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ON ONE-SIDED PRIMITIVITY OF BANACH ALGEBRAS

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Abstract Let S be the semigroup with identity, generated by x and y, subject to y being invertible and $yx = xy^2$. We study two Banach algebra completions of the semigroup algebra $\mathbb{C}S$. Both completions are shown to be left-primitive and have separating families of irreducible infinite-dimensional right modules. As an appendix, we offer an alternative proof that $\mathbb{C}S$ is left-primitive but not right-primitive. We show further that, in contrast to the completions, every irreducible right module for $\mathbb{C}S$ is finite dimensional and hence that $\mathbb{C}S$ has a separating family of such modules.

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1. Introduction

In 1964 Bergman [**1**] gave the first example of a ring which is right-primitive (has a faithful irreducible right module) but not left-primitive. Further examples were given in 1969 by Jategaonkar [**9**]. In 1979 Irving [**8**] gave a whole family of examples of rings, even linear algebras, which are left-primitive but not right-primitive. The question of whether or not there exists a Banach algebra with this property was raised by Bonsall and Duncan [**2**], and remains open. With a view to examining the Banach algebra situation we look in detail at one of the examples of Irving, namely the semigroup algebra $\mathbb{C}S$, where S is the semigroup with identity, generated by x and y, subject to y being invertible and $yx = xy^2$. Thus,

$$S = \{x^m y^n : m \in \mathbb{Z}_+,\ n \in \mathbb{Z},\ yx = xy^2\},$$

where $\mathbb{Z}_+ = \{k \in \mathbb{Z} : k \geqslant 0\}$. In Appendix A we present a proof of Irving's result for $\mathbb{C}S$ that may be more amenable to Banach algebraists. In fact, we prove further that every irreducible right module of $\mathbb{C}S$ is finite dimensional. Since $\mathbb{C}S$ is left-primitive and hence semi-simple, it follows that $\mathbb{C}S$ has a separating family of irreducible right representations on finite-dimensional spaces. In the language of representation theory this says that S is residually finite. It follows that the algebra $\mathbb{C}S$ has the property of direct finiteness: that is, for $a, b \in \mathbb{C}S$ we have $ba = 1$ if and only if $ab = 1$.

Our principal aim in § 2 is to study two Banach algebra completions of $\mathbb{C}S$. The obvious completion, A, say, is given by $\ell^1(S)$, where we replace finite sums by absolutely convergent sums with the usual ℓ^1-norm. It will be convenient to view A as a graded algebra; thus,

$$A = \left\{ a = \sum_{m=0}^{\infty} x^m \phi_m(y) : \phi_m \in \ell^1(\mathbb{Z}),\ \sum_{m=0}^{\infty} \|\phi_m\|_1 < \infty \right\},$$

the norm being given by $\|a\| = \sum_{m=0}^{\infty} \|\phi_m\|_1$. The convolution Banach algebra $\ell^1(\mathbb{Z})$ is isometrically isomorphic to the Wiener algebra, W, of all continuous complex functions on $\mathbb{T}$, the unit circle, with absolutely convergent Fourier series (the norm being the absolute sum of the Fourier coefficients). On occasions it will be helpful to regard the ϕ_m as functions on $\mathbb{T}$. We shall also consider the completion B given by

$$B = \left\{ b = \sum_{m=0}^{\infty} x^m \phi_m : \phi_m \in C(\mathbb{T}),\ \sum_{m=0}^{\infty} \|\phi_m\|_\infty < \infty \right\},$$

the norm being given by $\|b\| = \sum_{m=0}^{\infty} \|\phi_m\|_\infty$. The product in A and in B is determined by the formula

$$x^m \phi(y) x^n \psi(y) = x^{m+n} \phi(y^{2^n}) \psi(y).$$

We shall show that both A and B are left-primitive and also residually finite and hence also have the property of direct finiteness. We show that, in contrast to the situation for $\mathbb{C}S$, the Banach algebras A and B have many non-faithful irreducible infinite-dimensional right modules: enough, in fact, to separate the points of the algebra. We are still unable to prove that either A or B fails to be right-primitive. A key step in the proof that $\mathbb{C}S$ is not right-primitive involves a technique that has no analogue for *infinite* series. It is amusing to note that any answer to the right-primitivity of A or B will be interesting. If either A or B is *not* right-primitive, then we have a desired example. If they *are* right-primitive, then we have a dramatic difference between the purely algebraic $\mathbb{C}S$ and two of its natural Banach algebra completions.

