5

Playing the Novikov Game

5.1 Overview

It turns out that the topological Novikov conjecture is only the first example of a more general phenomenon wherein the fundamental group of a manifold (or variety or . . .) plays an extremely large role on the geometry of the manifold – often mediated through analysis. And, as is clear from Chapter 4.1, this theme also extends to noncompact manifolds where the role of the fundamental group is supplemented by the quasi-isometry type of the manifold.

This chapter is about the "Novikov game": what it is, how to play, and what are the typical things that happen when you play.

One starts with a theorem about characteristic classes (or an index) true for all closed manifolds and interprets as being merely the simply connected version of a more general statement, hopefully true for all groups, where one augments the simply connected statement by the cohomology of the fundamental group.

As far as I can tell, the first player of this game was Reinhardt Schultz, in the mid-1970s. One of the nice topological applications of the index theorem is:

Theorem 5.1 (Atiyah and Hirzebruch (1970)) *If M is a closed smooth spin manifold, and M admits a nontrivial smooth S^1-action, then*

$$\langle A(M), [M] \rangle = 0.$$

For the definition of the A-genus of a spin manifold, see Borel and Hirzebruch (1959a,b, 1960): it is an analogue of the L-genus that played such an important role in Chapter 4.

Schultz asked whether this theorem is true for the "higher A-genus" for non-simply connected manifolds that have smooth circle actions.

One has to be a little careful with this statement. After all, the higher A-number associated to the fundamental class of any $K(\pi, 1)$-manifold is nonzero, but the torus $\mathbb{T}^n$ has a circle action! The way around this issue is to not consider

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all of $\pi_1(M)$, but rather the part that is "orthogonal to the circle," i.e. the following:

Theorem 5.2 (Schultz's conjecture; see Browder and Hsiang, 1982) *If M is a smooth spin manifold admitting a smooth S^1-action, and $f\colon M \to K(\pi,1)$ is any map, then for any $\alpha \in H*\big(K(\pi/\mathrm{orb},1);\mathbb{Q}\big)$, the higher A-genus vanishes; that is $\langle f*(\alpha) \cup A(M),[M]\rangle = 0$.*

Here orb is the class in the fundamental group represented by any orbit. (This class is clearly independent of the orbit. It always lies in the center of the fundamental group so we can quotient by the subgroup it generates.)

Similarly, there is another result for (smooth) S^1-actions:

Theorem 5.3 (Atiyah–Singer) *If S^1 acts on a compact manifold M, then* $\operatorname{sign}(M) = \operatorname{sign}(F)$*, where F is the fixed set of the action (if F is suitably oriented).*[1]

This has the expected generalization to the non-simply connected case.

Theorem 5.4 (Weinberger, 1985b, 1987) *If S^1 acts on a compact manifold M, $f\colon M \to K(\pi,1)$ is any map, and F is the fixed set of the action (if F is suitably oriented), then*

$$\langle f^*(\alpha) \cup L(M),[M]\rangle = \langle f^*(\alpha) \cup L(F),[F]\rangle$$

for all $\alpha \in H^*(K(\pi/\mathrm{orb},1);\mathbb{Q}$ (which equals $H^*(K(\pi,1);\mathbb{Q})$ if $F \neq \varnothing$).

However, when we begin examining the same story for $\mathbb{Z}/n$ actions, the situation is more complicated.

Theorem 5.5 (Consequence of the G-signature theorem) *Suppose that $\mathbb{Z}/n$ action acts homologically trivially on M, that $f\colon M \to K(\pi,1)$ is any map, and that F is the fixed point set of the action. Then there is a characteristic class $c(\nu)$*[2] *of the equivariant normal bundle to F, so that*

$$\operatorname{sign}(M) = \langle c(\nu) \cup L(F),[F]\rangle$$

(so that, if $F \neq \varnothing, \operatorname{sign}(M) = 0$).

Theorem 5.6 (Weinberger, 1985b, 1987) *Suppose that $\mathbb{Z}/n$-action acts homologically trivially on M,*[3] *that $f\colon M \to K(\pi,1)$ is any map, and that F is*

[1] This formula makes sense and is true even for topological actions – at least if the fixed set is an ANR. That it makes sense is due to the fact (see Borel, 1960) that the fixed set of a circle action is automatically a rational homology manifold.

[2] Here $c(\nu)$ is the average over the generators of $\mathbb{Z}/n$ of the characteristic classes arising in the formula for tr_g G-signature in Atiyah and Singer (1968b).

[3] This means that the $\mathbb{Z}/n$-action lifts to the universal cover, commutes with the action of the covering translates, and acts trivially on the rational homology there.

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the fixed-point set of the action. Then for the characteristic class $c(\nu)$ of the equivariant normal bundle to F mentioned above, the formula for the higher signature of M:

$$\mathrm{Sign}_{\alpha}(M) = \langle f*(\alpha) \cup c(\nu) \cup L(F), [F]\rangle$$

is true iff the Novikov conjecture is true for the group π.

We shall discuss more the interaction between the Novikov conjecture and group actions below in this and the succeeding chapters – because it turns out to be actually a somewhat different problem, and it takes some work[4] to find an equivariant version that is provably exactly equivalent to the original problem! (The first things one thinks of seem to be of the same depth as the Novikov conjecture – i.e. proofs of the Novikov conjecture usually affirm these as well – but not quite provably equivalent to it.)

Let me mention one last example of an equivariant problem that we will see works out rather differently:

Theorem 5.7 *Define an action of G on X to be* pseudo-trivial *if* $(X \times \mathrm{EG})/G \cong X \times \mathrm{BG}$. *If $G = S^1$ or $\mathbb{Z}/p\mathbb{Z}$ acts pseudo-trivially on a (noncompact) manifold (or manifold with boundary) homotopy equivalent to a closed manifold M, then, if the fixed set X^G is a compact manifold, we have*

$$\mathrm{sign}\,(M) = \mathrm{sign}\,(X^G).$$

The proof of this follows from Smith theory[5]. The map $X^G \to M$ can be seen to be a rational homology equivalence, and *a fortiori* preserves signatures. Later, we will discuss what happens in the non-simply connected case is a provocative problem.

As we move from topology to differential geometry and beyond, the problems we study do not seem to have direct implications for the original Novikov conjecture. They are in the *spirit* of the problem; they are analogues and can be studied simultaneously and profitably.

The most prominent example is the question of "which manifolds have metrics of positive scalar curvature?" Recall that if M is a Riemannian manifold, then the scalar curvature is a function on M that measures infinitesimally the

[4] This is all meant philosophically. Conceivably all the currently unknown versions of the Novikov conjecture are true, and then they will be equivalent to each other . . . However, we will see that working on the equivariant version very quickly leads one to introducing coefficients and other refinements and extensions of the original problem.

[5] One can improve this to where G is a p-group or an extension of a torus by a p-group. But for non-p-groups or Lie groups with nonabelian identity components, the relationship between M and M^G is much more tenuous, even for pseudo-trivial actions. Indeed, in the noncompact case, one can always arrange for M^G to be empty (as the reader should be able to prove after reading §7.1). In the compact case, achieving this also requires a condition on $\chi(M)$.

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extent to which the Riemannian volumes of balls of radius r deviate from the Euclidean volumes of balls of the same radius:

$$\mathrm{Vol}(\mathrm{B}(r)) = \omega_n r^n - [K(p)/6(n+1)]r^{n+2} + O(r^{n+3})$$

so positive scalar curvature means that balls are infinitesimally smaller than they "should be." The Gauss–Bonnet theorem implies that the only connected oriented surface with positive scalar curvature is the sphere.

Using the index theorem for the Dirac operator and a Bochner-type formula that Lichnerowicz discovered, Atiyah, Lichnerowicz, and Singer gave the first obstructions to any manifold of dimension greater than 2 having positive scalar curvature in the following theorem:

Theorem 5.8 (Atiyah and Singer, 1968b) *If M is a compact spin manifold with a Riemannian metric of positive scalar curvature, then $\langle A(M), [M]\rangle = 0$.*

This suggests, according to the same pattern:

Conjecture 5.9 (Gromov–Lawson–Rosenberg; see Gromov and Lawson (1980a, 1980b)) *If M is a compact spin manifold*[6] *with positive scalar curvature, and $f\colon M \to K(\pi, 1)$ is any map, then*

$$f * (\alpha)\langle \cup A(M), [M]\rangle = 0,$$

for all $\alpha \in H^*(K(\pi, 1)\mathbb{Q})$.

In particular, no closed (spin) $K(\pi,1)$-manifold should admit a metric of positive scalar curvature. The special case of tori of dimension ≤ 7 was established by Schoen and Yau (1979a). Gromov and Lawson (1980a,b) observed that, for all dimensions, the torus cannot have a positive scalar curvature metric by combining the Atiyah–Lichnerowicz–Singer argument with the argument of Lusztig's thesis. Further, they proved the non-existence for closed non-positively curved manifolds with residually finite fundamental group, and in Gromov and Lawson (1983) they removed, by developing enough index theory on the universal cover, the residual finiteness.[7]

Rosenberg (1983, 1986a,b) directly connected this problem to the work of Kasparov (and Mischenko and Fomenko) on the Novikov conjecture, greatly clarifying the situation and showing that more than analogies were involved here – this chapter owes a great debt to him.

The third important operator studied by Atiyah and Singer is the $\bar{\partial}$ operator on a complex manifold, whose study leads to the Hirzebruch Riemann–Roch

[6] It turns out to be reasonable to only ask for a spin structure on the universal cover of M.
[7] Surely this resonates with earlier discussions.

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theorem. It, too, gives rise to a characteristic class statement in the simply connected case that one can try to generalize.

If M is a complex manifold and $E \downarrow M$ is a holomorphic vector bundle, then the Hirzebruch Riemann–Roch theorem calculates

$$\sum (-1)^i \dim H^i(M; E) = \langle \mathrm{ch}(E) \cup \mathrm{Td}(M), [M] \rangle.$$

(Here $\mathrm{ch}(E)$ is the Chern character of E, and $\mathrm{Td}(M)$ is a graded characteristic class in the Chern classes of M.) The arithmetic genus is the alternating sum of the dimensions of the space of holomorphic k-forms.

Theorem 5.10 (Corollary to Hirzebruch Riemann–Roch) *If M and M' are birational smooth algebraic varieties, then*

$$\langle \mathrm{Td}(M), [M] \rangle = \langle \mathrm{Td}(M'), [M'] \rangle.$$

A birational equivalence is an almost-everywhere-defined isomorphism that is locally a quotient of polynomials. In fact, it is automatically defined in the complement of a complex codimension-2 subvariety[8] (of domain and range); this implies, in light of Hartog's theorem, that holomorphic functions on the complement extend over the subvariety – and that even the individual (holomorphic) cohomology groups are isomorphic.

A consequence of the fact that the singularities of a birational map being complex codimension-2 is that, if M and M' are smooth birational varieties, then they have the same fundamental groups. This led Rosenberg (2008) to conjecture the following theorem:

Theorem 5.11 (Block and Weinberger, 2006; Borisov and Libgober, 2008; Brasselet *et al.*, 2010) *If M and M' are birational smooth varieties and $f\colon M \to K(\pi, 1)$ is continuous, then for any $\alpha \in H^*(K(\pi, 1); \mathbb{Q})$, we have*

$$\langle f(\alpha) \cup \mathrm{Td}(M), [M] \rangle = \langle f(\alpha) \cup \mathrm{Td}(M'), [M'] \rangle.$$

The goal of this chapter is to explain more about how to play the Novikov game, and to give some feeling for when the result of playing the game is a conjecture that tends to be a theorem (as in the examples of the Schultz conjecture, the signature localization theorem for S^1-actions, and the Rosenberg conjecture) and when the conjecture seems to be deeper than this – e.g. implying the Novikov conjecture, or at least only being currently provable for some class of fundamental groups. And then there are sometimes when you play and you lose: the new "Novikov conjecture" is just plain false.

In doing this, we will need to broaden our perspective from topology to

[8] In the smooth case.

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index theory (as must surely be obvious) and develop the analogy between these fields. In doing this, it becomes possible to improve the various rational statements that we have been focusing on to more precise integral ones.

5.2 Anteing Up: Introduction to Index Theory

As might have been obvious in the examples of §5.1, almost all of the examples (except perhaps the one about pseudo-trivial actions) involve the Atiyah–Singer index theorem in some fashion. (That one involved Smith theory, although an index theorem would be involved to translate the posited equality to be one of characteristic numbers.)

The characteristic class (while perhaps rational) represents the index of an operator and our goal is to somehow boost the power of this result in the presence of a fundamental group.

Here we'll give a brief indication; more references can be found in §5.6.

Suppose that D is an *elliptic complex* on a manifold; that is, suppose that we have a sequence of vector bundles, $E_i, \downarrow M$, and $D_i\colon C^\infty(E_{i+1})$ are linear operators acting on the smooth sections of the E_i, given as differential operators (in local coordinates), so that $D_i D_{i-1} = 0$. Ellipticity is the condition that the Fourier transforms are exact away from the 0-section.

Concretely, let's consider the case of a single operator (i.e. a complex with just two bundles) on functions on the circle $\mathcal{S}^1$. The operator $d/d\theta$ (acting on sections of the trivial line bundle) is elliptic; its Fourier transform is everywhere $\times\xi$, which is invertible (and hence gives an acyclic complex) when $\xi \neq 0$. Similarly the Laplacian on functions on the flat 2-torus, i.e., $\mathbb{T}^2$, given by $\partial^2/\partial X^2 + \partial^2/\partial y^2$ is elliptic, but the wave operator on functions given by $\partial^2/\partial X^2 - \partial^2/\partial y^2$ is not. (Its Fourier transform vanishes on the lines $\xi_x = \pm\xi_y$.)

By the "elliptic package," i.e. Sobolev space theory together with the theory of the Fredholm index, for any elliptic complex on a compact manifold, the cohomology groups

$$H^i(D) \cong \operatorname{Ker} D_i/\operatorname{Im} D_{i-1}$$

are all finite-dimensional. The individual groups can be quite subtle and depend on more information than just the symbol of the operators. For example, the Laplacian ∇ on $C^\infty(S^1)$ has Ker $\cong \mathbb{C}$, but the perturbation by a small zeroth-order term $f \to \nabla f + \lambda f$ has no kernel unless λ is in the spectrum of ∇ (which then depends on the length of the circle!).

A similar but more topological example is the de Rham complex on a compact manifold M, but instead of considering real-valued forms, consider instead

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ξ-valued forms, for a flat bundle ξ. The symbol will only see $\dim(\xi)$, but, say $\dim H^0(M,\xi)$ is exactly the dimension of the part of ξ that has trivial monodromy.

The index theorem gives a topological calculation, though, of the index of D, $\operatorname{ind}(D)$ which is, by definition, $\sum(-1)^i \dim H^i(D)$. By taking an Euler characteristic, the subtleties of the individual cohomology groups are largely erased. The reader might enjoy seeing how this happens in these two examples. In the first case, one should use some Fourier series to see what's happening on both the kernels and cokernels, and in the second, the hint might be to consider why Euler characteristic is independent of the field used to define it (while the cohomology vector spaces have dimensions dependent on the characteristic of the field).

That the index is independent of "lower-order perturbations" is exactly the key property of the Fredholm index in functional analysis – invariance under compact perturbations.

The topological formula for the index involves the construction of a "symbol complex" (the analogue of the Fourier transform) over T^*M. It defines an element of $K^0(\mathrm{T}^*M)$.[9] Noting that T^*M is an almost-complex manifold, and that the Bott–Thom isomorphism theorem (i.e. Bott periodicity for complex bundles, the form explained in the first chapter of Atiyah and Singer, 1968a) then says that almost-complex manifolds are oriented for K-theory, we have $K^0(\mathrm{T}^*M) \cong K_{2m}(\mathrm{T}^*M) \cong K_{2m}(M)$, where we are now dealing with the dual homology theory to K-theory.[10] The index is then given by pushing forward the symbol homology class under the map $M \longrightarrow *$:

$$\begin{aligned} K_{2m}(M) \longrightarrow K_{2m}(*) &\cong \mathbb{Z}, \\ [D] &\longrightarrow \operatorname{ind}(D). \end{aligned}$$

The equivariant index theorem holds for G a compact Lie group acting on the complex and is essentially exactly the same result! In that case the cohomology vector spaces are G-representations, and the relevant K-theory is equivariant K-theory, $K^G(M)$. The pushforward to a point is now the equivariant index, which is a (virtual) G-representation (G acts on both the kernel and cokernel).

9 We follow the standard convention that K-groups of noncompact spaces are assumed to be with compact supports. (Despite this, we do not have a standard convention for ordinary homology or cohomology.)

10 We have shifted our point of view away from the original (Atiyah and Singer, 1968a,b, 1971). They pushed forward a cohomology class to a point using a "wrong-way map" that was induced by Bott periodicity. Atiyah (1970) later gave a K-homology class associated to an elliptic complex. Brown, Douglas, and Fillmore and Kasparov later still gave a development of K-homology where elliptic operators on manifolds form its cycles. See Higson and Roe (2000), and references therein, for a very lucid account.

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A similar situation arises when the (manifolds and the) elliptic complex is part of a family $(M_p D_p)$, with $p \in P$, a parameter space. In that case the cohomology groups $H^i(D_p)$ form a bundle over "most of" P. Atiyah and Singer show that it is possible to add a small perturbation d to the family so one has $D + d$, to repair this. In that case, the cohomology vector spaces all become vector bundles $H^i(D_p + d)$ over P, and the index is an element of $K^0(P)$. Atiyah and Singer (1971) explain how to compute this (and its Chern character).

These examples will interest us considerably in what follows (both have already been applied in earlier discussion – Lusztig[11] making use of the families index theorem to prove the Novikov conjecture for $\mathbb{Z}^n$, and the equivariant index theorem is relevant to the Atiyah–Hirzebruch and signature localization theorems mentioned in §5.1.) However, for now, we would like to focus on a formal algebraic aspect of all these theorems.

Notice the groups that the indices in these theorems take values in: for the index theorem it's in $\mathbb{Z}$, in the G-index theorem it is an element of $\mathrm{R}(G)$, the representations of G, and in the families index theorem it is $K^0(p)$.

Indeed, even the $\mathbb{Z}$ in the index theorem is just $K^0(C)$, the Grothendieck group of finitely generated projective as $\mathbb{C}$-modules (i.e. finite-dimensional vector spaces). $\mathrm{R}(G)$ is the same thing as projective modules over CG, if G is finite, and over C^*G (the C^* completion – to be discussed below) if G is compact. $K^0(p)$. can be thought of, thanks to a classical theorem of Swan, as the (Grothendieck group of finitely generated) projective modules over $C(P)$, the ring of continuous functions on P.

