *through* P *with centre* A. *Our initial statement of de Longchamps' chain remains true if* $L(A)\ L(B)$.... *are replaced throughout by* $S(A)\ S(B)$.....
The analogues of Steiner's and of Bath's theorems do not hold; in

fact in the latter case $P(ABCDE)$ is $M(ABCDE)$, the *centre* of $SM(ABCDE)$.

2. *The Families of Ancillary Circles.* Let us return to the standard de Longchamps' chain (the considerations of this paragraph apply with equal force to Richmond's extension), and let us deal first with the 5-line case. At the final stage of the chain we have five circles $S(ABCD) \ldots . S(BCDE)$ which meet in a point $P(ABCDE)$ and whose centres $(ABCD) \ldots . (BCDE)$ lie on a circle $S(ABCDE)$. From these we may therefore construct a complete circular 5-point. Consequently, if $S(A'B'CDE)$ be the circle through (ABC) (ABD) (ABE) there are ten such circles, and $S(A'B'CDE)$ $S(A'BC'DE)$ $S(A'BCD'E)$ $S(A'BCDE')$ *all pass through a point* $(A'BCDE)$. There are five such points and *they all lie on a circle* $S_1(ABCDE)$. Furthermore the 16 centres of these circles determine 16 other circles, the only one appearing in the original enunciation being $S(ABCDE)$; *there being also ten circles typified by say* $S(A'B'C'DE)$ *through* $(ABCD)$ $(ABCE)$ $(AB'C'DE)$ $(A'BC'DE)$ $(A'B'CDE)$ *and five circles typified by say* $S_1(A'BCDE)$ *through* $(A'B'CDE)$ $(A'BC'DE)$ $(A'BCD'E)$ $(A'BCDE')$ $(ABCDE)_1$. The centres of these give yet another 5-point and so on indefinitely.

In the 6-line case we have, by omitting one line at a time six series of 5-points such as are described above. These do not combine into a single 6-point. In fact through say (ABC) go twelve circles $S(A'B'CDE)$ $S(A'BC'DE)$ $S(AB'C'DE)$ $S(A'B'CDF)$ $S(A'BC'DF)$ $S(AB'C'DF)$ $S(A'B'CEF)$ $S(A'BC'EF)$ $S(AB'C'EF)$ $S(ABCD)$ $S(ABCE)$ $S(ABCF)$, and these do not belong to any *general* n-point, since $S(A'B'CDE)$ $S(A'B'CDF)$ $S(ABCD)$ for instance all go also through (ABD). There is also a series of 6-points based on the fact that $S(ABCDE) \ldots . S(BCDEF)$ meet in a point and have concyclic centres. For the first member, if the circle through $(ABCD)$ $(ABCE)$ $(ABCF)$ is called $S(A'B'C'DEF)$, then $S(A'B'C'DEF)$ $S(A'B'CD'EF)$ $S(A'B'CDE'F)$ $S(A'B'CDEF')$ *all meet in a point* $(A'B'CDEF)$, *the points* $(A'B'CDEF)$ $(A'BC'DEF) \ldots . (A'BCDEF')$ *all lie on a circle* $S(A'BCDEF)$ *and the circles* $S(A'BCDEF) \ldots . S(ABCDEF')$ *meet in a point* $(ABCDEF)_1$.

These results extend to a 7-point and so generally.

3. *Steiner's and Bath's Theorems.* The ancillary circles discussed in the last section do not arise in the 4-line case of de Longchamps' chain (though they do in Richmond's extension). But here Steiner's

https://doi.org/10.1017/S0013091500024433 Published online by Cambridge University Press

theorem states that $P(ABCD)$ lies on $S(ABCD)$, so through $P(ABCD)$ pass five circles. Let us built up the resulting complete 5-point. It will not be circular since $(ABCD)$ will not lie on $S(ABCD)$.