We remark here that it is much easier to establish primitivity than to establish non-primitivity. To prove that an algebra $\mathcal{A}$ is right-primitive, it is sufficient to construct one faithful irreducible right module. To prove that $\mathcal{A}$ is not right-primitive we have to show that *every* irreducible right module fails to be faithful. Equivalently, we have to identify every maximal modular right ideal K of $\mathcal{A}$ and show that the quotient ideal $K : \mathcal{A}$ is always non-zero. Bergman was able to do this for his example. For Banach algebras, identifying all maximal modular right ideals is usually a hopeless task. We are thus forced (along with Irving) to assume the existence of some faithful irreducible right module and look for some contradiction. We note also that a candidate for a left-primitive Banach algebra which is not right-primitive appeared in [**2**]. The Banach algebras A and B considered below are much more tractable.

2. Some representation theory for the Banach algebras A and B

As in § 1, A is the Banach algebra $\ell^1(S)$ which we may regard as all elements of the form $a = \sum_{m=0}^{\infty} x^m \phi_m$ with $\phi_m \in \ell^1(\mathbb{Z})$, where we identify $\ell^1(\mathbb{Z})$ with the Wiener algebra W. We have $\|a\| = \sum_{m=0}^{\infty} \|\phi_m\|_1$, the latter norm being the ℓ^1 norm. Also, B is the Banach algebra of all elements of the form $b = \sum_{m=0}^{\infty} x^m \phi_m$ with $\phi_m \in C(\mathbb{T})$ and with the norm $\|b\| = \sum_{m=0}^{\infty} \|\phi_m\|_\infty$.

Our first task is to show that A and B are left-primitive. It is not difficult to give continuous extensions of Irving's faithful irreducible left representation of $\mathbb{C}S$ to both A and B. In passing we are able to simplify dramatically the provision in [**2**] of an example of specific dual representation behaviour. We then consider an averaging construction on irreducible right representations and thereby obtain equivalent irreducible right representations on classical Banach spaces. This enables us to describe all the irreducible matrix right representations of A and B. Also, we introduce a family of non-faithful irreducible infinite-dimensional right modules parametrized by a rich collection of subsets of $\mathbb{T}$, namely infinite compact sets which are square-closed, and minimal with respect to set inclusion. The argument for the case of A requires us to generalize the classical Wiener Lemma on the invertibility of continuous functions on the circle with absolutely convergent Fourier series.

We recall a fundamental construction for dual representations of Banach algebras (see [**2**] or [**3**]). Let $\mathcal{A}$ be any Banach algebra; for convenience, we suppose $\mathcal{A}$ has a unit, 1. Given $f \in \mathcal{A}'$ we get left and right ideals given by

$$L_f = \{a : f(\mathcal{A}a) = (0)\} \quad \text{and} \quad K_f = \{a : f(a\mathcal{A}) = (0)\}.$$

We have associated left and right regular representations on the quotient spaces $X_f = \mathcal{A}/L_f$ and $Y_f = \mathcal{A}/K_f$, respectively. The quotient spaces X_f and Y_f are in normed duality with $\langle a', b' \rangle_f = f(ba)$ for any choice of a and b in the cosets, and the representations are linked by

$$\langle a\xi, \eta \rangle_f = \langle \xi, \eta a \rangle_f.$$

The left and right representations have the same kernel $\{a : f(\mathcal{A}a\mathcal{A}) = (0)\}$. The representations are irreducible if and only if L_f and K_f, respectively, are maximal one-sided ideals. Every irreducible left representation of $\mathcal{A}$ is equivalent to the representation on some X_f (but then we know nothing in general about the irreducibility of the right representation on Y_f), and similarly for right representations.