Thus we are led to thinking of the ordinary index theorem as a theorem about complex-valued elliptic operators, and the other index theorems as being about elliptic operators over other C^*-algebras, depending on the situation.

Commutative C^*-algebras correspond to functions on a space (the Gel'fand–Naimark theorem) – and one can[12] think of the general noncommutative case as some kind of index theorem for families "parameterized by a noncommutative space" – including situations where there is a group action as a very special case – and what then follows as a chapter in Connes, 1994).

Instead let's recall that, for elliptic operators P, there are parametrices, or "pseudo-inverses," Q, so that $PQ = \mathrm{I} + \text{Compact}$, and similarly for QP. So, for operators over any algebra, we should think that we have a ring R of operators, and J an ideal, and then Fredholm (also known as elliptic) operators are those that are invertible modulo compacts, i.e. modulo J. In algebraic K-theory, there

[11] The need for perturbations for families of operators was already implicit in our discussion of Lusztig's method: even the cohomology of S^1 with coefficients in a flat bundle is not constant even for the Lusztig family of line bundles.

[12] If one chooses to.

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is an exact sequence (see Milnor, 1971, pp. 33–34):

$$K_1(R) \to K_1(R/J) \to K_0(J) \to K_0(R)\,.$$

The boundary is essentially the index map – one has $\partial[p] = \mathrm{Ker}(P) - \mathrm{cok}\,(P)$.

In the operator-theoretic setting of "A-algebras," the same is true, except that one has to add on a suitable "A-compact operator" to the elliptic operator to make the kernel and cokernel projective A-modules (e.g. vector bundles without singularities).

Critically important for our story is the following basic C^*-algebra associated to a representation. Suppose π is a group, and $\pi \to \mathrm{U}(H)$ is a representation. We can therefore think of $C\pi$ as an algebra of operators on H. We then obtain a *C^*-algebra* by completing with respect to the operator norm.

There are two extreme cases of this construction that are of fundamental importance. The first is $H = L^2\pi$, and the completion is called $C_r^*\pi$, the *reduced C^*-algebra of π*. The other is when H is the sum of *all* irreducible unitary representations of π, and this yields $C^*_{\max}\pi$, the maximal C^*-algebra of π.

The latter choice has better functorial properties: any group homomorphism gives a map between the associated algebras, but Property (T) shows that $C^*_{\max}\pi$ can have rather large K-theory: $K_0(C^*_{\max}\mathrm{SL}_3(\mathbb{Z}))$ is an infinitely generated group with an infinitely generated subgroup generated by irreducible representations that factor through finite groups $\mathrm{SL}_3(\mathbb{Z}/N)$.

Basically, if our interest is in the Novikov game, i.e. injectivity type statements,[13] this is not a problem and we can live cheerfully with $C^*_{\max}\pi$, but when we begin playing the Borel game (which in this setting is called the Baum–Connes conjecture) and look for isomorphism theorems, $C_r^*\pi$, despite its functorial defects, will play the starring role. We will often be cavalier and just use $C^*\pi$ as notation when these details aren't important (or the completion is obvious).

With the above preliminaries, we have enough of a buy-in[14] to begin playing the Novikov game.

We do not look at every possible characteristic class formula, but rather ones associated to an elliptic operator.

Coupling this operator to an infinite-dimensional flat bundle $\times_\pi C^*\pi$, we try to get information from $K(C)^*\pi$ to improve the formula to one involving the fundamental group. Analogous to the assembly map in surgery,

$$H_*(K(\pi,1)L(e)) \to L_*(\mathbb{Z}\pi),$$

[13] See §4.7.

[14] For example, we have enough resources to be able to get a seat at the table.

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there is a map (also called the assembly map, but mainly by reason of analogy[15] – it would be better called the index map[16])

$$K_*(K(\pi, 1)) \to K_*(C^*\pi).$$

Both of these maps "tend to be injective" at least rationally – as we have discussed in the case of L-theory and shall yet discuss more. These are Novikov-type statements and can be sometimes proved by expanding our point of view to consider L-groups of categories associated to and K-groups of algebras of operators associated to metric spaces (and other controlled settings).

Indeed, the usual Novikov conjecture can be studied from the operator-theoretic point of view by restricting attention to the signature operator. Thus, the operator-theoretic setting thus is a large extension of perspective.

Note that we can now try to extract some integral information, like the precise tangential information present in the Borel conjecture, and not just the rational that we have focused on in our discussion of Novikov. Note that the conclusion of the form of the Novikov conjecture that we have just proposed (and had proposed in Chapter 4 via surgery) contains some integral information. It asserts the vanishing of the pushforward of some operator under the natural map

$$K_*(M) \to K_*(K(\pi, 1)).$$

For example, it will include the statement that if we push forward the signature operator[17] (viewed as an element of $K_*(M)$ as above) into the K-homology of the fundamental group, we get an oriented homotopy invariant.

Even if we are interested in just the rational problem, working integrally is a good way to probe approaches, but for some applications, this information is absolutely necessary. The map $L \to K$ (for a point, for example) is split injective *away from the prime 2*, so away from 2 it is possible to deduce a topological injectivity from the analytic results. However, at 2, one cannot obtain strong L-theoretic results analytically (at least, not in any too direct a fashion).

[15] In a recent paper (Weinberger *et al.*, 2020) the analytic map is defined in a way that really looks like assembly.

[16] Or the higher index map, if one wants to emphasize how the higher cohomology of the group is implicated in this story.

[17] Here we are taking for granted the quite nontrivial point that the symbol class of the signature operator can be defined for topological manifolds. (See Rosenberg and Weinberger, 1990, for a discussion of this issue. In Weinberger *et al.*, 2020, a functorial map is built using just smooth manifolds, and a more substantial contribution from controlled topology.) In any case, this class can be defined in another way that we will explain in the next section.

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Appendix: A Glimpse through the Looking Glass

... At a Parallel Universe whose Arrows are Reversed[18]

While we have focused on manifolds (and orbifolds), and in this setting there are very nice analogies between surgery theory and index theory, both subjects naturally encompass more territory where the analogies are not as evident (and one might imagine pessimistically that they don't extend or, if they do, they don't help[19]).

This mini-section is a brief about the noncommutative geometric perspective.

Many different kinds of things have L-theory: rings with (anti-)involution,[20] pairs (i.e. relative L-groups), additive categories, stratified spaces, etc. Each of these opens up new ranges of application.

On the index theory side, the object one takes the K-theory of is always a C^*-algebra. Rather than generalize the setting of K-theory, which was the surgical route, one instead creates innovative constructions of these algebras in various geometric situations, as is explained lovingly and inspiringly in Connes (1994) and Connes and Marcolli (2008).

More generally, in noncommutative geometry[21] one tries to take seriously noncommutative algebras as analogues of spaces, and then one mimics important geometric constructions in this setting.

The basic example from which everything generalizes is that of X, a locally compact Hausdorff space, to which one associates $C^*(X)$, the ring of continuous complex-valued functions on X, with respect to complex conjugation and the uniform norm as the norm.[22] This is a *contravariant* functor that defines an equivalence of categories. K-theory enters by Swan's theorem that projective modules over $C^*(X)$ are the same thing as vector bundles over X. On the other hand, K-homology is associated to extensions[23] (by the algebra of compact

[18] That humans are remarkably symmetric makes it the case that what we see through the looking glass is the same kind of object as we are. Of course, the looking glass of C^*-algebras assigns an infinite-dimensional algebra to a beautiful finite-dimensional space. It might be fun to create a mixture of a mirror with night goggles, so that one gets a similar feeling of strangeness on looking through the mirror.

[19] Even for positive-dimensional group actions on manifolds, the two theories have very different flavors. I am not a pessimist, however.

[20] I'm old fashioned: I stick in the "anti-" but many writers don't bother and use the word involution for the same concept.

[21] We ignore the algebraic geometric side of this philosophy, just as in the commutative world we went continuous from differentiable, not analytic.

[22] This itself is in analogy to the very beginnings of algebraic geometry (initially, at least, over an algebraically closed field) where one associates to each affine variety the coordinate ring of polynomial functions on the variety.

[23] See Higson and Roe, 2000 for an excellent exposition of the Brown–Douglas theory and the beginnings of Kasparov's KK-theory.

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operators on a separable Hilbert space) and (therefore) to generalized elliptic operators over X.

Having K-theory, one wants then to generalize the Chern character $K(X) \to \oplus H^{2i}(X, \mathbb{Q})$ to get (computable) invariants of elements[24] of these groups.

Cyclic homology was introduced by Connes (1985) to be the target of such a generalized Chern character from K-theory to something more immediately computable and definable algebraically without the commutativity: it is closely related to de Rham cohomology in the commutative case (see Loday, 1976, 1998, for an excellent treatment; needless to say, having computable invariants for K-theory is important in many situations where the ring[25] whose K-theory is taken is not a C^*-algebra.) It is thus an excellent example of the noncommutative philosophy: "commutative" invariants (i.e. geometric invariants of spaces) that can be interpreted noncommutatively are much more powerful and natural.[26]

In this very short appendix, I will describe just a few of the noncommutative algebras that arise in geometric situations.

Example 5.12 ($A = B$) Suppose we start with two points, A and B. We form $X = \{A, B\}$ and $C^*(X) = \mathbb{C}^2$, where addition and multiplication is coordinatewise. A function on X requires two values, one for each of A and B, and there is no communication between them.

Suppose we now set $A = B$; then the functions need to assign the same values to A and B and we obtain the algebra $\mathbb{C}$.

But suppose we just had an equivalence relation $A \sim B$: then A would communicate with B and vice versa. It would not be crazy to consider associated to this system $M_2(\mathbb{C})$, the 2×2 matrices that have off-diagonal entries, that reflect the communication between A and B.

We can consider this as being governed by the groupoid[27] associated to the equivalence relation. Then we get $C(A \cup B)$ acted on by the bounded operators, one for each arrow in this category. (Assuming boundedness[28] will make the convolution product of the operators defined in this way to be defined.)

Interestingly $\mathbb{C}$ and $M_2(\mathbb{C})$ are closely related algebras: they are Morita

[24] This is very relevant to the Novikov conjecture, which is, after all, a lower bound on K-theory – so a suitable Chern character, rich enough to detect the image of the assembly map, would be a dream come true.

[25] Randy McCarthy's thesis (see McCarthy, 1994) generalized the definition of cyclic homology and the trace map to the setting of exact categories.

[26] This is also true of working with stratified spaces. Frequently, arguments that must work in full generality are constrained, and therefore easier to find, than ones that just apply to very particular classes.

[27] Recall that a groupoid is a category in which every arrow is invertible.

[28] The boundedness is automatic in this setting, but it seemed like a good idea to say it explicitly anyway.

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equivalent – they have equivalent categories of projective modules. So in this case $A \sim B$ and $A = B$ have very similar effects. But for more complicated equivalence relations, it is indeed important to remember that equivalent does not mean equal, and that the noncommutative perspective includes the price of seeing that points are equivalent in a quotient space and keep track of these "transaction costs" when one does further analysis.[29]

The Morita equivalence of various ways of producing quotient objects (and, in particular, when there's a reasonable commutative choice) is common in tame situations.

Example 5.13 (Group actions) If G is a finite group acting nicely on a space X, one can form X/G. The continuous functions on this are $C(X)^G$. But the G-space X has much more information than X/G.

Even from the theory of vector bundles, one knows that it's much more exciting to consider equivariant bundles on X as opposed to bundles over X/G. So, what kind of space is this?

So for X that is a point, we want a vector space with a self-identification associated to each self-identification of X given by $g \in G$. This means that we need modules over $\mathbb{C}[G]$ which are representations of G. In general, we should have modules over the semidirect product $C(X) \rtimes G$. For G a compact Lie group, one needs more general convolution algebra.

Of course the K-theory of this convolution algebra is exactly the Grothendieck group of equivariant vector bundles over X. A similar construction can also be made for locally compact groups acting properly on X.

When the action on X is not proper, then things become less clear geometrically. However, one can still form convolution algebras and study their properties. This could well lead you to the Baum–Connes conjecture (with coefficients) if you were bold enough.

Example 5.14 (Tilings, bounded geometry, etc.) Suppose[30] that every day

[29] Mathematicians are trained from early years to take equivalence classes, and form quotient objects. Try explaining this to a non-mathematician! It is not at all easy to do. (Maybe you remember wondering whether real numbers were really equivalence classes of sequences of rational numbers or whether they were "really" numbers.)

Logicians know well the importance of distinguishing between = and ~, and in model theory have special rules for the interpretation of =. Because (as Bill Clinton famously explained) it is not always clear what it is.

Respecting that one should have to pay for every use of an equivalence relation is a good idea (and is behind things like Dehn functions in geometric group theory). The C^*-algebra approach to quotients does this naturally.

[30] This example is the result of many conversations with Jean Bellissard and Semail Ulgen-Yildirim and inspired by the work of Bellissard (1995) and Abert *et al.* (2017), among others.

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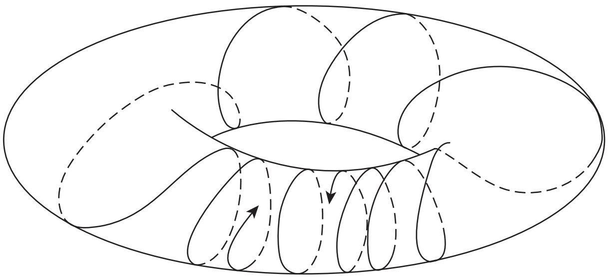

Figure 5.1

were exactly like the previous and the next. So 9:00 a.m. today would be like 9:00 a.m. any other day. Then my experience of time would be indistinguishable from a circle. It would be a matter of debate for the metaphysicians (who would come to either no conclusion or the very same conclusion every day) whether time was "really" a line – with certain regularities holding – or whether it was a circle.

I think it could be an amusing project to write a (series of) novel(s) that would have such a periodic structure. But it would be a contrived project in that even if the last page of the last volume were identical to the first page of the first, in the absence of determinism one would not expect "the continuation" to be the same as the first time around.[31] (Very interesting are the random samples from a periodic distribution.)

Returning to the real world, we can consider the experience of a creature with bounded memory and resources on various spaces. Suppose that there were one glitch in the periodic "time-line" universe. Then for all of early eternity till that glitch, time would be a circle, but from the glitch on, depending on how good the creature's memory was, it would seem like there was non-periodicity – after all, there was some time from "the beginning" or "the change" but that it would asymptote also to a circle. (Figure 5.1 describes this – assuming the stable states at $\pm\infty$ are the time-reversed for entertainment. Without the identification, there would be a similar picture taking place on a subset of the cylinder.)

This topological space is the space where experience takes place. Geometric processes that weight local geometry and involve large scales in increasingly damped fashions should extend to this space that compactifies the time-line.

Now let's think still about a time-bound creature, but time, while a line, is

[31] It might be more interesting to double the number of volumes and have a second series not identical to the first. I'm not sure what I want the last page of the last volume of the second series to be like.

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tiled by tiles A and B, with some rules about how the A and B are put down. If all the A were in positive time and all B in negative, then one would get the two-circle space mentioned above. But if instead we can put our tiles down in a random fashion then time would formally be $\mathbb{R}$: there is no periodicity, but if our memories get weak, each time will have been anticipated infinitely often, and indeed history will yet repeat itself infinitely often. We can embed this $\mathbb{R}$ in $\mathbb{R} \times \{A, B\}^{\mathbb{Z}}$/the disagonal action of $\mathbb{Z}$ where the product is bi-infinite. A real number is mapped to r (this being the label of the tile of which it's a part), with boundary points resulting in the identification on the right. The space most appropriate for modeling the experience of our creature will be the same as this limit space, i.e. the closure of the $\mathbb{R}$-orbit in this space. If the placement of tiles were truly random, the closure would be the whole space. (Almost all $\mathbb{R}$-orbits in this space are dense.)

I like this example a lot. It is easy to build into topology a theory that takes into account only balls of some, perhaps unspecified, size. However, allowing far-away points to have influence that is decaying seems much better suited to analysis.

This is an example of a foliated space, which is a slight generalization of the notion of a foliation. And, there is a C^*-algebra associated to such spaces which, in the case of a fiber bundle foliated by fibers, would be Morita-equivalent to the continuous functions on the quotient.

This can be done for tilings of $\mathbb{R}^n$ rather than just on $\mathbb{R}^1$.

A lot of the information of a tiling on $\mathbb{R}^n$ can be described by giving the centers of gravity of the tiles.[32] (In higher dimensions, we can get interesting examples without the expedient of labeling as we did in one dimension.) We are interested in tilings where these point sets are:

(1) C-dense for some C (every point is within C of some center);
(2) sparse, i.e. no two points are within δ of each other; and
(3) repetitive, so the pattern of every ball of radius R repeats in a $C(R)$ dense way everywhere.

From conditions (1) and (2), one can see that the set of patterns has an embedding in a nice locally compact space. This space of tilings has an $\mathbb{R}^n$-action on it (actually the whole Euclidean group acts on it, and when we take that into account with condition (3) one gets a broader notion). Condition (3) is of a different nature, and guarantees that the set of patterns arising forms a minimal dynamical system with respect to this action.

[32] If one is just interested in convex tiles, then one can use a slightly different point set S: a set of points such that the tile containing p is defined as the set of points closer to p than to any of the other points of S.

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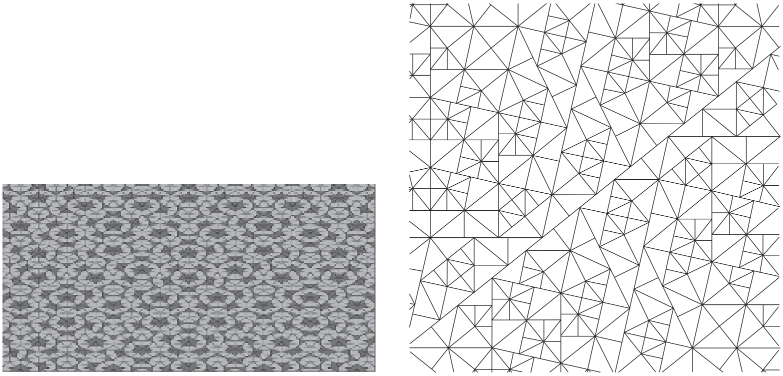

(a) (b)

Figure 5.2 (a) The Penrose tiling. (b) A pinwheel tiling. (From Sadun, 1998. Reprinted by permission of Springer Nature.)