Write $S(a) \equiv S(BCD)$, $S(b) \equiv S(ACD)$, $S(c) \equiv S(ABD)$, $S(d) \equiv S(ABC)$, $S(e) \equiv S(ABCD)$. Then $(ab) \equiv (CD)$, ..., $(cd) \equiv (AB)$; (ae) is the intersection of $S(BCD)$ and $S(ABCD)$. Call it $P(BCDE)$ and define similarly $P(ACDE)$ $P(ABDE)$ $P(BCDE)$. $S(abc)$ is the circle through (ab) (ac) (bc), *i.e.*, through (CD) (BD) (AD); so $S(abc) \equiv L(D)$, and similarly $S(abd) \equiv L(C)$, $S(acd) \equiv L(B)$, $S(bcd) \equiv L(A)$. $S(cde)$ is a circle through (ce) (de) (cd), *i.e.*, through $P(ABDE)$ $P(ABCE)$ (AB). This circle has not appeared earlier in the figure; call it $S(ABE)$ and define similarly $S(ACE)$ .... $S(CDE)$. $P(abcd)$ is the common point of $S(abc)$ $S(abd)$ $S(acd)$ $S(bcd)$, *i.e.*, of $L(D)$ $L(C)$ $L(B)$ $L(A)$, so $P(abcd) \equiv \infty$, the point at infinity. $P(bcde)$ is the common point of $S(bcd)$ $S(bce)$ $S(bde)$ $S(cde)$, *i.e.*, of $L(A)$ $S(ADE)$ $S(ACE)$ $S(ABE)$. So these meet in a point which we shall call (AE). Similarly we have points (BE) (CE) (DE). Now $P(abcd)$ $P(abce)$ $P(abde)$ $P(acde)$ $P(bcde)$ lie on a circle $SM(abcde)$, so ∞ (AE) (BE) (CE) (DE) lie on a circle, that is, (AE) (BE) (CE) (DE) lie on a line which we shall call $L(E)$.

The resulting configuration is symmetrical in the five letters $ABCDE$; if for instance we begin with $L(B)$ $L(C)$ $L(D)$ $L(E)$ the line appearing at the end is $L(A)$. *Given four lines they determine a de Longchamps configuration containing five circles. Associated with this are six further circles and a further line, and the same entire configuration is obtained from whatever four of these five lines we begin.*

$P(ABCD)$ $P(ABCE)$ $P(ABDE)$ $P(ACDE)$ $P(BCDE)$ all lie on $S(ABCD)$, so they also lie on $S(ABCE)$ $S(ABDE)$ $S(ACDE)$ $S(BCDE)$. So in fact these circles all coincide in a circle on which lie 15 points, viz., all the points (ABC) .... (CDE) and $P(ABCD)$ .... $P(BCDE)$. *Any four of the five lines $L(A)$ .... $L(E)$ have the same "final" circle in their appendent de Longchamps configuration.*

In exactly the same way in the 5-line case Bath's theorem states that the six circles $S(ABCD)$ .... $S(BCDE)$ $SM(ABCDE)$ meet in a point, and may thus be used to form a complete 6-point, leading in this way to still further circles connected with the 5-line. In the case of Steiner's theorem the 5-point developed contained the Clifford chain circles and points, including ∞, and there is thus introduced a new line; but the 6-point based on Bath's theorem contains not the

Clifford chain circles and points, but their centres and the circles through them.

We shall conclude with an elementary plane proof of Bath's theorem, showing its dependence on Steiner's theorem. This implies that $P(ABCDE)$ $P(ABCD)$ (ABC) (ABD) all lie on $S(ABCD)$. Then angle $P(ABCD)$ $P(ABCDE)$ (ABC) = angle $P(ABCD)$ (ABD) (ABC) $= \frac{1}{2}$ angle $P(ABCD)$ (ABD) (AB) [since $S(ABD)$ and $S(ABC)$ cut in (AB) and $P(ABCD)$] = angle $P(ABCD)$ $P(ABDE)$ (AB) [since $P(ABDE)$ also lies on $S(ABD)$]. Similarly angle (ABC) $P(ABCDE)$ $P(ABCE)$ = angle (AB) $P(ABDE)$ $P(ABCE)$. And therefore, adding, angle $P(ABCD)$ $P(ABCDE)$ $P(ABCE)$ = angle $P(ABCD)$ $P(ABDE)$ $P(ABCE)$, whence $P(ABCDE)$ is concyclic with $P(ABCD)$ $P(ABCE)$ $P(ABDE)$.

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