We write δ_n for the usual point mass function on $\mathbb{Z}$ (with value 1 at the point n of $\mathbb{Z}$). Irving's faithful irreducible left representation of $\mathbb{C}S$ on $\mathbb{C}\mathbb{Z}$ is then determined by

$$x\delta_{2n-1} = 0, \qquad x\delta_{2n} = \delta_n, \qquad y\delta_n = \delta_{n-1}.$$

It is routine to verify that the above formulae determine a bounded left representation of A on $\ell^1(\mathbb{Z})$. We write V_A for $\ell^1(\mathbb{Z})$ with this left module action. For the case of B we rewrite δ_n as the function ζ^n on the unit circle. We take the corresponding left module for B to be $C(\mathbb{T})$. The left action by y is given by

$$yf(\zeta) = \zeta^{-1} f(\zeta), \quad \zeta \in \mathbb{T}.$$

More generally the action by $\phi(y)$ is given by

$$\phi(y)f(\zeta) = \phi(\zeta^{-1})f(\zeta), \quad \zeta \in \mathbb{T}.$$

The left action by x is well defined by the formula

$$xf(\zeta) = \tfrac{1}{2}[f(\zeta^{1/2}) + f(-\zeta^{1/2})], \quad \zeta \in \mathbb{T}.$$

We denote this left module by V_B.

Here we introduce the natural dual representations (see [**3**, p. 141]) associated with V_A and V_B. There is a dual pairing on $V_A \times V_A$ given by

$$\Big\langle \sum \alpha_n \delta_n, \sum \beta_n \delta_n \Big\rangle = \sum \alpha_n \beta_n$$

and there is a dual pairing on $V_B \times V_B$ given by

$$\langle f, g\rangle = \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta}) g(\mathrm{e}^{-\mathrm{i}\theta})\,\mathrm{d}\theta.$$

In each case we get a dual representation in which the right actions are (both) given by

$$f(\zeta)x = f(\zeta^2), \qquad f(\zeta)\phi(y) = \phi(\zeta)f(\zeta). \tag{2.1}$$

In the case of A, (2.1) holds for all $f \in W$, so we may identify V_A with W. For each module we have

$$\langle av, w\rangle = \langle v, wa\rangle$$

and so the left representation is faithful if and only if the right representation is faithful. This allows us to choose whichever leads to a simpler argument.

Theorem 2.1. *A and B are left-primitive Banach algebras.*

Proof. For the left module V_A we again write $\{\delta_j\}$ for the usual normalized basis. Since $y^k\delta_0 = \delta_{-k}$, $k \in \mathbb{Z}$, it follows that δ_0 is strictly cyclic. We show next that any non-zero $\xi = \sum \xi_j \delta_j$ can be mapped arbitrarily close to δ_0 by the left module action. It then follows that A is topologically irreducible on V_A. Since it has one strictly cyclic vector it is then strictly irreducible on V_A (see, for example, [**5**, Lemma 1]). We have $\xi_k \neq 0$ for some $k \in \mathbb{Z}$. Note that $y^k\xi = \sum \xi_n \delta_{n-k}$ and that the coefficient of δ_0 is ξ_k, which is non-zero. Since $\xi \in \ell^1(\mathbb{Z})$, it is straightforward to show that $\xi_k^{-1}x^N y^k \xi \to \delta_0$ as $N \to \infty$.