Two examples of aperiodic tilings are shown in Figure 5.2. The pinwheel tiling does not have recurrence entirely on the basis of translations, but that is irrelevant to the story we are discussing, which allows all isometries.

The closure of this orbit is called the *hull* of the tiling. It consists of the pointed Gromov–Hausdorff limits of centered balls in the tiling.[33] It consists of the tilings that cannot be locally distinguished from the tilings that occur within the original one, but is usually much larger. If the tiling is aperiodic, then it has countably many "subtilings" (just recenterings, essentially) but there are always uncountably many possible limits.

For aperiodic tilings, the hull usually has the local structure of the product of a manifold with a Cantor set, but if one relaxes condition (3) then very different kinds of structures can occur. (The reader might enjoy considering the hulls of graphite versus diamond.)

This can also be done with respect to any manifold with bounded geometry. These limits can be thought of as doppelgangers, whose properties restrict the original manifolds. They are like the way a novelist can use pieces of our personalities to create characters that resemble us yet are more extreme than we are – in order to shed light on what we are like. More subtle than just the

[33] Following an unpublished note with Bellissard and Ulgen-Yildirim, it can be defined as the closure of the natural map of the manifold into the inverse limit of the pointed Gromov–Hausdorff space of centered Riemannian balls of various radii (under the natural map that sends a centered ball of radius R to one of radius r if $R > r$). Each point in M is mapped to the consistent set of balls in M centered at that point. Generically this map is an embedding, and the closure is a foliated space, but for special manifolds that have a lot of symmetry, this mapping can have some strange properties.

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individual limit points within this construction are the properties that describe the size of the hull.

Periodicity, like regular covers, gives rise to compact hulls with the manifold not embedded in the hull. The difference between crystals, quasicrystals, and glass is apparent in the structure of the hull. One is led very naturally to fascinating problems like when such constructions have transverse measures (e.g. when one has a reasonable notion of "typical behavior" on a manifold), etc.

Example 5.15 Connes introduced C^*-algebras associated to foliations (as in the previous examples); these are variations on the ones associated to groupoids – and they are all essentially convolution algebras acting on functions (or half-densities[34]) on the total space. If the foliation were a fibration, this algebra is Morita-equivalent to the continuous functions on the quotient, but interesting foliated spaces can frequently have all leaves dense (like the foliation of a torus by irrational lines or planes, and the example of aperiodic tilings above).

There are many more examples that come up from physics (e.g. the standard model) or number theory (e.g. spaces of $\mathbb{Q}$-lattices, etc.). I won't discuss these, but see Connes and Marcolli (2008), and references therein.

When one has a C^*-algebra/noncommutative space, one begins using it to do mathematics. At first, there are questions about which algebraic properties of the algebra hold, or how are they reflected in the geometry of the situation. For example, a trace on the C^*-algebra of a foliation corresponds to a transverse measure.

After such a dictionary is established, it becomes possible to implicate various functors such K-homology and K-theory and develop and apply appropriate index theorems in this setting. Foliations (say, of compact manifolds) always have an implicit dynamics as noncompact leaves recur, and such results can have profound implications.

Cyclic homology was invented to be a noncommutative analogue of de Rham cohomology. It's close, but doesn't quite agree with this. Nevertheless, it provides an important invariant of algebras and a place to map K-theory to.

In dimension 0, it captures the idea of a trace, and these are associated to the simplest index theorems. Higher homology reflect higher index theorems (with the important, seemingly technical, issue that one frequently has to go to dense subalgebras[35] to get nontrivial higher homology).

Let's color in the outlines of the picture we tried to draw. Here's how the dream would go in the special case of $R = \mathbb{C}\pi$), a group ring: we are interested

34 Half-densities are useful for creating appropriate L^2-spaces.

35 Such algebras describe types of smoothness – and, once one says this, the idea of passing to such algebras is not at all shocking if we are to use these for doing analysis.

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in invariants on projective modules over R (i.e. $K_0(R)$) generalizing the notion of dimension. Consider the formula

$$\dim P = \mathrm{rank} P,$$

which almost looks too tautologous for words! This formula exploits our habit of writing the same letter P for a projective module and a projection $P^2 = P$ whose image is that module. For projections we have

$$\mathrm{rank} P = \mathrm{trace} P.$$

The key defining property of a trace is that for all matrices $\mathrm{trace}(AB) = \mathrm{trace}(BA)$ (from which follows the key property that it is an invariant of an automorphism, not merely of a matrix). This suggests considering

$$R/[R, R] = R/\{ab - ba\}$$

as the best target for a trace that we could possibly hope for (and why hope for any less?).

Here we are considering a quotient abelian group in this formula. (The set of elements $ab - ba$ is not closed under addition, and we consider the subgroup generated by such commutators.) In the case of a group ring, the right-hand side breaks up into pieces, $\bigoplus \mathbb{C}(g)$, where (g) is the equivalence class of g under conjugation. A little thought shows that taking the "ordinary" trace of a matrix and collecting all the coefficients of the elements within a conjugacy class actually is a well-defined operation.

This quotient is exactly $\mathrm{HC}_0(\mathbb{C}\pi)$, and this algebraic trace is called the Hattori–Stallings trace. The trace map from K_0 is quite nontrivial, as the case of π finite readily indicates. Indeed, it shows that for any non-torsion-free group, the modules induced from finite subgroups can be non-free projective modules for the group. (The converse, of course, is part of the Borel package to which we have alluded several times.)

It also clearly has connections to Nielsen fixed-point theory (where we generalize the Lefshetz number just by changing the meaning of the word trace).

In the K_0 setting it is an interesting and natural question whether the coefficient of a conjugacy class of infinite order can ever be nonzero. (That this is impossible is sometimes called the Bass conjecture: see Bass, 1976.) This is known in many cases – and it seems worth noting here that the theory of cyclic homology and the higher trace maps has been effectively deployed in this direction (see Eckmann, 1986)).

In passing to the completion there is very serious (analytic) trouble. If an element g has infinitely many conjugates, perhaps an element in $\mathbb{C}^*\pi$ might

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want to give that trace an infinite value.[36] That there are homomorphisms from $K(\mathbb{C}^*\pi)$ to $\mathbb{C}$ corresponding to conjugacy classes of finite order does not seem any less deep than the Novikov conjecture.[37]

Cyclic homology actually closely resembles the homology of $\mathrm{E}\mathcal{S}^1 \times_{\mathcal{S}^1} \Lambda X$, where Λ denotes the free loop space, and $\mathcal{S}^1$ acts on this space by rotation of loops (see, e.g., Burghelea, 1985; Goodwillie, 1985). The case relevant to us is $\mathrm{B}\pi$: $\Lambda\mathrm{B}\pi$ has components corresponding to the conjugacy classes of elements of π – the HC_0 that we saw earlier. The component of loops freely homotopic to a given g is itself aspherical, and is a $K(Z(g), 1)$, where $Z(g)$ is the centralizer of g.

When we take the Borel construction on the action of $\mathcal{S}^1$ by loop rotation, we get different behavior when g is finite order and when it's infinite. In the finite case we get a $K(Z(g), 1) \times \mathbb{CP}^\infty$, but in the infinite case, it is $K(Z(g)/\langle g\rangle, 1)$. Rationally, the Baum–Connes conjecture asserts that $K(\mathbb{C} * *\pi)$ is isomorphic to $\bigoplus K(C(g))$ where the sum is taken over g with finite order.[38]

The general theory provides for maps $\mathrm{HC}_n \to \mathrm{HC}_{n-2}$ dual to Bott periodicity, which here is dual to the cup product with the Euler class of the circle bundle, and the trace with target HC_{n-2k} factors through this. This is used in Eckmann's work on the Bass conjecture, and is also completely reasonable in the C^*-setting where K-theory has a Bott periodicity isomorphism. Of course, the $K(Z(g), 1) \times \mathbb{CP}^\infty$ results in certain homology groups being counted many times, but, inverting the periodicity, they each are rationally counted once.

5.3 Playing the Game: What Happens in Particular Cases?

In §5.2 we explained that the formal framework of index theory is rather similar to that of surgery theory,[39] and, as a result, the Novikov phenomenon applies more broadly to other operators.

This point of view is fine as a starting point, but it is way too formal to be a

36 On the other hand, if g has only finitely many conjugates, then there is no trouble defining a trace associated to (g) and this can be exploited for geometric gain. This, for example, arises for groups with a finite normal subgroup whose quotient is torsion free.

37 More precisely, it actually does seem less deep, but I have never found an *a priori* argument that doesn't involve Novikov technology. Weinberger and Yu (2015) is a failed struggle with this problem.

38 This is compatible with what one would expect from the topological conjectures if one replaced $\mathbb{Q}\pi$ or $\mathbb{R}\pi$ with $\mathbb{C}\pi$ (with the involution being complex conjugation on $\mathbb{C}$).

39 Indeed, the signature operator gives rise to a *functorial* mapping of surgery theory into operator K-theory, as mentioned above (see Higson and Roe, 2005a,b,c). It is important, though, to note that the infinite loop space structures on the two theories are different at the prime 2 (see Rosenberg and Weinberger, 2006) because of the nature of the boundary map in the exact sequences of pairs.

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stopping point – it oversimplifies, missing the exquisite texture of the subject and the true benefit of unification.

Some methods develop naturally within the context of one problem, and the informal parallels between subjects leads to a search for cognate results for other problems – ultimately leading to multiple analogous theorems. In some sense, every time we play the Novikov game, we get a new test, a new area that is suggestive of techniques internal to it – that afterwards we can hope will shed light on the original problem, or, if not to it, perhaps to some other analogues, and it also leaves us with the puzzle of understanding why we didn't succeed in this export.

Surely the richest two cases were the original Novikov problem, which we have already discussed and shall have to review in light of the index-theoretic perspective, followed by the problem of positive scalar curvature.[40] We shall first discuss this latter problem before returning to the first and then to the general discussion.

The first results on the positive scalar curvature problem, after the Atiyah–Lichnerowicz–Singer vanishing result, was the proof by Schoen and Yau (1979a) of the nonexistence of positive scalar curvature metrics on the torus and related manifolds (e.g. those that resemble Haken manifolds or have maps of nonzero degree to them) and have dimension ≤ 7.[41] This method has the feel of the original methods on the Novikov conjecture using codimension-1 splitting, and it has consequences that we do not yet know how to approach by Dirac operator techniques (coupled to the fundamental group).[42] The geometric nature of this method makes it possible to describe it before any discussion of applications of the index theorem.

The very elegant idea in Schoen and Yau (1979a), in its embryonic form, is this. Let M be a manifold with positive scalar curvature and a nonzero degree map to $\mathbb{T}^n$. Then one finds in M an area-minimizing minimal hypersurface dual to any class in H^1. This hypersurface is smooth if M has dimension ≤ 7 (and this is where the dimension hypothesis enters) and has a map of nonzero degree to $\mathbb{T}^{n-1}$.

40 Needless to say, for many people the positive scalar curvature problem is the more interesting one, because, for example, it has important connections to general relativity (the positive mass conjecture).

41 See also Lohkamp (2006) for an announcement of a method for getting around the fact that minimal surfaces can develop singularities in dimension greater than 7, and a more recent paper of Schoen and Yau (2017) that gives a different approach.

42 Results using the Dirac operator require some spin structure. Indeed, for non-spin simply connected manifold of dimension greater than 4, Gromov and Lawson (1980a,b) have shown that positive scalar curvature metrics always exist! On the other hand, the Schoen–Yau result shows that if V is any manifold of dim < 8, then $V\#\mathbb{T}$ never has positive scalar curvature. This manifold's universal cover is, of course, not spin.

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A calculation then shows that the induced metric on a minimal hypersurface in a positive scalar curvature manifold is naturally conformally equivalent to a positive scalar curvature metric. (The conformal factor is a power of the eigenfunction associated to the first eigenvalue (necessarily positive) of $\Delta - (n-3)K/4(n-2)$, where k is scalar curvature on the hypersurface.)

One repeats this argument till one gets down to dimension 2, and the result follows from the Gauss–Bonnet theorem.

Let us now turn to the Dirac operator method. We shall not review the definition of the Dirac operator, leaving the reader to standard references for it and its theory (e.g. Atiyah *et al.*, 1964; Atiyah and Singer, 1968a,b, 1971; Lawson and Michelsohn, 1989; Roe, 1998; Berline *et al.*, 2004; Higson and Roe (20XX)).

Lichnerowicz showed the following "Bochner type" formula relating the Dirac operator on a manifold and the Laplacian on forms: $\mathbf{D}^2 = \nabla^*\nabla + k/4$ (see e.g. Atiyah and Singer, 1968b; Roe, 1998). This implies, assuming that the scalar curvature is everywhere positive,[43] that $\mathbf{D}$ and $\mathbf{D}^*$ can have no kernel (as the Laplacian is semidefinite). In other words "M can have no harmonic spinors," and, in particular, $\mathrm{ind}(\mathbf{D}) = 0$. The index theorem then gives (see, e.g., Atiyah and Singer, 1968b for the calculations of the symbol of the Dirac operator, and how the index theorem works out in this case) that

$$\langle A(M), [M]\rangle = \mathrm{ind}(\mathbf{D}) = 0.$$

In every dimension that is a multiple of 4, there is a spin manifold whose A-genus is nonzero, and these give examples of simply connected manifolds with no positive scalar curvature metrics.

Actually, one can do somewhat better than this (without using the fundamental group at all). The Dirac operator naturally has a *real* structure, allowing more subtle *real* index theorems to be applied. The index then takes values in $\mathrm{KO}_i(*) = \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$ (depending on $i \bmod 8$) – thus providing a refinement at the prime 2. Hitchin (1974) showed that, under the assumptions of spin and positive scalar curvature, for an i-manifold

$$\mathrm{ind}(\mathbf{D}) = 0 \in \mathrm{KO}_i(*).$$

Thus spin manifolds of dimensions $1, 2 \bmod 8$ there is an extra $\bmod\ 2$ obstruction to having positive scalar curvature. There are even examples of manifolds homeomorphic to the sphere that do not have positive scalar curvature metrics.

43 Kazdan and Warner (1975) showed that if M has a metric that is nonnegative everywhere, and positive somewhere, then *any* smooth function on M is the scalar curvature function of some metric. Therefore, we can weaken the curvature conditions in this argument.

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Remarkably, for spin manifolds of dimension greater than 4,[44] Stoltz (1992) has shown that every simply connected manifold with vanishing ind(**D**) (in $\mathrm{KO}_i(*)$) actually has a positive scalar curvature metric.[45]

Lusztig's proof of the Novikov conjecture (§4.4) applies: when we couple the Dirac operator to a bundle ξ, the formula $\mathbf{D} * \mathbf{D} = \Delta * \Delta + K/4$ gets another term coming from the curvature of the bundle, but if the bundle is flat, then the same argument gives the vanishing of $\mathrm{ind}(\mathbf{D}_\xi)$. As we vary ξ over a parameter space, especially over the dual torus $\mathbb{T} = (\mathrm{Hom}\,(\pi_1(M)\colon S^1)$, we get a zero-dimensional trivial bundle as the index $\in K^0(\mathbb{T})$.

This then gives the result that, if M is a spin manifold with positive scalar curvature, then for any $f\colon M \longrightarrow \mathbb{T}$, and $\alpha \in H^*(\mathbb{T})$,

$$\langle f * (\alpha) \cup A(M), [M] \rangle = 0.$$

Gromov and Lawson (1980a,b) suggested a beautiful alternative, using families.

If we use only a single (finite-dimensional) flat bundle, then we gain nothing from the index theorem; we still have $\mathrm{ind}(\mathbf{D}_\xi) = \mathrm{ind}(\mathbf{D}) = 0$, but the index theorem just has on the topological side

$$\langle \mathrm{ch}(\mathbf{D}_\xi) \cup A(M), [M] \rangle.$$

But $\mathrm{ch}(\mathbf{D}_\xi) = \dim\xi$ has no positive-dimensional component.[46] However, if we allow ξ to be a bundle with *very small* curvature (in comparison to $\inf k/4$ that we assume is $> \varepsilon > 0$), then conceivably $\mathrm{ch}(\mathbf{D}_\xi) = \dim\xi \neq 0$, but the new curvature term in the Lichnerowicz–Bochner formula is still positive enough to give the vanishing of the index.

So, how do we get bundles with *arbitrarily small curvature* and nontrivial Chern class c_n so we can implement this idea?

This is impossible on a single compact manifold (because the cohomology classes represented by Chern classes are *integral*, so, if they are sufficiently small,[47] they will integrate to 0 on all cycles). However, we can find (sometimes) a sequence of bundles ξ_i on covers M_i of M with nontrivial c_n, and whose curvatures tend to 0.

For example, suppose $M = K\backslash G/\Gamma$, where Γ is a uniform lattice, and suppose (without loss of generality for our current purposes) that $\dim(G/K) = 2n$ is

[44] In dimension 4, the Seiberg–Witten invariants give additional obstructions to the existence of positive scalar curvature metrics (see, e.g., Morgan, 1996).

[45] Gromov and Lawson had earlier used cobordism methods to show that *every* simply connected non-spin closed n-manifold ($n > 4$) has a positive scalar curvature metric, and Stoltz extended their cobordism arguments using very clever algebraic topological arguments.

[46] Chern classes are integral, so sufficiently small curvature implies that they vanish rationally. (Note that it's only the real Chern classes that have a description in terms of curvature, so even zero curvature is compatible with a torsion Chern class.)

[47] And of a fixed dimension.

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even. Then, by the residual finiteness of Γ we can find finite normal covers $K\backslash G/\Gamma_i$ whose injectivity radii R_i are arbitrarily large. We can use the logarithm map, i.e. the inverse of the exponential map, followed by the pinch map that wraps the Euclidean ball of radius $R < R_i$ (for i large) onto the standard round sphere $\mathcal{S}^{2n}$ of curvature equal to $+1$. Let $L_i\colon K\backslash G/\Gamma_i \to \mathcal{S}^{2n}$ be this logarithm map, associated to some base point where there is an embedded geodesic R-ball. As $i \to \infty$, the Lipschitz constant of $L \to 0$ (because of the rescaling by R; the logarithm map itself is 1-Lipschitz for non-positively curved manifolds).

Note that by Bott periodicity there is a bundle ξ over $\mathcal{S}^{2n}$ with $C_n(\xi) \neq 0$ (indeed, the "Bott element" has $c_n = (n-1)!$). Let $\xi_i \downarrow K\backslash G/\Gamma_i$ be $L_i * (\xi)$. It is an almost-flat family yet $c_n(\xi_i) = \langle c_n(\xi), [S^{2n}]\rangle [K\backslash G/\Gamma_i]$, a nontrivial multiple of the fundamental class. Given a manifold M with fundamental group Γ we can use a map inducing the isomorphism of fundamental groups $M \to K\backslash G/\Gamma$ to pull back the almost-flat bundle and see that at least the higher A-genus associated to the fundamental class of Γ obstructs positive scalar curvature.