To prove faithfulness we shall use the right representation. Thus, we have

$$\delta_n x = \delta_{2n}, \qquad \delta_n y = \delta_{n+1}.$$

Suppose that $V_A a = 0$ with $a = \sum_{m=0}^{\infty} x^m \phi_m(y)$. This is equivalent to saying that $\delta_j a = 0$ for all $j \in \mathbb{Z}$: that is, $\sum_{m=0}^{\infty} \delta_{2^m j}\phi_m(y) = 0$. Rewrite this in the notation of the Wiener algebra W and we have $\sum_{m=0}^{\infty}\zeta^{2^m j}\phi_m(\zeta)=0$ for all $\zeta\in\mathbb{T}$ and all $j\in\mathbb{Z}$. Alternatively, we may write this as

$$\sum_{m=0}^{N}\zeta^{2^m j}\phi_m(\zeta)+M(\zeta)=0,$$

where $\|M\|_1$ is arbitrarily small. Multiply through by ζ^{-j-k}, integrate around the circle and let $|j|\to\infty$ to get the kth Fourier coefficient of ϕ_0 arbitrarily small and hence zero for all k. This forces $\phi_0=0$. Now repeat the argument with a slight modification to get $\phi_1=0$. Similarly, we obtain that all $\phi_m=0$ and so $a=0$, as required.

The above arguments require only minor modifications for the module V_B. □

Remark 2.2. Let T be the semigroup generated by x and y subject only to the relation $yx=xy^2$; thus, we may regard T as a subsemigroup of S. Irving [**7**] proved that $\mathbb{C}T$ is both left-primitive and right-primitive. He also proved that all non-faithful (left and right) irreducible representations of $\mathbb{C}T$ are finite dimensional. We get Banach algebra completions of $\mathbb{C}T$ by replacing A by $\ell^1(T)$ and modifying B by replacing $C(\mathbb{T})$ by the disc algebra $\mathcal{A}(\mathbb{D})$. It may be verified that each completion is both left-primitive and right-primitive; the modules are then $\ell^1(\mathbb{Z}_+)$ and $\mathcal{A}(\mathbb{D})$.

In [**2**] the authors presented a rather complicated example of a dual representation of a Banach algebra in which the left representation is irreducible, while the right representation is invariant on a chain of closed subspaces with zero intersection. We can now present a simpler example of this behaviour.

Theorem 2.3. *For the dual representation of both A and B as given above, the left representation is irreducible and the right representation is invariant on a chain of closed subspaces with zero intersection.*

Proof. The argument is essentially the same in each case. We present it for A. For $p\in\mathbb{N}$, define

$$V_p=\{f\in V_A : f(\zeta)=0 \text{ whenever } \zeta^{2^p}=1\}.$$

Clearly, each V_p is a closed subspace and is non-zero since it contains polynomials. Also $V_{p+1}\subseteq V_p$ and $\bigcap_{p\in\mathbb{N}}V_p=(0)$ since any f in the intersection vanishes on a dense subset of $\mathbb{T}$. Clearly, V_p is invariant under right action by y and it is also invariant under x since $\zeta^{2^{p+1}}=1$ whenever $\zeta^{2^p}=1$. Then V_p is invariant under right action by A, as required. □

We turn now to an averaging construction for irreducible right representations. Recall that any irreducible module for a Banach algebra may be assumed to be a Banach space, with continuous action. Let the Banach space V be an irreducible right module for A or B. We may suppose without loss that y and y^{-1} have operator norm 1 on V (apply [**3**, Theorem 4.1]), and hence the operator y has spectrum contained in $\mathbb{T}$. Suppose that $vy=\alpha v$ for some $\alpha\in\mathbb{T}$ and some non-zero $v\in V$. Choose $\rho\in V'$ with $\rho(v)=1$, and define $f\in A'$ (or $f\in B'$) by

$$f(a)=\operatorname*{LIM}_{n\to\infty}\alpha^{-n}\rho(vay^n),$$

where LIM is a Banach limit. Note that $f(1) = 1$ and, for all a in A or B,

$$f(ay) = \operatorname*{LIM}_{n\to\infty} \alpha^{-n}\rho(vayy^n) = \operatorname*{LIM}_{n\to\infty} \alpha\alpha^{-n-1}\rho(vay^{n+1}) = \alpha f(a).$$