If you are unhappy with the ξ_i being over different manifolds (although this is completely irrelevant to the application of the argument!), we can push them forward with respect to the covering map $K\backslash G/\Gamma_i \to K\backslash G/\Gamma$. These pushforward bundles will still be almost-flat – in the sense of having decreasing curvature approaching 0 – and have nontrivial Chern class, but increasing dimension.

See Hanke and Schick (2006) and Hanke *et al.* (2008) for papers that explain how this argument fits into the assembly map perspective.

In Gromov and Lawson (1983), the linearity condition on the fundamental group of the nonpositive curvature manifold were removed. Above, we used this condition to produce the sequence of finite covers with arbitrarily large injectivity radius – unfortunately, as far as we know, there might be a non-positively curved manifold whose fundamental group is simple![48]

However, there's always the universal cover, which has infinite injectivity radius – or alternatively $\mathbb{R}^{2n}$ has a bundle with compact support, which has connections on it which are trivial outside a ball, and have arbitrarily small curvature (simply rescaling of the Bott element on $\mathbb{C}^n$) – that can be used if only you are bold enough to give up on compactness in index theory.[49] Gromov and Lawson develop the relevant index theory and, using this one almost-flat bundle with compact support, proved the result without any residual finiteness hypothesis.

[48] No such Riemannian example is known, but Burger and Mozes (2001) have given simple finitely presented groups that act properly discontinuously on a product of trees.

[49] This is a different type of noncompact index theory than the L^2 index theorem of Atiyah for infinite regular covers that we had discussed in Chapter 3.

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These almost-flat ideas were later turned around and applied to define the notion of the signature of a manifold with coefficients in an almost-flat bundle (or family) (Connes *et al.*, 1990). Then the relevant index theorem applies to give the Novikov conjecture in the nicest possible way: it gives a simple-to-understand homotopy invariant that expresses the reason that the higher-signature characteristic classes are homotopy invariant.

The reader must have noticed that the discussion above sufficed to explain the higher A-genus (and signature) associated to the fundamental class – but not the other cohomology classes. In Gromov and Lawson (1983) and Connes *et al.* (1993), various approaches to the other cohomology classes are given, via families (very similar to the "descent argument" given in §4.9).

Rosenberg (1983, 1986a,b) modified this argument by using infinite-dimensional flat bundles in place of finite-dimensional almost-flat bundles – this is essentially the work described in §5.2. (He also explained in Section 2 of Rosenberg (1991) the relevant aspects of the *real C^*-theory* and its K-theory, to get the correct obstructions for the prime 2 to include the Hitchin obstructions.)

In the infinite-dimensional setting there is relatively little difference between a single operator or a family. Asserting that one can fill $K(K(\Gamma, 1))$ by Chern classes of flat $C^*\Gamma$ bundles is essentially a form of the Novikov conjecture.

The next step in continuing our development of the parallelism between surgery and operator theory is to develop analogues of the topological theories of noncompact manifolds, and, perhaps most importantly, the analogue of bounded control. We will see that the replacement for $L^{\mathrm{Bdd}}(\Gamma)$ is the K-theory of a C^*-algebra (the "Roe algebra" of the discrete metric space Γ, often denoted by $|\Gamma|$). I refer the reader to Roe (1993, 1996) and Higson and Roe (2000) for detailed discussions.

This Roe algebra is the algebra of "bounded propagation speed operators" on $|\Gamma|$. The idea is this (and is entirely analogous to the algebraic description of $L^{\mathrm{Bdd}}(\Gamma)$ that we did not give!): Imagine that at each point of Γ we attach a Hilbert space, and we only allow operators that map the Hilbert space at p to (the sum of) ones that are at most some distance away (where the distance bound is independent of p).

Except that in order to make a C^*-algebra it is necessary to take the closure of such operators, and this allows some amount of infinite speed, but "very little that is going very fast." An example is the heat flow $e^{-t\nabla}$, where ∇ is the Laplacian, for $t > 0$, which is bounded propagation speed in the most naive sense: the combinatorial model of ∇ propagates at a scale of 1 unit, and when we truncate the exponential after finitely many terms we get a norm converging sequence of bounded propagation speed operators, but whose speed keeps growing.

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The algebra is denoted by $C^*|\Gamma|$ or for general metric spaces[50] by $C^*(X)$.

If X is a complete manifold, then the "geometric operators" on X have bounded propagation speed (see Cheeger *et al.*, 1982). Elliptic operators give rise to elements of $K_x^{\mathrm{lf}}(X)$ and there is an "index map" (that is, a cousin of the "assembly map" in bounded surgery theory) that assigns an index to each elliptic operator over X an element of the group $K(C^*(X))$:

$$K_x^{\mathrm{lf}}(X) \to K_x(C^*(X)).$$

This index contains, for example, for spin manifolds, an obstruction to the existence of a positive scalar curvature metric on X so that the map $(X, g) \to X$ is Lipschitz (in the large) (so the propagation speed still is finite from the $|X|$ point of view).

Note that if X is uniformly contractible, then one can conjecture that the index map is an isomorphism, and that, even without this,

$$KX_x^{\mathrm{lf}}(X) \to K_x(C^*(X))$$

is. (The manifold in Dranishnikov *et al.* (2003) gives a counterexample to the first statement, but it makes use of the map from $KX_x^{\mathrm{lf}}(X)$ and remarkably, even though X is uniformly contractible, the left-hand sides of these seemingly identical constructions do not coincide.) Guoliang Yu (1998) gave a very elegant example showing that this latter map is not even injective. Let X be the disjoint union of rescaled copies of the sphere:

$$X = \bigcup \sqrt{n}\mathcal{S}^{2n}.$$

This is a spin manifold with positive scalar curvature bounded away from 0. The rescalings mean that, at any scale, only finitely many of the spheres can be ignored. We have an isomorphism

$$KX_x^{\mathrm{lf}}(X) \approx \prod(\mathbb{Z})/\bigoplus \mathbb{Z}$$

with the Dirac operator representing the element $(1, 1, 1, \ldots)$ which is nontrivial, and hence an element of the kernel of the assembly map.

This is a very striking example[51] but (like the Dranishnikov *et al.*, 2003, example) it involves unbounded geometry. I'd be curious to know what $K(C^*(X))$ is in this example.

Despite these counterexamples, the reverse is true: there are many situations (such as complete non-positively curved manifolds) where one can show that

[50] This only depends on the coarse quasi-isometry type of X, or even just of the bounded category of X (in the sense of §4.8).

[51] And it would be very interesting to know if it gives an example for noninjectivity of the coarse assembly map in bounded L-theory.

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the coarse index map is an isomorphism, and then the method of descent applies to give the Novikov conjecture for the group Γ (from the metric space $|\Gamma|$). See §5.6 for more details and further extensions that take into account the fundamental group of the noncompact manifold.

Now let us turn to the problems of group actions, where there are different phenomena in the case of the circle and the case of finite groups, and then to the birational invariance of higher Todd genus, where the result is actually true unconditionally (and integrally!). In all of these cases[52] it is not too hard to promote the simply connected argument to a proof that's conditional on the injectivity of the index homomorphism $K(\mathrm{B}\Gamma) \to K(C^*\Gamma)$ (also known as the strong Novikov conjecture[53]). However, what we would like to understand is "why" these are now known to be true unconditionally. The mechanism is actually different in the two cases, but the "reason" seems to be the same.

Let us start with the results about S^1-actions, and, for simplicity, let's assume that the action is "semi-free," namely that every orbit is either trivial (corresponding to fixed points) or free,[54] leaving the reader to refer to the original papers for the general case (which is philosophically the same, but does show some different aspects in detail).

Let's start with the case of the higher A-genus:

Proposition 5.16 *Suppose $\mathcal{S}^1$ acts on M semi-freely with nonempty fixed point set* F*; then the map $M \to M/\mathcal{S}^1$ is split injective on the fundamental group.*

We suppose that codimension F is at least 4: this can be achieved by taking the product with $\mathbb{C}^2$, giving the latter the obvious circle action thinking of $\mathcal{S}^1$ as unit complex numbers. In that case, the map is actually an isomorphism on the fundamental group by a simple application of Van Kampen's theorem (and general position: in that removing F will not change the fundamental group).

Now, recall from §4.5 that to prove the vanishing of the image of $A(M) \cap [M]$ in $\bigoplus H_{m-4i}(K(\pi,1);\mathbb{Q})$, it just suffices to show that, for all "subcycles" X of $K(\pi, 1)$ that have a trivial normal bundle neighborhood, the transverse inverse image $f^{-1}(X)$ has vanishing A-genus. But this is true, since by first taking the

[52] Except for the case of invariance of higher signatures for fixed sets of pseudo-trivial actions.

[53] So-called because it implies the homotopy invariance of higher signatures – i.e. the original Novikov conjecture. It also implies, away from 2, integral refinements that we will discuss in the next section.

[54] Actually, the free case is even easier: M bounds a $\mathcal{D}^2$ bundle over $M/\mathcal{S}^1$ and then we can use cobordism invariance of characteristic classes to see that, for any $\alpha \in H*(M/\mathcal{S}^1)$, the higher signature of M and the higher A-genus of M vanish. Alternatively, dually, and somewhat more precisely, the pushforwards of [Sign] and [$\mathcal{D}$] in $K_m(M/\mathcal{S}^1)$ vanish. (They are the boundary of the natural classes in $K^{\mathrm{lf}}_{m+1}(E)$, where E is the total space of the vector bundle whose unit sphere bundle is the circle bundle defined by $M \to M/\mathcal{S}^1$.)

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transverse inverse image of X in $M/\mathcal{S}^1$ and then taking its inverse image in M, we obtain an inverse image for X that is still spin (it has a trivial normal bundle in a spin manifold) and has a nontrivial $\mathcal{S}^1$-action. Thus, the ordinary Atiyah–Hirzebruch theorem gives the conclusion.

The result about higher-signature localization for $\mathcal{S}^1$-actions follows from similar reasoning. There is a cobordism (see below) from M to a union of $\mathbb{CP}^{c/2-1}$ bundles over the components of F, where c is the codimension of the component. We need the following lemma that tells us about the L-classes of this total space in terms of the L-class of F.

Lemma 5.17 *If $\pi\colon \mathrm{E} \to \mathrm{B}$ is a (block) bundle with (homotopy) fiber (a homotopy) $\mathbb{CP}^k$, whose monodromy is trivial on $H^2(\mathbb{CP}^k)$, then $\pi_*\big(L(\mathrm{E})\cap[\mathrm{E}]\big) = \operatorname{sign}(\mathbb{CP}^k)L(M)\cap[M]$.*

This boils down (by the same reasoning as before) to the fact that for all such bundles $\operatorname{sign}(\mathrm{E}) = \operatorname{sign}(\mathbb{CP}^k)\operatorname{sign}(M)$. This is a theorem of Chern, Hirzebruch, and Serre (1957).[55]

The cobordism is explicit: it is $M\times[0,1]/\sim$ where we identify points on $M\times 1$ that are on the same orbit if that orbit does not touch a tubular neighborhood of F. The structure comes from the equivariant tubular neighborhood theorem (in the smooth case, and the proof of the existence of block bundles in the PL case[56]). Explicitly, this proves:

Theorem 5.18 *If $\mathcal{S}^1$ acts on a manifold M with nonempty fixed set* F, *then the higher signatures of M are those of* F, *i.e. for all $\alpha \in H^*(\mathrm{B}\pi)$, one has*

$$\langle f^*(\alpha)\cup L(M),[M]\rangle = \langle f^*i^*(\alpha)\cup L(\mathrm{F}),[\mathrm{F}]\rangle.$$

For $\mathbb{Z}_p$-actions, the cobordism argument fails (the inverse image has a $\mathbb{Z}_p$-action, but its homological properties are not restricted). We will give one argument now about the connection to the Novikov conjecture – another one can be made based on the ideas from Chapter 6 when we study the equivariant Novikov (and Borel) conjectures more systematically. The current argument is based on preliminary remarks about rational homology manifolds.

The reasoning given in §4.5 for the definability of L-classes for PL-manifolds actually produces homology L-classes for oriented rational homology mani-

[55] Their theorem assumes the monodromy of the bundle is trivial – for the $\mathbb{CP}^k$ case one could have monodromy of order 2 (i.e. inducing complex conjugation on the fiber). However, in that case, the 2-fold cover of this fibration has trivial monodromy, and signature is multiplicative for all finite sheeted covers (as a consequence of the Hirzebruch signature theorem) and the result follows again.

[56] The topological case requires some form of Smith theory to make these arguments (that are essentially locally homological and sheaf-theoretic, rather than completely geometric).

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folds[57] (that agrees with the Poincaré dual of the L-cohomology class): all that one needs to produce L-classes is a cobordism-invariant definition of signature – and one has this using rational cohomology.

Moreover, this characteristic class can be pushed into group homology to give a "higher signature." It turns out (see immediately below) that if the Novikov conjecture is true for manifolds, then it is true for rational homology manifolds in the sense that the higher signature in the homology of $K(\pi, 1)$ will be preserved by maps $f\colon X \to Y$ that are orientation-preserving and induce $\mathbb{Q}$-homology isomorphisms on the regular covers induced from the map $Y \to K(\pi, 1)$.

The most straightforward argument for this uses Ranicki's algebraic theory of surgery (Ranicki, 1980a,b) (mentioned earlier in §4.7) or its predecessor (Mischenko, 1976). Ranicki views the L-groups of surgery as cobordism groups of certain chain complexes with duality over $\mathbb{Z}\pi$. Inverting 2,[58] we can view $X \cup X \to X$ as a surgery problem over the ring $\mathbb{Q}\pi$ (since we are only assuming a $\mathbb{Q}$-homology manifold) and thus gives an element $\sigma^*(X^n) \in L_n(\mathbb{Q}\pi)$. This element is homotopy invariant since it is defined using chain complexes (the chain complex of a homotopy equivalence gives a cobordism in the appropriate sense). Under the assembly map,

$$H_n(K(\pi, 1); L(\mathbb{Q})) \to L_n(\mathbb{Q}\pi),$$

the homology L-class gets mapped to $\sigma^*(X^n)$.[59] It is a general and remarkable algebraic result of Ranicki (1979a) that, for any π, the map $L(\mathbb{Z}\pi) \to L(\mathbb{Q}\pi)$ is an isomorphism away from the prime 2 – indeed, has kernel and cokernel annihilated by multiplication by 8 – so the injectivity of this assembly map – away from 2 – is equivalent to the injectivity of the usual one and thus the

[57] In the PL case: for the topological situation, this can be done using controlled topology (see Cappell *et al.*, 1991 – as a topological definition of L-homology classes that works for manifolds implies Novikov's theorem on Pontrjagin classes, such a definition cannot be too trivial.

[58] In the coefficient ring, since this is a degree-2 map, so we need to multiply the Poincaré duality isomorphism by 2 in the range – which is OK if 2 is inverted in the coefficient ring. Ranicki actually does something different and better. He defines a slightly different cobordism group of chain complexes with self-duality for rings with anti-involution, and then observes that the chain complex of X is naturally an element of this different group. Wall's L-groups, the surgery L-groups, are denoted using subscripts, and Ranicki denotes these modified groups with a superscript: $L_s(\mathbb{Z}\pi)$, for instance (despite their remaining covariantly functorial) and christens them *symmetric L-groups* (and refers to Wall's as *quadratic L-groups*). In any case these only differ at the prime 2. This is not merely an academic issue though: alive versus dead, yes versus no, being and nothingness, are all mod 2 issues.

[59] This, while useful, is just unraveling all the definitions of the objects and morphisms involved. Of course, deducing the Hirzebruch formula from this approach then involves identifying two different homology L-classes!

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Novikov conjecture in this setting follows from (is equivalent to) the usual one.[60]

Now, if $G \times M \to M$ is an orientation preserving action of a finite group, then M/G is a $\mathbb{Q}$-homology manifold, and sign (M/G) is just the signature of the G-invariant part of the cohomology of M. In terms of the G-signature of M, which is a representation, we are looking at the multiplicity of its trivial component – which can be computed using character theory as

$$\mathrm{Sign}\,(M/G) = 1/|G| \sum \chi_g(G\text{-signature})$$

and the right-hand side can be computed by characteristic classes of the fixed sets M^g and their equivariant tubular neighborhoods. If $G = \mathbb{Z}_p$ the formula is

$$\mathrm{Sign}\, M/\mathbb{Z}_p = \frac{1}{p}\mathrm{sign}\,(M) + \frac{p-1}{p}\langle v^8(\xi) \cup L(F), [F]\rangle,$$

where $\nu(\xi)$ is $p/(p-1)$ times[61] the sum of the local contributions in the G-signature formula from all of the generators of $\mathbb{Z}_p$ of the characteristic class of equivariant normal bundle to F from Atiyah and Singer (1968b). If the action is homologically trivial, then Sign $(M/\mathbb{Z}_p) =$ sign (M), so we get

$$\mathrm{Sign}\,(M) = \langle \nu(\xi) \cup L(F), [F]\rangle.$$

If one combines the various formulas, one obtains[62] that, after applying the assembly map,

$$\sigma_*(M) = \sigma_*(M/\mathbb{Z}_p) = A_* f_*\big(\nu(\xi) \cup L(F) \cap [F]\big) \in \bigoplus H_{*-4i}\big(K(/\pi, 1); \mathbb{Q}\big),$$

so that assuming the Novikov conjecture, one gets the localization formula. Conversely, if M is a manifold with $\chi(M) = 0$,[63] so that $f_*(L(M) \cap [M])$ lies in the kernel of the assembly map,[64] then the surgery problem $M \to M \times K(\mathbb{Z}_p, 1)$ in $L_m(\mathbb{Q}[\pi \times \mathbb{Z}_p])$ will have vanishing obstruction[65] (after tensoring with $\mathbb{Q}$).

[60] Of course, this leaves some room for differences in the integral theory.

[61] We put this factor in to make later formulas more pleasant.

[62] Somewhat profligately, since it is possible to only invert 2 and p in this formula.