We easily verify that

$$K_f = \{a : f(ax^m y^n) = 0\,(m \in \mathbb{Z}_+,\, n \in \mathbb{Z})\} = \{a : f(ax^m) = 0\,(m \in \mathbb{Z}_+)\}.$$

Let $K = \{a : va = 0\}$ so that K is a maximal right ideal giving a right regular representation equivalent to the given representation. Clearly, $f(K) = (0)$ and so $K \subseteq K_f$. But K_f is proper since $f(1) = 1$, and hence $K = K_f$. For each a in A (or B), define the bounded sequence on $\mathbb{Z}_+$ by $g_a(m) = f(ax^m)$, and let G be the space of all such g_a. It is straightforward to check that our given right representation is equivalent to the right representation on G defined by $g_a b = g_{ab}$ (in fact, we simply map V to G by $va \to g_a$). Moreover, we have explicit formulae for the actions of x and y on G:

$$\left.\begin{aligned} (g_a x)(m) &= g_{ax}(m) = f(axx^m) = g_a(m+1),\\ (g_a y)(m) &= g_{ay}(m) = f(ayx^m) = f(ax^m y^{2^m}) = \alpha^{2^m} g_a(m). \end{aligned}\right\} \tag{2.2}$$

Notice what happens for the case $\alpha = 1$. Then y acts as the identity operator on G and so the irreducible image Banach algebra is commutative; but its centre is $\mathbb{C}$ and hence V is one dimensional. In general, the one-dimensional representations come from the multiplicative linear functionals χ, and we pause here to list them. They are determined by the complex numbers $\chi(x)$ and $\chi(y)$ in the closed unit disc subject to

$$\chi(y)\chi(x) = \chi(yx) = \chi(xy^2) = \chi(x)\chi(y)^2.$$

Since y is invertible, we have either $\chi(y) = 1$ and $0 < |\chi(x)| \leqslant 1$, or $\chi(x) = 0$ and $|\chi(y)| = 1$.

We return to the general case of $\alpha \in \mathbb{T}$ and show that this leads to the finite dimensionality of the module V. We have already dealt with the one-dimensional case.

Lemma 2.4. *Let V be an irreducible right module for A or B with $\dim V \geqslant 2$, and let α be an eigenvalue for y on V (and so $\alpha \in \mathbb{T}$). Then $\alpha^{2^k} = \alpha$ for some $k \in \mathbb{N}$ and $\dim V < \infty$. Moreover, x^k acts as a non-zero multiple of the identity on V.*

Proof. We use the equivalent representation on G which we constructed above (see (2.2)). We have a non-zero g with $gy = \alpha g$ and hence

$$\alpha g(m) = \alpha^{2^m} g(m), \quad m \in \mathbb{Z}_+.$$

If, for all $m \geqslant 1$, $\alpha^{2^m} \neq \alpha$, then we have $g(m) = 0$ for all $m \geqslant 1$. Then $G = \mathbb{C}g$ and the representation is one dimensional. So we must have $\alpha^{2^k} = \alpha$ for some $k \in \mathbb{N}$. It is routine to verify that $yx^k = x^k y$ as operators on G. It follows that x^k is in the centre of an irreducible Banach algebra of operators and hence is a complex multiple of the identity. Since $y^{2^k} = y$ as operators on G, it follows that the image algebra is a finite-dimensional

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irreducible algebra and hence V is also finite dimensional. Let k be minimal with the property that $\alpha^{2^k} = \alpha$. Then $g(m) \neq 0$ implies $\alpha^{2^m} = \alpha$, which implies that k divides m. Suppose that $x^k = 0$ on G. For $m \geqslant k$, $g(m) = gx^k(m-k) = 0$. Hence, $g(m) \neq 0$ implies $m = 0$ and again the representation is one dimensional. $\square$

Note that if V is any finite-dimensional irreducible right module for A or B, then y automatically has an eigenvalue on V. This enables us to list all the finite-dimensional irreducible right representations for A and B. We determine all the corresponding canonical matrices for x and y. These matrices essentially appear in [**7**].