[63] If the Euler characteristic is nonzero, then it is impossible to have a finite complex with a free homologically trivial action, by the Lefshetz fixed-point theorem. It occurs formally in our setting, in ensuring that the infinite complex $M \times K(\mathbb{Z}_p, 1)$ has rational chain complex chain equivalent to a finite complex. (The element of $K_0 \times (\mathbb{Q}[\pi \times \mathbb{Z}_p])$ represented by the complex $C_*\big(M \times K(\mathbb{Z}_p, 1)\big)$ is $\chi(M)[\mathbb{Q}\pi]$ where $\mathbb{Q}\pi$ is clearly a nontrivial projective module over $\mathbb{Q}[\pi \times \mathbb{Z}_p]$.

[64] Note that the map from bordism to group homology with coefficients in the (symmetric) L-spectrum, given by $[?] \to f_*(L(?) \cap [?])$ is onto; this is certainly elementary, and all we need, if one tensors with $\mathbb{Q}$.

[65] Surgery with coefficients in a subring of $\mathbb{Q}$ measures the obstruction of a degree-1 normal map being cobordant to one that is a local homology equivalence (in the universal cover) (see, e.g., Taylor and Williams, 1979b).

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The result of the surgery will be a manifold with free homologically-trivial $\mathbb{Z}_p$-action cobordant to a multiple of M, and hence with nontrivial higher signature. It will be a counterexample to the localization principle. To summarize, we have explained:

Theorem 5.19 *If $\mathbb{Z}_p$ acts on a manifold, trivially on π_1 and on twisted homology, then one gets a localization formula*

$$f_*(L(M) \cap [M]) = f_*(\nu(\xi) \cup L(F) \cap [F]) \in \bigoplus H_{*-4i}\big(K(\pi,1);\mathbb{Q}\big)$$

iff the Novikov conjecture is true for the group π.

Why is there no localization principle for (untwisted) higher signatures for pseudo-trivial actions? We note that the whole problem is anomalous from the point of view of the game: although there is an index equality in the simply connected case, it isn't based (solely) on index theory – but rather Smith theory, a homological result, played a key role.

That doesn't mean that one can't play the game – only that we don't see how to win. The actual failure is based on two principles.

The first is that Smith theory, i.e. the homologically trivial theory of group actions, essentially gives p-adic information for p-groups, rational information for tori, but almost nothing at all for non-p-groups or nonabelian compact Lie groups.

In any case, for p-groups, the only information one can hope for is $\mathbb{F}_p[\pi]$ information, with $\mathbb{F}_p$ some finite field with p elements, and, with some effort, for any manifold M' that is $\mathbb{F}_p[\pi_1(M)]$-homology equivalent to M, one can construct a quasi-trivial $\mathbb{Z}_p$-action on the product of M with a disk, with M' being the fixed-point set.[66]

The second is the very general[67] homological surgery theory of Cappell

[66] We have ignored some algebraic K-theory problems that actually arise even if $\pi = \mathbb{Z}$ (as we will show in forthcoming work with Cappell and Yan). As usual, such can be gotten rid of by the violent act of crossing with a circle.

In the smooth category, there are additional bundle-theoretic considerations even to obtaining actions in a neighborhood of M', since a real vector bundle can only admit a $\mathbb{Z}_p$ action if it has a complex structure. For the PL and topological situations, there are results that put a $\mathbb{Z}_p$ action on many neighborhoods (see Cappell and Weinberger, 1991a) and this is being tacitly invoked here. Extending the action outside a neighborhood in the semi-free case (even in low codimension) assuming suitable K-theory conditions is the main result of Assadi and Vogel (1987).

[67] Although, over the past decade there have been a number of occasions when I had wished for a yet more general theory: the Cappell–Shaneson theory is very well adapted to the problems for which it was invented, codimension-2 embedding theory, a.k.a. knot and link theory, but it does not describe the obstructions to doing surgery to obtain a map that is a homology equivalence so that the map is a homology equivalence with coefficients in rather general bundles, or to handle general Serre classes that are not associated to localizations.

Even $\mathbb{Z} \to \mathbb{R}$ is not "officially" part of their theory, although of course a map of $\mathbb{Z}$ chain

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and Shaneson (1974). They define obstruction groups to performing surgery in this setting, and the relevant group is $\Gamma_m(\mathbb{Z}\pi \to \mathbb{F}_p[\pi])$. The even-dimensional groups are quite hard to get one's hands on, and there are interesting phenomena to be unraveled, but the odd-dimensional groups are typically very small: there is a natural map

$$\Gamma_m(\mathbb{Z}\pi \to \mathbb{F}_p[\pi]) \to L_m(\mathbb{F}_p[\pi])$$

that is automatically one-to-one for m odd. Since $L_m(\mathbb{F}_p[\pi])$ is a module (by tensoring) over the Witt group of nonsingular quadratic forms $\mathrm{W}(\mathbb{F}_p)$, and $\mathrm{W}(\mathbb{F}_p)$ is always of exponent at most 4 (Milnor and Husemoller, 1973), these Γ groups are exponent 4. Consequently, we can easily produce manifolds in odd dimensions that are $\mathbb{F}_p[\pi]$ homotopy-equivalent to M, whose higher signatures deviate almost at will from those of M.[68]

Then it is necessary to produce group actions with these manifolds as fixed sets, and for this there is well-developed machinery (Assadi and Browder, 1985; Weinberger, 1985a, 1986; Jones, 1986; Assadi and Vogel, 1987; Cappell and Weinberger, 1991a).

We close this section with a *very* brief discussion of the argument given in Block and Weinberger (2006) for Rosenberg's algebraic–geometric Novikov conjecture on birational invariance of higher Todd genera. It follows the pattern we saw above for the S^1 localization formula (or vanishing of higher A-genus).

If V and V' are birational smooth varieties, then according to Abramovich *et al.* (2002)[69] one can move from V to V' by a sequence of blowings up and down. Thus one needs only to check that if

$$\pi : V' \to V$$

is a blowup, then $\pi_*(O_{V'}) = O_v$, and then rely on the topological Riemann–Roch theorem of Baum, Fulton, and MacPherson (1975) to map further (and give commutativity of the diagram) $K(V) \to K\big(K(\pi, 1)\big)$. This is a local result (and is essentially Hartog's argument given for the birational invariance of the Todd class).

complexes is an $\mathbb{R}$ isomorphism iff it is a $\mathbb{Q}$ isomorphism, so one can apply their theory by replacing $\mathbb{R}$ by $\mathbb{Q}$.

68 In even dimensions some higher signatures survive the map $L_m(\mathbb{Z}\pi) \to \Gamma m(\mathbb{Z}\pi \to \mathbb{F}_p[\pi])$ while many don't. For instance, for free abelian groups, only the ordinary signature survives, but for surface groups of genus greater than 2, the higher signature associated to the fundamental class also survives (as a consequence of the Atiyah–Kodaira fiber bundle).

69 This result is a kind of Hironaka theorem for maps.

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5.4 The Moral

What have we learned from playing several rounds of "the Novikov game"?

I think there are two lessons:

There really is a gap in level of depth between the problems that seem to be conjectural and the ones that we know how to prove. The latter tend to be essentially local statements, and the difficulty (nowadays) is to prove theorems from essentially *global* hypotheses.

Novikov's theorem is (as we have seen) the statement that L-classes can be preserved by *hereditary* homotopy equivalences (i.e., CE-maps). The whole problem with the Novikov *conjecture* is determining for which cycles homotopy equivalences "descend" or can be "inherited" and after how much work. Also, the birational invariance is of the same sort: the key to the proof is that a map that is birational is birational on all of its Zariski-open subsets.[70]

The group actions overwhelmingly ratify this perspective. Being a circle action is something that descends, by definition, to invariant submanifolds. However, being a homologically trivial group-action is a local condition.

The results about positive scalar curvature are apparently exceptional in this regard, but actually the point is that it is impossible to ever get a connection between the Dirac operator and its kernel from positive scalar curvature locally: this is a global phenomenon that requires completeness (and thus does *not* descend to open subsets). Indeed, on a manifold with "big" A-genus, nothing about the homological structure of this class is reflected in the structure of the negative scalar curvature set. The remarkable results of Kazdan and Warner (1975) imply that any closed manifold of dimension ≥ 3 has a metric whose scalar curvature is strictly positive outside of a ball – irrespective of fundamental group: the obstruction to positive scalar curvature doesn't "carry" to negative scalar curvature.[71]

On the other hand, the results about the Novikov conjecture all have global hypotheses. In order to play the game, we need an operator, and a hypothesis that combines well with flat bundles of arbitrary dimension (and that's what

[70] It is possible to write down modern proofs of these two theorems so that the diagrams look exactly the same, as can be specialized from the argument soon to follow. For the Novikov theorem, one thinks of the "canonical class" (that contains the signature operator, and the Poincaré dual of the L-class) as a self-dual sheaf that is preserved under hereditary homotopy equivalences. For the birational theorem, the canonical sheaf is a coherent algebraic sheaf that is preserved by birational equivalences.

[71] However, in some noncompact situations, something like this is true: see Roe (1988a,b) for a situation where a nonvanishing index on a noncompact manifold guarantees that the negative scalar curvature set is noncompact. This is also true for the results in Gromov and Lawson (1983) for the manifolds with bad ends. See also Chang and Weinberger (2010).

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fails for the pseudo-trivial group-action situation: in passing to covers, we do not get any information that holds in characteristic 0 generally).

In light of this analysis, we can now formulate some additional theorems of Novikov type. That is, we need situations where our global conclusion is local from the point of view of some alternative space. For example:

Proposition 5.20 *If $f: M \to N$ is a Riemannian fiber bundle with spin structure so that the fibers $f^{-1}(n)$ have positive scalar curvature, then $f_*([D_m]) = 0 \in \mathrm{KO}_*(N)$.*

Note that M does not immediately have positive scalar curvature in this situation: however, by rescaling the fibers to make them tiny, we can arrange for the vertical directions in this bundle to overwhelm the others and make the scalar curvature positive. Doing this indicates that the reason for the positive scalar curvature is *local from the point of view of the manifold* N, so the vanishing is to be expected, and it is not hard to prove.

Similarly we can generalize Novikov's theorem as follows:

Proposition 5.21 *If $f: M' \to M \to N$ is a homotopy equivalence over N, i.e. for all open subsets U of N, $f: f^{-1}\pi^{-1}U \to \pi^{-1}U$ is a proper homotopy equivalence, i.e. the map of pushforward sheaves $R\pi_* Rf_*(\mathbb{Q}) \to R\pi_*(\mathbb{Q})$ is a quasi-isomorphism of sheaves over N, then*

$$\pi_*[f_*([\mathrm{sign}_M]) = \pi_*([\mathrm{sign}_M]) \in K_*(N).$$

Note that Novikov's theorem is the special case of π = identity $M = N$.[72]

Moreover, both of the statements above can be coupled to statements about the Novikov conjecture if the fibers are non-simply connected, but we know the Novikov conjecture for them, and we inflate the K-theory of N to include this additional information.

For example:

Proposition 5.22 *If $f: M' \to M \to N \times K(\pi, 1)$ is a map that is a homotopy equivalence (locally) over N (but not necessarily over $K(\pi, 1)$), then one has, assuming the Novikov conjecture for π, the equality*

$$\pi_*[f_*([\mathrm{sign}_{M'}]) = \pi_*([\mathrm{sign}_M]) \in K_*\big(N \times K(\pi, 1)\big) \otimes \mathbb{Q}.$$

(One could work integrally if one takes the Novikov conjecture for π to mean an integral statement.)

[72] Strictly speaking, Novikov's theorem is the rationalization of this statement. This integral statement about the signature operators is the main result of Pedersen *et al.* (1995). And the above refinement is neither simpler nor more difficult than this result (just as the rational version of this statement follows *mutatis mutandis* from Novikov's argument – or the one we gave in §4.5).

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For N a point, this is the Novikov conjecture for π; for $\pi = e$, this is the generalized Novikov theorem.

These results can easily be understood from the point of view of controlled topology (see §4.8). We will explain this at the beginning of §5.5.

Another beautiful theorem that fits well into this philosophy is the following result of Borisov and Libgober (2008) that asserts that higher elliptic genera are invariants of K-equivalence. Recall that V and V' are K-equivalent if there is a (Z, ψ, ψ') with $V \leftarrow Z \rightarrow V'$ so that $\psi^* K v = \psi^* K_{V'}$ (where $K_?$ is the canonical divisor of '?'). (A motivating case is the Calabi–Yau case, where canonical divisors, by definition, vanish, so that one is asserting here the birational invariance of invariants of Calabi–Yau manifolds.)

Theorem 5.23 (Borisov and Libgober, 2008) *For any fundamental group, all of the rational higher elliptic genera agree for any K-equivalent smooth varieties.*

The second answer is perhaps more pragmatic. Although the original Novikov conjecture was phrased in terms of rational invariants, we have already seen that, for torsion-free groups, one can conjecture an integral injectivity result – and this implies that many of the characteristic class formulas or restrictions that we have developed have, for torsion-free groups, integral refinements (with more refined definitions of the characteristic classes necessary: e.g. using K-theory and the cycle associated to the defining elliptic operator[73]).

We will discuss in Chapter 6 what to do for groups with torsion, but, for now, let us note that it is necessary to do something.

This is readily apparent in the case of cyclic groups $\mathbb{Z}_p$. If we do not invert p, then simple examples involving homotopy-equivalent linear lens spaces (of high dimension[74]) show that higher signatures are not always invariant, i.e. the pushforward of the signature class in the K-homology of $K(\mathbb{Z}_p, 1)$ Similarly, for the positive scalar curvature problem, this is even more obvious: lens spaces also have nontrivial Dirac classes, yet they all have positive sectional curvature.

Moreover, in these two problems at least, things seem to be rather deeper. We know as a consequence of functoriality that any homology class in (the homology of) $K(\pi, 1)$ comes a manifold homotopy equivalent to M. One can

[73] Or in topology using things related to the Sullivan orientation inverting 2 (Sullivan, 2005), and the Morgan–Sullivan class at 2 (Morgan and Sullivan, 1974). These are subsumed in the *controlled symmetric signature*, i.e., the *symmetric signature* in the sense of Ranicki (1980a,b), of M, thought of as a Poincaré complex controlled over itself (see, e.g., Cappell *et al.*, 1991), amplifying our earlier discussion in the chapter of signature-type invariants of $\mathbb{Q}$-homology manifolds.

[74] Three-dimensional lens spaces are all parallelizable. However, the K-homology of $K(\mathbb{Z}_p, 1)$ fills up using the differences of the signature operators of higher and higher-dimensional linear lens spaces.

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show (using the Gromov–Lawson surgery theorem – see §5.6) that there is a similar statement possible for positive scalar curvature manifolds – there are no Dirac obstructions except for those detected in the $K(\pi, 1)$. However, in the local cases one can often refine the equalities to lie in more refined places than the group (K or L or Ell) homology. For example, note the refined Novikov and positive scalar curvature theorems discussed in this section that assert results in $K_*(N)$ for general (i.e. for not necessarily aspherical) N. (The result on Rosenberg's algebraic-geometric Novikov conjecture is also true integrally as follows from the argument of §5.3, showing how it fits into the realm of Novikov theorems rather than conjectures.)

Needless to say, until the Novikov conjecture is disproved, we do not know that there is a real difference between these classes of statements – and, moreover, the connection between statements about assembly maps and the geometric consequences are only tight in the topological case – conceivably the positive scalar curvature problem can work out differently than the higher-signature problem.[75]

5.5 Playing the Borel Game

It is time to take some stock again of where our journey has taken us so far.

Starting from the original Borel conjecture, we have seen how the geometry of lattices and ideas in geometric rigidity theory can lead to a great deal of information about the topological structure of these (and other much larger classes of aspherical) manifolds, if not their topological rigidity. We were inevitably led to consider the implications of functoriality (forced upon us by the π–π theorem of surgery).

In studying how the Borel conjecture restricts the variation of characteristic classes (and spectral geometry), we were led to the Novikov conjecture. This is a very broad phenomenon wherein the fundamental group of a manifold has strong implications for its global analysis, some of whose implications we studied in this chapter. The key to the breadth of what we've seen to this point always has involved elliptic operators and their properties – and was often the

[75] It is even conceivable that the Gromov–Lawson–Rosenberg conjecture is true, but the strong Novikov conjecture (i.e. the C^*-algebra version) fails and another mechanism is behind this truth. Nevertheless, the ordinary Novikov conjecture, not involving completions, has a definite chance of being true even if the strong Novikov conjecture is correct.

The main reason that one can imagine the first statement is that the work of Schoen, Yau, and Lohkamp gives methods *completely unrelated to Dirac operators* for the nonexistence of positive scalar curvature metrics on certain manifolds. The second statement can be suggested (to the authors of science fiction monographs) by some deviations between the topological analytic conjectures that will be discussed, for example, in Chapter 8.

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consequence of the general injectivity of (related) assembly maps in K-theory or a Hermitian cousin, L-theory.

But it is natural to play the Borel game, as well, not just the Novikov. What can we say about the isomorphism statement regarding the assembly map, say, for torsion-free groups? Does this problem have relatives with a substantial family resemblance – that might themselves have beautiful implications?

It is also time to expand our perspective to situations that are not mediated by elliptic operators, topological K-theory, but exist within *algebraic* K-theory,[76] clearly an analogue (if only because of its name), but also directly connected to the difference between homotopy and homeomorphism. So, it could, in principle, also obstruct the Borel conjecture.

This section is just a first pass at this project. We will return to it in Chapter 6 when we deal more seriously with groups with torsion (after all, in this chapter, we have only dealt so far with products of torsion-free groups with finite ones).

5.5.1 Fibering and Controlled Surgery

Let us recall, temporarily ignoring the Whitehead group (see §4.1), the Farrell fibering theorem.[77] It describes the obstruction of fibering a manifold over a circle; given $f: M \to S^1$ one first considers whether the associated infinite cyclic cover is a finite complex up to homotopy type (or even finitely dominated). If so, then the mapping torus of the covering translate $T(\tau)$ is a finite complex, homotopy equivalent to M.

Indeed $T(\tau) \to \mathcal{S}^1$ describes a *controlled Poincaré complex* over the circle. It is a Poincaré complex, and this is true for the inverse image of every open subset of the circle.[78] The homotopy equivalence $M \to T(\tau)$ is, among other things, a normal invariant for this Poincaré complex.

We can think of this situation in two different ways (that up to algebraic K-

[76] Actually, an analogue of the Novikov conjecture is known for the very large class of groups whose homology is finitely generated in every degree – according to a remarkable theorem of Bökstedt *et al.* (1993). (See also Dranishnikov *et al.* (2020) for a broader explanation of these ideas within the realm of algebraic K-theory.) It is conceivable that the correct Hermitian analogue of their technique could prove the Novikov conjecture for a similarly broad class of groups – although it is unlikely that the C^*-algebra version could ever succumb to such an approach.

[77] Taking the Whitehead group into account is more subtle than one might think. One quickly comes to the conclusion that Nil groups should be the source of non-approximate fibering, but Farrell *et al.* (2018) show that, in the presence of Klein bottles in the fundamental group of the base, there are a number of nil-type obstructions that all have to vanish.

[78] To be more precise, the inverse image of each open set is a proper Poincaré complex, satisfying the kind of Poincaré duality that open manifolds do – interchanging cohomology with support – having compact projection to the circle with ordinary homology. One could also define an *approximate* Poincaré complex, in an ε–δ fashion where deviations of duality at one scale are trivial in a somewhat larger one

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theoretic obstructions are equivalent). First of all, we have a Poincaré complex blocked over the circle $\mathcal{S}^1$. That is, we have a Poincaré complex for each vertex (in a triangulation) and a Poincaré cobordism between these over each edge. And associated to this we have a blocked surgery obstruction, which will be a map $[\mathcal{S}^1 : \mathbf{L}_{m-1}(\pi)]$ (where π is the fundamental group of the fiber, and $\mathbf{L}$ indicates the space which encapsulates the obstruction to blocked surgery).

Alternatively, we can try to do controlled surgery, which is to build a map $M \to T(\tau) \to \mathcal{S}^1$, so that over each open subset of $\mathcal{S}^1$ the map restricts to a proper homotopy equivalence.

Both alternatives are a little weaker than fibering: the s-cobordism theorem (or, alternatively, obstructions that naturally give an element in $H^1\big(\mathcal{S}^1; \mathrm{Wh}(\pi)\big)$ is used to straighten the h-cobordisms in the first theorem to be a fibration. For the second, there are issues related to $K_0(\mathbb{Z}\pi)$ as well – there is no guaranteed way to find the fiber over a point from the fiber over open intervals,

$$[S^1 : \mathbf{L}_{m-1}(\pi)] \cong \mathbf{L}_m(\Gamma),$$

where $\Gamma = \pi_1(M)$. Needless to say, the π on the left-hand side really means $\pi_1(F_\theta)$, the fundamental group of the fiber over a point θ in the circle. That means the left-hand side should be thought of as sections of a fibration rather than as a function space in general. (The monodromy of the bundle over the circle with fiber $\mathbf{L}_{m-1}(\pi)$ is induced by the covering translate on the infinite cyclic cover.)

In the controlled situation, cohomology is the wrong variance (block bundles, and their obstructions, pull back): we push forward a controlled surgery problem to obtain a problem with somewhat looser control. This leads to the conclusion – and it is one that we had earlier seen in some situations using the α-approximation theorem of Chapman and Ferry – that controlled surgery theory should be a homology theory (again twisted if $\Gamma \neq \mathbb{Z} \times \pi$),

$$\mathbf{L}^{\text{controlled}}(T(\tau) \to \mathcal{S}^1) \cong H_1(\mathcal{S}^1 : \mathbf{L}_{m-1}(\pi)) \cong \mathbf{L}_m(\Gamma),$$

where the first statement is a "general" calculation (and would be correct were S^1 replaced by some other space X) and the second statement is a consequence of the fibration theorem.

Note that in the Borel conjecture we had the assembly map

$$H_m(M^m : L(e)) \to \mathbf{L}_m(\Gamma),$$

being an isomorphism when M is a $K(\Gamma, 1)$-manifold.[79] But surely it is now irresistible to suggest that $H_m(M^m : L(e)) \to \mathbf{L}_m(\pi_1(E))$ is an isomorphism

[79] This statement does not require M to be closed if we use compact supports, as suggested at the end of Chapter 3 and §4.7.

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where $E \to M$ is a fibration, and where π is the fundamental group of the fiber (and thus the left-hand side should be interpreted in a (co-)sheaf-theoretic way).

These statements are conjecturally the case, and are included in what I call the *Borel package*, a collection of statements yet more general. They have an interpretation in terms of some kind of fibering of manifolds over aspherical ones.

Note, of course, that the Borel conjecture itself is the statement that if M is aspherical and M' is homotopy equivalent to it, we can find a homotopy of this homotopy equivalence $M' \to M$ to one that is a fibration over every open subset (which is, therefore, a controlled homotopy equivalence, which is the same a CE map – when we are mapping between manifolds of the same dimension – which in turn is a limit of homeomorphism, by the theorem of Siebenmann or Edwards).

Needless to say, also, that this is compatible with the Borel conjecture if the group π satisfies the Borel conjecture. One nice feature of this viewpoint is that it tautologously builds in a closure of the Borel under short exact sequences: $1 \to \pi \to \Gamma \to \pi' \to 1$ (i.e., the result for Γ follows from that for π and π').

The Borel package itself needs at least two further amplifications. The first is a modification or expansion to include algebraic K-theory; we give the modification immediately and the expansion later in this section. Whitehead groups are not always trivial; the product of h-cobordant manifolds with the circle are Cat-isomorphic,[80] so there can never be a uniqueness of fibering without taking algebraic K-theory into account. Moreover, there is also an algebraic K_0 condition to being able to compactify the infinite cyclic cover, which would surely be possible if the manifold fibered over the circle. However, if the statements we had written were correct "on the nose," then, for example, blocked surgery theory would indeed give existence and uniqueness of fibering.

However, it is pretty close. It turns out that all algebraic K-theoretic obstructions die after crossing with a torus, and that, by using tori, one can make the solutions essentially unique. (The uniqueness is typically another algebraic K-theory obstruction). As a result, one way to get around the algebraic K-theory issue is to "stabilize." We can just cross all the groups involved with $\mathbb{Z}^\infty$ and then the arguments would work as described above. This is a little awkward, and it is best to replace $L_k(\pi)$ by $\lim L^{\mathrm{Bdd}}_{k+d}(\pi \times \mathbb{R}^d \downarrow \mathbb{R}^d)$, where we map $L^{\mathrm{Bdd}}_{k+d}(\pi \times \mathbb{R}^d \downarrow \mathbb{R}^d) \to L^{\mathrm{Bdd}}_{k+d+1}(\pi \times \mathbb{R}^{d+1} \downarrow \mathbb{R}^{d+1})$ by crossing with $\mathbb{R}$.

This limit is referred to as $L_k^{-\infty}(\pi)$. The map $L_k(\pi) \to L_k^{-\infty}(\pi)$ is an isomor-

80 Recall that the obstruction to an h-cobordism being a product is the Whitehead torsion; torsions are multiplied by Euler characteristic in products (see e.g. Milnor, 1966; Cohen, 1973).

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phism away from the prime 2. (I don't know any example where the kernel and cokernel don't have reasonably small exponent,[81] but I can't imagine a proof of such a statement either given our current state of knowledge.)

The second comment and amplification is that L-groups have a completely algebraic definition, and all of the constructions, while we have made or explained them geometrically, can also be algebraicized. Note that $\mathbb{Z}[A \times B] \cong \mathbb{Z}[A][B]$ (and similarly with a twisted group ring for semidirect products, and a more complicated but obvious enough expression for the situation where one has an extension that is not split). It suggests therefore that, even for a nontrivial family of rings R over $K(\Gamma, 1)$, there should be an isomorphism

$$H_m(K(\Gamma, 1), L^{-\infty}(R)) \to L^{-\infty}(\text{“}R\Gamma\text{”}),$$

where “$R\Gamma$” is the group ring (twisted, when the family demands it). Frequently, geometric techniques for the Borel conjecture will initially apply directly to the assembly map where $R = \mathbb{Z}$, and then have an extension to fibered situations, allowing R to be a group ring – but, with some algebraic variation of the method, one can get this whole package.[82]

The above statement is equivalent to the statement that the forget-control map from “controlled L-theory (in the $-\infty$ sense) with coefficients in R” to $L^{-\infty}$ (“$R\Gamma$”) is an isomorphism.

This package can have useful applications geometrically that go beyond the Borel and Novikov conjectures themselves. We mention three examples that have bearing on issues that we've already discussed.

The first is the proof of the combined Novikov conjecture/Novikov theorem made in §5.4. If we have a controlled homotopy equivalence as in the hypothesis of that proposition, then we would get equivalence of the signature classes in

$$H_m\big(N, L^{-\infty}(\mathbb{Q}, \pi)\big).$$

However, if we know that the map $H_*(K(\pi, 1)L(\mathbb{Q})) \to L^{-\infty}(\mathbb{Q}\pi)$ is (rationally) split injective, then generalities about homology theories gives injectivity of composition,

$$H_*(N \times K(\pi, 1), L(\mathbb{Q})) \to H_m(N, L^{-\infty}(\mathbb{Q}\pi)),$$

and, therefore, controlled homotopy invariance in the domain of this map (rationally, if that's our assumption on π).[83]

[81] It's not hard, though, to give examples where these are infinitely generated.

[82] See Weinberger (1985b, 1987) and Bartels and Reich (2007).

[83] This argument is only being asserted for untwisted fundamental group situations. Frequently proofs of the Novikov conjecture are natural enough to accommodate twistings, but this is a stronger hypothesis.

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As a second application, we observe that the proper Borel conjecture for the $\mathbb{Q}$-rank-2 case follows from the Borel package of the underlying lattice. In this case the Borel–Serre boundary is aspherical, as we have already noticed, and proper rigidity of the original manifold follows from the rigidity of the compactified manifold.[84] This, in turn, by the exact sequence of a pair in surgery and group homology, reduces to the isomorphism statement for the lattice and the boundary separately. The first is the ordinary Borel conjecture, but the second is the twisted one in a situation where we have a (non-split) group extension (associated to the Borel–Serre boundary), and the relevant ring is the group ring $\mathbb{Z}[F_\infty]$, which is not associated to a finitely presented group (and so would require more effort to deal with geometrically, since the relevant fiber could not be a compact manifold).

Finally, in our discussion of the higher-signature localization formula for homologically trivial group actions on non-simply connected manifolds, we gave a particularly symmetric expression of the formula that ended up being equivalent to the Novikov conjecture. The characteristic class on the right-hand side of the formula was the average of classes introduced by Atiyah and Singer, as one goes over the generators of the cyclic group. However, the reasoning suggesting the formula suggests that one can use any generator to get a characteristic class formula: and all generators should give the same result – i.e., included in such a formula would also be a vanishing theorem for certain higher characteristic classes.

This is indeed feasible, except that the argument that one would naturally give would be phrased in the ring $L(\mathbb{Q}[\xi][\Gamma])$, where ξ is a primitive root of unity.[85] I do not see any way to deduce from a statement about the $\mathbb{Q}[\Gamma]$ the full necessary statement about $\mathbb{Q}[\xi][\Gamma]$.[86] However, the "Novikov package," which is also available in as wide a generality as the Novikov conjecture (at this point in time) would give this.

5.5.2 The C^*-algebra Setting (the Baum–Connes Conjecture, First Meeting)

In the C^*-algebra setting we also have an assembly map (interpreted as an index map)

$$K(K(\Gamma, 1)) \to K(C^*\Gamma),$$

[84] Modulo issues about vanishing of Whitehead groups that follow by the K-analogue of the L-argument we are now giving (and appropriate work on the K-theoretic Borel package for the lattice).

[85] $\mathbb{Q}[\xi]$ arises naturally as a piece of $\mathbb{Q}[\mathbb{Z}_n]$.

[86] This would not be an issue in the C^*-algebra framework – the abstract algebra $C^*(\mathbb{Z}_n \times \Gamma)$ is obviously a finite product of m copies of $C^*(\Gamma)$.

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with the Novikov conjecture being a statement about (rational) injectivity. The Baum–Connes conjecture is the isomorphism statement that goes along with this injectivity statement. Thus, the BC conjecture would assert (in its strong package form) that an assembly map involving Γ–C^*-algebras going to a cross-product algebra should always be an isomorphism for Γ torsion-free (again, leaving the discussion of groups with torsion to Chapter 6).

Recall that $C^*\Gamma$ is a completion of the group ring $\mathbb{C}\Gamma$ which we think of as an algebra of operators either on $L^2\Gamma$, in which case we get $C^*_{\mathrm{red}}\Gamma$ (the reduced C^*-algebra), or acting on arbitrary unitary representations, which then produce $C^*_{\max}\Gamma$. To have a mathematical statement, surely it is necessary to specify which completion should be used. (Note that there is a map $C^*_{\max}\Gamma \to C^*_{\mathrm{red}}\Gamma$ so injectivity for the reduced assembly map implies injectivity for the max.)

The problem is this. Given a homomorphism of groups $\Gamma \to \Delta$, there is a map between their Eilenberg–Mac Lane spaces, so there is functoriality of the left-hand side, but there is no induced map $C^*_{\mathrm{red}}\Gamma \to C^*_{\mathrm{red}}\Delta$ (except for the situation where the kernel of the map is amenable).

Using $C^*_{\max}\Gamma$, there is an induced map, so both parts of the picture do have the same functoriality. However, there is no chance that this map can be an isomorphism in general: If Γ has Property (T), then the trivial representation is a projective module over $C^*_{\max}\Gamma$, and it lies in the cokernel of the assembly map. (In fact, for a group like $SL_3(\mathbb{Z})$ – or a torsion-free congruence subgroup thereof – there are infinitely many $\mathbb{Z}$ summands in $K_0(C^*_{\max}\Gamma)$ coming from the infinitely many irreducible representations coming from finite quotients that are all isolated in the Fell topology. The domain of the assembly map is a finitely generated abelian group.)

So we have a dilemma for those who would make conjectures: To be true, one must work with $C^*_{\mathrm{red}}\Gamma$, else Property (T) immediately explodes the conjecture, yet doing so posits a highly non-obvious functoriality for the K-groups that does not appear to make any sense at the level of the algebras themselves.

The latter is what Baum and Connes (2000) boldly did in an influential paper (that appeared many years after its initial circulation!).[87] It was a major advance[88] when V. Lafforgue (2002) gave an example of a group that has Property (T) and satisfies the conjecture. Subsequently, building on these techniques, the Baum–Connes conjecture was verified for all hyperbolic groups by Lafforgue (2002) and Mineyev and Yu (2002) – with Lafforgue (2012) subsequently giving a proof in this situation with coefficients, as well (see Puschnigg, 2012).

[87] I have to admit to having been offended by this reckless behavior – as a penance for my timid skepticism.

[88] Thereby causing me to doubt my skepticism.

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However, we now know that the Baum–Connes conjecture with coefficients is false in general (we will discuss this further in Chapter 8). It remains an extremely important insight – injectivity is known for a very large class of groups, as we shall see – and understanding the extent of its full validity is a major problem, e.g. for lattices or linear groups.

5.5.3 Algebraic K-Theory

So, finally, let's turn to the long overdue issue of algebraic K-theory and how it connects to this story. This subject fills bookshelves in a library: we shall devote only a few pages to this.

Classical algebraic K-theory centered around two functors of rings (that were linked) $K_0(R)$ and $K_1(R)$. These have important applications in topology and are analogues of (say, complex) vector bundles over X and ΣX if R is the space of continuous functions on a compact Hausdorff space X (a C^*-algebra). (A bundle over ΣX can be viewed as two trivial bundles over each of the two cones, that are "clutched" or identified over X: this is a family of changes of bases – i.e. a map $X \to \mathrm{GL}_k(C)$, i.e. an element of $\mathrm{GL}_k(C(X))$.)

Even earlier, these functors, in the case of number rings, had important arithmetic interpretations, and consequently served as a bridge between topology and arithmetic.

Subsequently, the functors and their range of topological applications grew to include $K_i(R)$ both for i negative and for $i > 1$, and also deep connections to algebraic geometry and arithmetic developed. The negative groups having direct meaning using controlled (or bounded) algebra, the positive groups being related to the homeomorphism and diffeomorphism groups of manifolds.

The case $i = 0$, i.e., $K_0(R)$, is the Grothendieck group of finitely generated projective R-modules. It is thus the the group that contains the most general possible dimension for finitely generated projective modules.

It arises frequently in geometric topology as the Euler characteristic of a chain complex that has finiteness properties. Note that a chain summand of a chain complex of finitely generated free modules is such a chain complex; homological vanishing theorems can often detect that a chain complex is chain-equivalent to one of this form.

Thus $K_0(\mathbb{Z}\pi)/K_0(\mathbb{Z})$ contains an obstruction to a finitely dominated cell complex (e.g., a cell complex that's a retract of a finite complex) to being homotopy-equivalent to a finite complex, and indeed this is the whole obstruction according to the finiteness theory of Wall (1965). It also measures (according to Siebenmann, 1965) the obstruction to putting a boundary on a noncompact manifold that is "tame at ∞."

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Perhaps even simpler, consider this: if X is a finite complex with a PL G-action (for G finite), then the cellular chain complex is projective over $\mathbb{Q}G$. (All finitely generated modules over $\mathbb{Q}G$ are projective.) As Euler characteristic is the same on the chain and homology level (when the latter is projective), we can identify this invariant on homology, and then, via characters, with the invariant at the chain level. In other words, we see that

$$\mathrm{Tr}_g \chi(G, X) = \chi(X^g),$$

and the Lefshetz fixed-point theorem is thus encoded in this functor (i.e. the equivariant χ is a multiple of the regular representation – the image $K_0(\mathbb{Q})$, which is equivalent to the vanishing of the character of the representation on all nontrivial elements).

Whitehead actually defined $K_1(\mathbb{Z}\pi)$ earlier. We can think of it as the Grothendieck group of automorphisms of finitely generated projective modules. By adding on a complementary projective module with the identity automorphism (a complement to P is a finitely generated module Q so that $P \oplus Q$ is free), one can think of this as made from automorphisms of free modules, i.e. elements of $\mathrm{GL}_n(R)$ which are allowed to be stabilized.

This is the same as the Grothendieck group of finitely generated free based[89] acyclic chain complexes.

We can therefore think of K_1 as the universal target for determinants of invertible matrices (over the ring).

Thus for any finite-dimensional orthogonal representation of π, there is a Norm map that assigns to an invertible matrix over $\mathbb{Z}\pi$ the determinant of the associated matrix with real entries. Thus for $\pi = z_p$, with p an odd prime, the trivial representation gives nothing, but the remaining $(p-1)/2$ representations all give interesting invariants: however, the products of all of these determinants must be ± 1 (because it is the norm of a unit of an algebraic integer). That this is the complete dependency is the content of the Dirichlet unit theorem.