We may suppose that k is minimal with the property that $\alpha^{2^k} = \alpha$. It follows that the numbers $\alpha, \alpha^2, \alpha^4, \dots, \alpha^{2^{k-1}}$ are then distinct, for otherwise $k \geqslant 2$ and we get $\alpha^{2^i} = \alpha^{2^j}$ with $1 \leqslant i < j \leqslant k$. Raise both sides to the power 2^{k-j} and we find that $\alpha^{2^{k+i-j}} = \alpha$ with $1 \leqslant k + i - j < k$, which is impossible. We continue to work with the equivalent representation on the subspace G of ℓ^∞, so that $x^k = \beta 1$ on G for some $|\beta| \leqslant 1$. Consider first the case when $\beta = 1$. Since $x^k = 1$ on G, it follows that each $g \in G$ is periodic with period k. Let $K = \{\alpha^{2^m} : m \in \mathbb{Z}_+\}$, a finite set. For $g \in G$, let $h(\alpha^{2^m}) = g(m)$, and let H denote all such functions h. This is well defined because of the periodicity of each g. It is easy to verify that we get an equivalent right representation on H defined by

$$(hx)(\alpha^{2^m}) = (gx)(m) = g(m+1) = h(\alpha^{2^{m+1}}),$$
$$(hy)(\alpha^{2^m}) = (gy)(m) = \alpha^{2^m} g(m) = \alpha^{2^m} h(\alpha^{2^m}).$$

When we write these in standard function terms we get, for $\zeta \in K$,

$$f(\zeta)x = f(\zeta^2), \qquad f(\zeta)y = \zeta f(\zeta) \tag{2.3}$$

(cf. (2.1)). By Theorem 2.8, below, this right representation is irreducible for any such K. Alternatively, a computational proof of irreducibility can be constructed using the matrices below. First verify that H contains the characteristic functions of the singletons $\{\alpha^{2^j}\}$ for $j = 0, 1, \dots, k-1$. Take these functions as a basis for H, and let X and Y be the matrices (with right action) corresponding to the right actions of x and y, respectively, on H. It is straightforward to check that

$$X = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} \alpha & 0 & 0 & \cdots & 0 \\ 0 & \alpha^2 & 0 & \cdots & 0 \\ 0 & 0 & \alpha^4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha^{2^{k-1}} \end{bmatrix}.$$

Return now to the general case with $x^k = \beta 1$. Write $\beta = \gamma^k$. The right representations with x and y mapped to γX and Y, respectively, are equally all irreducible with $0 < |\gamma| \leqslant 1$. Write $\pi_{\alpha,\gamma}$ for this right representation.

Theorem 2.5. *Up to equivalence, the irreducible finite-dimensional right representations of A and B are given by the one-dimensional representations listed above and the*

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family $\pi_{\alpha,\gamma}$ (subject to $\alpha^{2^k} = \alpha$ for some $k \in \mathbb{N}$, and $0 < |\gamma| \leqslant 1$). These representations separate the points of A and B.

Proof. It remains to prove the final assertion. Suppose that $a\pi_{\alpha,\gamma} = 0$ for all these representations. Let $a = \sum x^m \phi_m(y)$. Then we have

$$\sum \gamma^m X^m \phi_m(Y) = 0$$

for all complex γ with $0 < |\gamma| \leqslant 1$. It follows that $X^m \phi_m(Y) = 0$ for each m. But X is invertible and so $\phi_m(Y) = 0$ for each m. We now have $\phi_m(\alpha) = 0$. Since α can be any $(2^k - 1)$th root of unity, this shows that $\phi_m = 0$, giving $a = 0$, as required. □

Corollary 2.6. *Both A and B have the property of direct finiteness.*

Proof. Let $ba = 1$ in A or B. Let π be any finite-dimensional irreducible right representation from the above family. Then $\pi(b)\pi(a) = \pi(1) = I$. Since $\pi(a), \pi(b)$ are complex matrices, we have $\pi(a)\pi(b) = I$. Thus, $\pi(ab - 1) = 0$ for all such π, and hence $ab = 1$, as required. □

It is clear by dual representation theory that the family of irreducible finite-dimensional left representations of A and B may be parametrized by mapping x and y to the matrices γX and Y, respectively, where the matrices now act on the left on column vectors.