Just as $K_0(\mathbb{Z}\pi)$ measures existence of finite complexes within a homotopy type, $K_1(\mathbb{Z}\pi)$ measures the uniqueness of the finite complex. Given two finite homotopy-equivalent complexes, the mapping cone of the homotopy equivalence is almost a based acyclic $\mathbb{Z}\pi$-complex – the chain complex under discussion uses cells of the universal cover, but each cell has two orientations, and there is no canonical lift of a cell to the universal cover, so we also have an indeterminacy by multiplying by elements of π. So we get in this situation a "torsion" which is an element of

$$\mathrm{Wh}(\pi) \equiv K_1(\mathbb{Z}\pi)/(\pm\pi).$$

[89] A based chain complex is a chain complex where each chain module is given a specified basis.

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The equivalence relation this puts on finite complexes is called *simple homotopy equivalence*: it is the equivalence relation generated by viewing any finite L as equivalent to $L \cup e$, where e is a cell and the attachment is along a face in its boundary (i.e. elementary expansion). See Figure 5.3 for a schematic elementary expansion.

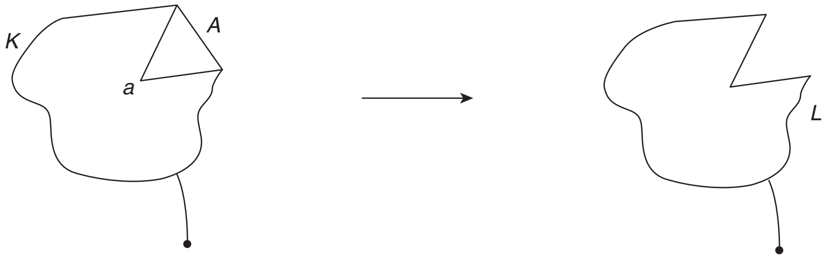

Figure 5.3 A schematic elementary expansion. Reproduced from Cohen (1973) with permission of Springer.

The s-cobordism theorem connects this to manifold theory. A manifold with boundary that deform-retracts to each of its two boundary components is an h-cobordism. It is a product – if the dimension is at least six – iff one of (and therefore both[90] of) the boundary inclusion(s) is (are) a simple homotopy equivalence.

If W is an h-cobordism, then $W - \partial\_W \cong \partial_+ W \times [0,1)$, so $\mathrm{Wh}(\pi)$ can be thought of as measuring the uniqueness of the solution to the problem of putting a boundary on an open manifold.

There is an important relationship (Bass *et al.*, 1964) between K_1 and K_0 called the fundamental theorem of K-theory. It asserts that

$$K_1(R[t,t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus \mathrm{Nil}(R) \oplus \mathrm{Nil}(R),$$

where $\mathrm{Nil}(R)$ is the Grothendieck group of nilpotent automorphisms of free modules. It frequently vanishes (e.g. when R is a regular ring) but is nontrivial, and indeed infinitely generated, when $R = \mathbb{Z}[\mathbb{Z}_p \times \mathbb{Z}_p]$.

The way to get a map from the right-hand side to the left is like this. On K_1 it's obvious. From K_0 consider assigning to P a finitely generated projective module with a complement Q, the isomorphism $P \oplus Q$ to itself, sending (p,q) to (tp,q), and to a nilpotent automorphism A of R^k, $\mathrm{I} + tA$ or $\mathrm{I} + t^{-1}A$ (hence two copies of the Nil terms).

[90] In general there is a formula, called the *Milnor duality formula* (see Milnor, 1966), relating $\tau(W, \partial_+ W)$ to $\tau(W, \partial\_W)$. It depends on the dimension, the orientation character, and the involution on $\mathrm{Wh}(\pi)$ induced by $g \to g^{-1}$.

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Since algebraic K-theory forms a spectrum, we can write this isomorphism as

$$K_1(R[\mathbb{Z}]) \cong H_1(S^1; K(R)) \oplus \mathrm{Nil}(R) \oplus \mathrm{Nil}(R).$$

Loday (1976, 1998) defined an assembly map in algebraic K-theory, and thus the fundamental theorem of algebraic K-theory[91] can then be interpreted as the statement that, for $\pi = \mathbb{Z}$, the assembly map is always split injective (a Novikov conjecture) and is an isomorphism if R is a regular ring.

There are transfer maps associated to the self-covers $S^1 \to S^1$. On the $K_1(R)$ summand, this map is multiplication by the index of the cover. On the $K_0(R)$ factor, this map is the identity (i.e. $K_0(R)$ is the transfer-invariant part of $K_1(R)[\mathbb{Z}])$), and on the Nil terms, the transfer is nilpotent, i.e. each element dies on passing to sufficiently high covers.[92]

Considering $K_0(R[\mathbb{Z}])$ and insisting that the fundamental theorem holds gives rise to a definition of $K_i(R)$.[93] We can go further, with $\mathbb{Z}$ replaced by $\mathbb{Z}^d$ to get negative K-groups. These have interpretations in terms of controlled topology. The controlled Whitehead group of $\mathbb{Z}\pi$ over $\mathbb{R}^d$ is $K_{i-d}(\mathbb{Z}\pi)$ – and it obstructs controlled h-cobordisms from being products (or controlled simple homotopy equivalent complexes from having controlled homeomorphic thickenings).

Higher algebraic K-groups, when introduced by Quillen, were also discovered to satisfy a fundamental theorem. Thus we can hope for a statement like:

Conjecture 5.24 *The assembly map*

$$H(K(\pi,1); K(R)) \to K(R[\pi])$$

is always split injective for π torsion-free (we assume that the nonconnective spectrum K is used; the extension to the general case will be given in Chapter 6*) and is an isomorphism if, in addition, R is regular.*

Of which the case of $\pi = \mathbb{Z}^d$ would be the theorem.

By the way, when we actually apply K-theory to topology as in the examples above, we use the reduced class group, i.e. we mod out by the image of $H_0\big(K(\pi,1); K(\mathbb{Z})\big)$, and the Whitehead group, where we mod out by $H_1\big(K(\pi,1); K(\mathbb{Z})\big)$, which is $H_1(\pi) \times \{\pm 1\}$. As a result, we are often interested in the cofiber of the assembly map – Whitehead theory more than the K-groups.

Further, note that the Borel conjecture actually implies (exercise, using the h-cobordism theorem) the vanishing of $\mathrm{Wh}(\pi)$ at least when $K(\pi,1)$ is a finite

91 Farrell and Hsiang (1970) gave the generalization to twisted polynomial extensions.

92 However, there are always elements that live arbitrarily long.

93 Ferry (1981) gives a very nice geometric approach to the Wall finiteness theory via the Whitehead simple homotopy theory and this perspective on K_0.

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complex, which is the above conjecture for $R = \mathbb{Z}$ and for homotopy groups in dimension ≤ 1. In particular, the above conjecture would imply that $\mathrm{Wh}(\pi) = 0$ for π torsion-free, and that, when this holds, the Borel conjecture for π really boils down to the isomorphism of the L-theory assembly map.[94]

Needless to say, it is important to understand what happens when R is not regular. In that case, let me mention a beautiful special case that shows what one can hope for.

Theorem 5.25 (Farrell and Jones, 1993a; Bartels *et al.*, 2008) *If π is a torsion-free hyperbolic group, then the above conjecture is true. Moreover, in general, there is an isomorphism*

$$H_i\big(K(\pi,1); K(R)\big) \oplus \bigoplus \mathrm{Nil}_i(R) \to K_i(R[\pi]),$$

where the sum is over conjugacy classes of nontrivial elements of π that are not proper powers.

The split injectivity result is not that difficult: it follows from the principle of descent, just like other Novikov conjecture results that we've discussed (see Ferry and Weinberger, 1991, 1995; Carlsson and Pederson, 1995), together with a trick (transferring to the infinite cover corresponding to the various Zs in π and thinking of this in a suitably controlled at ∞ way) for detecting the Nil terms.

The surjectivity statement is much deeper, and we have not yet seen any mechanism (other than codimension-1 splitting methods[95]) that can yield it.

In Chapter 8, I will explain where these summands come from, at least in the original situation of closed hyperbolic manifolds, where Farrell and Jones proved this using dynamical properties of geodesic flow. The Farrell–Jones conjectures describe in both K-theory and L-theory a more comprehensive picture of what happens that goes beyond the cases predicted by the Borel conjecture.

The higher K-groups have a close topological cousin invented by Waldhausen, called A-theory. Waldhausen's $A\big(K(\pi,1)\big)$ is a kind of group completion of $\mathrm{BGL}\big(\Omega^\infty\Sigma^\infty K(\pi,1)^+\big)$ (which surely looks close to $\mathrm{BGL}(\mathbb{Z}\pi)$), and the assembly map for them enters into an understanding of the higher homotopy of diffeomorphism and homeomorphism groups. We cannot do justice to this here, but instead refer the reader to Cohen (1987); Waldhausen (1987); Weiss

[94] One can think of $\mathrm{Wh}(\pi)$ as being an analogue in K-theory of $S(\mathrm{B}\pi)$ in surgery. (Early on. historically, because the difference between K_1 and Wh is so small, this point was obscured, and people would think of $\mathrm{Wh}(\pi)$ and $L(\pi)$ as being analogues.)

[95] Waldhausen (1978) developed such methods and, for example, proved that Whitehead groups vanish for fundamental groups of Haken manifolds.

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and Williams (2001) and Rognes and Waldhausen (2013) and just discuss a little piece of the story that directly bears on the Borel philosophy.

We discussed in §1.2 the notion of pseudo-isotopy, and observed that part of the Borel conjecture should be the statement that homotopic homeomorphisms are pseudo-isotopic.

A pseudo-isotopy is essentially a homeomorphism of $M \times [0,1]$ and we can ask whether it is isotopic to an isotopy, i.e. a level-preserving homeomorphism of $M \times [0,1]$.

This is kind of like uniqueness of the product structure in the s-cobordism theorem, and so should involve K_2. This is the case. It is a beautiful theorem of Cerf (1970) that for simply connected manifolds pseudo-isotopies are always isotopic to isotopies (in high enough dimensions), but Hatcher (1973) showed that it is *never* true in the non-simply connected case. This starts already on $\mathcal{S}^1 \times \mathcal{D}^n$ (rel ∂) and gives rise to homeomorphisms pseudo-isotopic to the identity but not isotopic on all aspherical manifolds. The space $A(\mathcal{S}^1)$ is quite complicated, and the fundamental theorem is already not unobstructed in this case: the cofiber of the assembly map is an analogue of Nil; and $\Omega^\infty \mathcal{S}^\infty$ is not a regular ring.

5.6 Notes

The notes in this chapter really divide up by problem more than by section.

Regarding the index theorem and K-theory, which plays a critical role in this chapter, good references, from various points of view, are the original papers of Atiyah and Singer (1968a,b, 1971), and the more recent Lawson and Michelsohn (1989), Roe (1998), Berline *et al.* (2004), Higson and Roe (2010), and Bleecker and Booss-Bavnbek (2013). The basic relevant functional analysis and understanding of elliptic operators can be found in many more places, such as Zimmer (1984), Evans (2010), and Taylor (2011a,b,c).

Topological K-theory, and the K-theory of C^*-algebras cannot be separated from index theory. For example, for compact Lie groups, equivariant Bott periodicity still only has the analytic proof given by Atiyah (1968), as far as I know. For the thrilling initial chapters of this story, nothing beats Atiyah's collected works.

Good sources for K-theory of C^*-algebras are Wegge-Olsen (1993), Blackadar (1998), and Higson and Roe (2000). At some point, you will surely want to look at Connes' (1994) masterpiece for a view of the world, indeed of the universe, centered at this mathematics. I hope my brief appendix is not useless in stimulating an interest in doing this sooner rather than later.

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We started the chapter by mentioning the celebrated theorem of Atiyah and Hirzebruch on circle actions on spin manifolds. This has had a celebrated extension, conjectured by Witten by arguing heuristically about the equivariant Dirac operator on ΛM (the free loop space, thought of as an S^1 space by rotating loops) and proved by Bott and Taubes (1989), with another proof by Liu (see Liu, 1995; Liu and Ma, 2000) that gives a K-theoretic refinement.

Stoltz (1996) has observed that Witten's heuristic can be developed to give a vanishing theorem for the so-called Witten genus, for manifolds of positive Ricci curvature that have $W_1 = W_2 = 0$ and $p_1/2 = 0$ ("string manifolds"). Unfortunately, this has never been proved.

It is not hard to see that all of this work has the expected non-simply connected generalization, by the method of Browder and Hsiang.

The twisted higher-signature localization theorem for $\mathbb{Z}_n$-actions discussed is equivalent to the Novikov conjecture for any fixed n. However, for other choices of characteristic class $c(\nu)$, this seems to be related to the Novikov conjecture with coefficients in a ring other than $\mathbb{Z}$ or $\mathbb{Q}$. It is for this reason that I was led to introduce this problem in Weinberger (1988a). The "simplest" formula is the Galois invariant one (i.e., invariant under change of generator of $\mathbb{Z}_n$). Rosenberg and Weinberger (1988) is an attempt to understand these equivariant geometric and topological phenomena in a coherent way.

Weinberger (1988a) also gives other formulations of the Borel conjecture with suitable coefficients in terms of being able to solve transversality problems in the setting of "homologically trivial group actions." Thus, if one has a free homologically trivial $\mathbb{Z}_p$-action on a manifold with fundamental group π satisfying the Borel conjecture, then, with a suitable dimension restriction, one can arrange for an equivariant map from $M \to K(\pi, 1)$ to have the transverse inverse image of any cycle (with manifold normal bundle) to have inverse image homologically trivial.

For smooth SU(2)-actions (or in general any smooth nonabelian connected group actions) there is a connection between the equivariant index theory and the positive scalar curvature problem: Lawson and Yau (1974) showed that any manifold with effective SU(2)-action has an invariant metric of positive scalar curvature. As a consequence we get a vanishing result of the (equivariant) Dirac operator for such manifolds. When combined with Hitchin's theorem, we discover that certain exotic spheres, for example, of dimensions 1, 2mod 8, do not have any (positive-dimensional) nonabelian group actions. For SU(2)-actions on non-simply connected manifolds, we then get a vanishing result in

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KO($K(\pi, 1)$ for all fundamental groups.[96] (Note that this integral statement is true even for groups with torsion!)

It is interesting to note that some of these exotic spheres do possess S^1-actions (by work of Schultz, 1975). Thus the Atiyah–Hirzebruch vanishing phenomenon is indeed quite subtle – the vanishing is not due to the vanishing of the KO-theoretic index of the Dirac operator.[97]

Understanding positive scalar curvature metrics has many parallels to surgery theory, but also some essential differences. As mentioned in the text, the most striking difference is the mysterious role of the spin condition: for simply connected manifolds of dimension greater than 4, every non-spin manifold has positive scalar curvature (Gromov and Lawson, 1980a), but in the spin case, according to Stoltz (1992), the necessary and sufficient condition is the triviality of the index of the Dirac operator in $\mathrm{KO}_n(*)$ (i.e. the Atiyah–Lichnerowicz–Singer and Hitchin conditions).

Besides the appearance of indices of Dirac operators that are analogues of the indices of signature operators (or symmetric signatures of Poincaré complexes), a key role is played by the surgery theorem of Gromov and Lawson (1980a) (see also Gajer, 1987; Rosenberg and Stoltz, 2001) – it gives rise to the analogue in the positive scalar curvature problem of the π–π theorem (that has suitable relative versions, as well): a spin manifold with boundary $(M, \partial M)$ of dimensions ≥ 6, that satisfies the π–π condition (namely, $\pi_1\partial M \longrightarrow \pi_1 M$ is an isomorphism) always has a positive scalar curvature metric that is a product in a neighborhood of ∂M. Having a positive scalar curvature metric is thus a spin cobordism invariant (with respect to the fundamental group of the relevant manifold). This is often referred to as the "surgery theorem" since it is proved by showing that it is possible to do surgery on spheres of codimension ≥ 3 and maintain positive scalar curvature.[98]

In this analogy, the concordance classes of positive scalar curvature metrics is closer to the surgery group than to the structure set. Thus, in the π–π setting,

[96] Note that since SU(2) is simply connected, the SU(2)-action lifts to an action on the universal cover of M. The group of all lifts of this action is then $\pi \times \mathrm{SU}(2)$ (since all automorphisms of SU(2) are inner). Thus, we can build an equivariant map $M \longrightarrow K(\pi, 1)$ where we give the latter the trivial action. This implies the vanishing of the Dirac class.

[97] In this way, playing the Novikov game for the Atiyah–Hirzebruch theorem is a more bold departure than playing it for the positive scalar curvature problem (or for birational invariance).

[98] This theorem is behind the positive results of Gromov and Lawson and of Stolz mentioned above. In the non-spin case, Gromov and Lawson use surgery to reduce the problem to special generators of oriented bordism. Stoltz uses spin cobordism, which is not fully analyzed, but shows that there are enough classes that are total spaces of $\mathbf{H}\mathbb{P}^2$ bundles (with its usual isometry group as structure group) to produce, by scaling the fibers to be very small, positive scalar curvature metrics on the kernel of $\mathrm{ind}D\colon \Omega^{\mathrm{spin}} \longrightarrow \mathrm{KO}(*)$.
In the missing case of dimension 4, where surgery methods fail, positive scalar curvature has additional obstructions that come from Seiberg–Witten theory.

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there is a unique concordance class of metrics. It is an important open problem whether there is a unique isotopy class.

The analogue of the "main result" of Chapter 3 – the problem of existence and nonexistence of complete positive scalar curvature metrics on arithmetic manifolds in the noncompact case – was settled much earlier in Block and Weinberger (1999). The low $\mathbb{Q}$-rank case (rank $\leq$ 2), where one is looking for obstructions, is settled using Novikov conjecture technology (*not* Borel conjecture technology, as is necessary for the rigidity statement, see §5.5.1). We made use of a souped-up version of an index theorem of Roe (1988c): this index theorem is a special case of the index theorem for bounded propagation speed operators on a metric space, adapted to the situation where the "corona" (i.e., the space at ∞) is disconnected. The higher-rank case follows from the surgery theorem.

For more on this theme as it refers to closed manifolds, I highly recommend Stoltz (1995) and Rosenberg and Stoltz (2001).

The situation for noncompact manifolds is much stickier. As indicated throughout our text, the parallels continue into the noncompact setting. One key difference is caused by the problem that C_r^* is not functorial, so we are forced to make use of $C^*_{\max}$ – where one is surely dealing with an algebra that is "further away" from geometry than feels reasonable. "Surely" the extra elements of $K(C^*_{\max}\pi)$ coming from, say, Property (T), should not arise, e.g. from relative indices associated to a pair of positive scalar curvature metrics on M with fundamental group π? In any case, for noncompact manifolds that are tame at ∞, one defines an index that lies in a relative group, $K(C^*_{\max}\pi, C^*_{\max}\pi')$. The assembly map

$$\mathrm{KO}(K(\pi,1), K(\pi',1)) \to K(C^*_{\max}\pi, C^*_{\max}\pi')$$

seems to have a tendency to be injective (rationally, or for torsion-free groups), although there is no legitimate 5-lemma reason to believe that this should be true.

Even in the absence of tameness, one can define an algebra that gives a *prima facie* place for index-theoretic obstructions (Chang *et al*., 2020). This includes, $\lim^1$ type obstructions to the existence of positive scalar curvature metrics, and other "phantom" phenomena in the theory.

Another problem in global analysis that has been connected to the Novikov conjecture is the "zero in the spectrum" problem.

Conjecture 5.26 (Gromov) *If M is a compact aspherical manifold, then the Δ on forms on the universal cover of M always has 0 in its spectrum. Indeed, it should be non-zero in the "middle dimension."*

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Middle dimension means dimension k if $\dim M = 2k + 1$. At the moment the only evidence for this is of the following form:

(1) It is observed to be true for $K \setminus G$, so it is true for the classical aspherical manifolds.
(2) It follows from the Novikov conjecture, for otherwise the index of the signature operator in $K(C * \pi)$ would vanish, but the "1" in dimension 0 of the L-class should give rise to a nonzero image in $K(C^*\pi) \otimes \mathbb{Q}$ if the Novikov conjecture were true.
(3) The chain complex of M (thought of as $R\pi$-modules) is not chain-equivalent to one with a zero morphism in the middle (this would contradict the cohomological dimension of $\pi_1 M > K$). We are alright, therefore, for arbitrary group rings. The issue is entirely one caused by completion.

It is also likely true for all uniformly contractible Riemannian manifolds (with bounded geometry[99]). In any case, neither of these problems is known to imply anything about the original Novikov conjecture, but both of them can be studied jointly with the Novikov conjecture.

As mentioned in the text, the analytic version of the Novikov conjecture can be proved, just like we did in Chapter 4 in L-theory, by a principle of descent. The analogue of the bounded category is the Roe algebra.

Other related ideas are the Dirac–dual Dirac argument (see Kasparov, 1988, for the exemplar of this), and the use of almost-flat bundles (as in the text), which are not completely unrelated. There are three ways of getting around the basic fact that the (rational) Chern classes of a finite-dimensional bundle on a compact polyhedron are trivial. The first is the use of families, as in the Lusztig method. The second is to use families of almost-flat bundles with increasing dimensional fiber, following Gromov and Lawson (1980a,b) and Connes *et al.* (1993), as we explained in the text. This naturally could lead one towards using infinite-dimensional fibers – which is essentially the problem of understanding $K(C * \pi)$!

The third method for producing almost-flat bundles on finite-dimensional spaces, which allows one to keep the same ground manifold and not increase fiber dimension, is to use compact support. (This is like the use of the Bott element on $\mathbb{R}^n$ with compact support in Gromov and Lawson, 1983.)

This is quite similar in spirit, if not completely in detail, to the use of

[99] A good test of the depth of this question is whether one can construct a complete uniformly contractible manifold with $0 \notin \operatorname{spec}(\Delta)$ even with bounded geometry. Currently one doesn't know any example, even, of a complete contractible manifold without 0 in its spectrum – or of such a manifold that is homotopy-equivalent to a finite complex (although surely these must exist!).

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the K-theory of Higson corona to obtain useful indices in Roe (1993). The Higson compactification of a locally compact metric space is an analogue of the Stone–Čech compactification, but one does not require that *all* bounded continuous functions extend – rather only those whose variation decays at infinity ($\operatorname{diam} f\big(B(R,p)\big) \to 0$ as $p \to 0$ for any fixed radius R).

Any reasonable compactification, i.e. one where restrictions to the interior have decaying variation (such as the ideal boundary of G/K or the Gromov–Tits compactification of a word hyperbolic group), admits a map from the Higson compactification, so objects on any of these can effectively be pulled back to the Higson compactification. In any case, bundles on the Higson corona (i.e. the ideal points of the Higson compactification) can be paired with bounded propagation speed operators to give useful obstruction indices. It is as if there were a Lipschitz map to the cone on the Higson corona, and a rescaling construction would produce tiny curvature (although this is not literally the case).

Needless to say, all of these techniques can be viewed as the simply connected versions of a more general phenomenon. Thus one can study, on a non-simply connected manifold, the bounded propagation speed operators taking values in $C^*\pi$ and get more subtle and useful information, just as we can do in the situation of bounded L-theory. The small-scale version of this is precisely what we discussed in controlled K- and L-theories in giving, for example, a Novikov theorem for situations where we have controlled homotopy equivalences.

We used this added flexibility in proving that there are no complete positive scalar curvature metrics on $\mathbb{Q}$-rank-2 lattices. Stanley Chang (2001) proved by this method (i.e. marrying the Roe algebra to a fundamental group) that for *no* $K \setminus G/\Gamma$ is there a coarse quasi-isometric metric of positive scalar curvature and thus the metrics of Block and Weinberger when $\mathbb{Q}$-rank > 2 must be quite distorted.

In §5.3, the method of Thom, Milnor, Rochlin, and Schwartz gives rational Pontrjagin classes for PL homology manifolds. Sullivan (2005) gave a refinement which gives (anachronistically describes) a class

$$\sigma^*\big(X\big) \in H_x(X; L^*(Q)\big).$$

(Note that since 2 is inverted in the coefficient ring Q, we have no issues regarding the difference between quadratic and symmetric L-theory.) This class, when we invert 2, then lies in $\mathrm{KO}_x(X)\otimes\mathbb{Z}[1/2]$ (see Taylor and Williams, 1979a, for an explanation of the work of Sullivan on the relation between L-spectra and K-theory away from 2, and the structure of L-spectra at 2 to see what we're

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throwing away by making this discussion somewhat crude). This is essentially the class of the signature operator on X.[100]

This class assembles to $\sigma^*(X) \in L^*(\mathbb{Q}\pi)$ just like in the case of manifolds. This can be viewed as forgetting control, or can be viewed along assembly lines (for the PL case, see e.g. Siegel, 1983, and Weinberger, 1987, for how such arguments go). It is important to note that there is an issue for the $\mathbb{Q}$ situation that we don't have for $\mathbb{Z}$ – namely, that homotopy equivalences can have degrees other than ± 1. Thus, the assembled characteristic class cannot be expected to be an oriented homotopy invariant in this setting – integrally. However, since the quadratic form $(d) \oplus (-1)$ is torsion (of exponent at worst 4) in $L^\circ(\mathbb{Q})$ (which equals Witt($\mathbb{Q}$)), this only affects the prime 2 – so, assuming that the assembly map with coefficients is injective, we get $\mathbb{Q}$-homotopy equivalence of higher signatures, with respect to orientation preserving maps of arbitrary (positive) degree – if the usual assembly map is an injection (away from the prime 2). Note that we are making use of Ranicki's localization result (Ranicki, 1979a) that tells us that the integral and $\mathbb{Q}$ assembly issues are equivalent (for all π) away from the prime 2.

The integral statement, allowing for 2, must take into account the degree of the map. Also, as far as I can tell, the $L^*(\mathbb{Q})$ injectivity statement is not equivalent to the $L^*(\mathbb{Z})$ – although both are part of the "Novikov package."

If one moves from the PL setting, then assuming that X is an ANR, controlled methods – e.g. following Yamasaki (1987) and Cappell *et al.* (1991) – allow the same statements to be made for topological $\mathbb{Q}$-homology manifolds.

Turning to §5.5, first of all let me call attention to Rosenberg (1996), a book that is a very useful introduction to many of the ideas of K-theory of all flavors, with many hands-on examples. There are still a few things that need to be discussed in view of our (belated) discussion of torsions and algebraic K-theory.

The first is that we have ignored all along "decorations" in surgery theory, and we now have the ingredients to set this straight. If X is a finite complex which satisfies Poincaré duality, then there are two natural questions to ask: (1) Is X homotopy-equivalent to a closed manifold? (2) Is X simple homotopy-equivalent to a closed manifold? The second takes advantage of the finite complex structure that X has – and is *not* a homotopy-invariant question.

However, (2) is not a reasonable question without some additional condition.

[100] This is literally the case for X smooth; for X a PL or Lipschitz manifold, this makes sense by the work of Teleman (1980). Recently Albin *et al.* (2018) have taken off on the seminal work of Cheeger on the L^2-cohomology of stratified spaces and the duality induced by $*$ (a variant of intersection homology) and have used microlocal analysis to give a suitable signature class on Witt and "Cheeger spaces."

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If M is a manifold, then the Poincaré duality isomorphism $C_*(M) \to C^{n-*}$ is actually a *simple* chain equivalence. If we change basis on $C_*(X)$ via A and dually to $C^{n-*}(X)$, then the torsion of the equivalence is change by $[A] + (-1)^n[A^*]$; here $*$ is induced by $g \to w(g)g^{-1}$ (where w is, as usual, the orientation character). Note, by the way, that the self-duality of the cap product tells us that the isomorphism $C_*(M) \to C^{n-*}$ is "self-dual."

Thus, there is a "simplicity obstruction" lying in

$$\{\tau \in \mathrm{Wh}(\pi) | \tau = (-1)^n \tau^*\}/\{\sigma + (-1)^n \sigma^*\};$$

this is the Tate cohomology of the involution $*$ on $\mathrm{Wh}(\pi)$, i.e. $H^n(\mathbb{Z}_2; \mathrm{Wh}(\pi))$.

This is the obstruction in the category of finite complexes; that is, can we take a given X with a chain-level Poincaré duality map that is not a simple equivalence homotopy-equivalent to one where the duality map is a simple isomorphism? If this obstruction is non-zero, then there's no chance of X being homotopy-equivalent to a manifold!

In the relative situation this is very simple to appreciate. For the mapping cylinder of a homotopy equivalence between closed manifolds, the τ of the duality map for the relative Poincaré chain complex is essentially the torsion of the homotopy equivalence.

However, more fundamentally, this discussion suggests that, for question (2) above, we only ask it for X a *simple Poincaré complex*, i.e. one for which the duality map is a simple equivalence.

Surgery theory, as we have discussed it, makes sense in both settings and gives slightly different obstruction groups. If X is a Poincaré complex, or (Y, X) is a Poincaré pair, we can ask if X or (Y, X) is homotopy-equivalent to a manifold (pair) – and the obstruction is finding a degree-1 normal map with vanishing surgery obstruction that lies in $L_n^h\big(\pi_1(X)\big)$ or $L_n^h\big(\pi_1(Y), \pi_1(X)\big)$.

If X is a simple Poincaré complex, or (Y, X) is a simple Poincaré pair (which implies that X is a simple Poincaré complex), then we can ask if X is simple homotopy equivalent to a manifold or (Y, X) to a manifold pair $(M, \partial M)$ – and the obstruction lies in $L_n^s\big(\pi_1(X)\big)$ or $L_n^s\big(\pi_1(Y), \pi_1(X)\big)$.

These groups have a π–π theorem, and fit into the obvious exact sequences, and further satisfy a Rothenberg sequence (Shaneson, 1969):

$$\begin{aligned} \cdots \to H^{n+1}\Big(\mathbb{Z}_2; \mathrm{Wh}\big(\pi(X)\big)\Big) &\to L_n^s\big(\pi_1(X)\big) \\ &\to L_n^h\big(\pi_1(X)\big) \to H^n\Big(\mathbb{Z}_2; \mathrm{Wh}\big(\pi(X)\big)\Big) \to \cdots . \end{aligned}$$

Thus the L-groups only differ at the prime 2, and only if the Whitehead group is nontrivial. Thus, conjecturally for torsion-free groups, for example, these

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groups are isomorphic. However, in general they are different – even for cyclic groups.

Even for manifolds, the choice of decoration makes important sense: $S^h(M)$ measures how unique the manifold homotopy equivalent to M is in the "h-sense," i.e. up to h-cobordism. The version $S^s(M)$ measures the manifolds simple homotopy equivalent to M up to s-cobordism, i.e. up to homeomorphism.

Note that, as a formal consequence of the Rothenberg sequence and a diagram chase, we get an exact sequence

$$\cdots \to H^{n+1}\Big(\mathbb{Z}_2; \mathrm{Wh}\big(\pi_1(M)\big)\Big) \to S^s(M) \to S^h(M)$$
$$\to H^n\Big(\mathbb{Z}_2; \mathrm{Wh}\big(\pi_1(M)\big)\Big) \to \cdots .$$

However, it is not so hard to understand it directly. The map $S^h(M) \to H^n(\mathbb{Z}_2; \mathrm{Wh}\big(\pi_1(M)\big))$ measures whether a homotopy equivalence can be h-coborded to a simple homotopy equivalence. The map

$$H^{n+1}(\mathbb{Z}_2; \mathrm{Wh}\Big(\pi_1(M)\big)\Big) \to S^s(M)$$

also comes out of the s-cobordism theorem. If I take an h-cobordism from M, then the torsion of the homotopy equivalence $M' \to M$ is $\sigma - (-1)^n\sigma^*$. If this vanishes, then I get a new simple homotopy equivalence.

If I take an h-cobordism from M to M' and "turn it upside down" to get one from M' to M, the torsion is changed by $\tau \to (-1)^{n+1}\tau^*$. I can glue these together to get a nontrivial s-cobordism for M to itself. These torsions – the obviously self-dual ones – never change the structure!

Finally, with these concepts, we can properly describe what happens for the product formula:

$$L_n^S(\pi \times \mathbb{Z}) \cong L_n^s(\pi) \times L_{n-1}^h(\pi).$$

For $L_n^h(\pi \times \mathbb{Z})$ we would need to introduce a new group $L_{n-1}^{\mathrm{p}}(\pi)$ to obtain a formula: the p indicating the use of projective modules in the definitions rather than free modules in the quadratic forms used to define the L-groups. And so on. We will be forced to descend into negative K-theory to give a comprehensive approach.

For example, for fibrations over the circle with nontrivial monodromy α, we would be led to "intermediate L-groups" between L^s and L^h. Instead of allowing arbitrary torsions of homotopy equivalences in L^h, we should use the theory associated to allowing torsions that are elements of $\mathrm{Ker}(1-\alpha_*) \subset \mathrm{Wh}(\pi)$.

We have simplified, and will continue to simplify, our discussion by working with the $L^{-\infty}$-theory – which has the interpretation as having to do with being

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able to obtain a homotopy equivalence after crossing with some (unspecified-dimensional) torus. This will, by a sequence of Rothenberg sequences, only affect the prime 2.

Our discussion of Waldhausen's work in §5.5 was highly inadequate. We will leave the question of whether a pseudo-isotopy is pseudo-isotopic to an isotopy, just saying that it is analogous to the question of whether an h-cobordism is h-cobordant to a product. The analysis requires consideration of an involution on pseudo-isotopy theory and its action on the homotopy types of the topological groups, $\mathrm{Homeo}(M)$ and $\mathrm{Diff}(M)$. Waldhausen's work gives a kind of description of these in a stable range that grows linearly with $\dim(M)$. (Unstably, we know very little about these groups: one could hope – although I believe that this is dubious – that the components of $\mathrm{Homeo}(M)$ are $\mathbb{Q}$-acyclic for closed aspherical manifolds with centerless fundamental group.[101])

More importantly, but this is a direction that has not yet been well integrated into the Novikov/Borel philosophy, $A(X)$ is a deformation or extension of $K(R)$ and allows the modification of problems involving $\mathbb{Z}$ to ones involving $\Omega^\infty S^\infty$, which has a lot of internal structure. In this analogy, one obtains that the analogue of the result mentioned about $K(R\pi)$ for π the fundamental group of a hyperbolic manifold is that for such a manifold:

$$\mathrm{Wh}(M) \cong \prod \mathrm{Wh}(\mathcal{S}^1),$$

where here Wh is the cofiber of the A-theory assembly map, and where the product is taken over primitive closed geodesics.

Similarly, the basic trace

$$K_0(R) \to R/[R, R]$$

that assigns to a projection the trace (i.e. the sum of its diagonal elements thought of as lying in R – as an additive group, modulo the additive subgroup generated by elements of the form $[r, s] = \{rs - sr\}$), of any matrix representing it, has a two-stage generalization. The first is Connes's trace map[102]

$$K_n(R) \to \mathrm{HC}_n(R)$$

from K-theory to cyclic homology. The second leaves the world of rings, and

[101] And, if there's center, rationally equivalent to a torus. Later we will see that there is not, in general, a homomorphism $\mathbb{T} \to \mathrm{Homeo}(M)$ inducing such a putative rational homotopy equivalence.

[102] Connes's interest in the trace was to deal with K-theory of C^*-algebras and then prove the operator-theoretic Novikov conjecture. Needless to say, executing this involves analytic difficulties in addition to the algebraic ones – however, in several important examples, this has been achieved – and the issues involved are in any case central for proving isomorphism conjectures (see Chapter 8.)

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moves into stable homotopy theory (i.e. of spectra) and is an analogue of this, called the cyclotomic trace, developed by Bökstedt and Madsen, and used in Bökstedt *et al.* (1993) to detect $K(\mathbb{Z})$ rationally[103] and therefore a proof for all π with finitely generated homology of the algebraic K-theory Novikov conjecture for the ring $\mathbb{Z}$. Hesselholt and Madsen (2003) have applied this to give a great deal of information on the algebraic K-theory of, for example, the ring of integers in a local number field.

Alas, these methods do not prove an integral version, and are highly sensitive to the ring $\mathbb{Z}$ – it is not at all routine to replace $\mathbb{Z}$ by another ring of integers. Currently, such a modification would require deep number-theoretic conjectures[104] so that, for example, certain p-adic L-functions would be guaranteed to have nonvanishing properties.

We refer the reader to the book by Dundas *et al.* (2013) that explains trace technology, and the Goodwillie calculus that shows that the trace is not just accidentally successful in these problems: the trace gives a calculation of relative K-theory $K(R, S)$ if the map $R \to S$ is "1-connected," so the trace is an effective linearization of K-theory.

I believe that the ideas of Waldhausen's K-theory, concordance theory, and traces are related to the embedding theory calculations in Chapter 6 that give rise to a class of counterexamples to the equivariant Borel conjecture, and that there should be some unification of all these – but, at the moment, this is too vague.

[103] Recall that this space was rationally analyzed by Borel by relating the cohomology of lattices to the Lie algebra cohomology of the Lie group containing them.

[104] See Lück *et al.* (2017) on this. We will discuss this paper somewhat in Chapter 6 – to avoid misconception, I note that some of its implications are indeed unconditional.

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