Suppose now that K is any compact subset of $\mathbb{T}$ which is square-closed: that is, $\zeta^2 \in K$ whenever $\zeta \in K$. It is easily verified that

$$f(\zeta)x = f(\zeta^2), \qquad f(\zeta)y = \zeta f(\zeta)$$

give a right representation of B on $C(K)$ (cf. (2.3)).

We can think of $C(K)$, above, being derived from $C(\mathbb{T})$ by the restriction mapping $f \to f|_K$, but it is better to regard it as the quotient of $C(\mathbb{T})$ by the submodule $N = \{f \in C(\mathbb{T}) : f(K) = (0)\}$. To get the corresponding right module for the Banach algebra A, we start with the analogous right module W and take the submodule $N = \{f \in W : f(K) = (0)\}$. We then get the quotient module W/N, which we denote by $W(K)$. Of course $W(K)$ can be identified with the restrictions to K of all the functions in W. Since N is a closed ideal in the Banach algebra W we can also consider $W(K)$ as a Banach algebra. The classic Wiener Lemma adapts to $W(K)$.

Lemma 2.7. *Let $f \in W(K)$. Then f is invertible in $W(K)$ if and only if f is never zero on K.*

Proof. This can be deduced from results in [**6**, § 4.1]. Alternatively, the argument in [**3**, Example 19.4] can be adapted as follows. Write $u(\zeta) = \zeta$ for $\zeta \in \mathbb{T}$, and write $u|_K$ for the restriction of u to K. It is sufficient to show that $\mathrm{Sp}(W(K), u|_K)$, the spectrum of $u|_K$ in $W(K)$, is K. It is clear that $K \subseteq \mathrm{Sp}(W(K), u|_K) \subseteq \mathbb{T}$. For each $\beta \in \mathbb{T} \setminus K$, we have to show that $u|_K - \beta$ is invertible in $W(K)$. Let the interval J be the component of the open set $\mathbb{T} \setminus K$ which contains β. We can modify the function $u - \beta$ only on J so that it never vanishes on J, and we can do so infinitely smoothly. This modified function then has an absolutely convergent Fourier series and never vanishes on $\mathbb{T}$, and so it has an inverse in W by Wiener's Lemma. Restrict this inverse to K and we see that $u|_K - \beta$ is invertible in $W(K)$. $\square$

When K is a proper subset of $\mathbb{T}$, the kernel of this right representation of B intersects the subalgebra $C(\mathbb{T})$ of B in the space of all functions that vanish on K, and so this representation is certainly not faithful. A similar remark applies for A. But we show below that the representation is irreducible if and only if K is minimal, with respect to set inclusion, amongst all compact square-closed subsets of $\mathbb{T}$.

Theorem 2.8. *Let K be a compact square-closed subset of $\mathbb{T}$. Then the above right representation of B (respectively, A) is irreducible on $C(K)$ (respectively, $W(K)$) if and only if K is minimal.*

Proof. If K is not minimal, then any smaller compact square-closed set produces an invariant subspace for the representation. Suppose now that K is minimal. The function 1 is cyclic. Let f be any non-zero function in $C(K)$ and let $g = |f|^2 = ff^*$ (here, $f^*(\zeta)$ is the complex conjugate of $f(\zeta)$). Then $g \in fB$. Let $h = \sum_{n=0}^{\infty} 2^{-n} g x^n$. Then $h \in fB$. Note that

$$h(\zeta) = \sum_{n=0}^{\infty} 2^{-n} g(\zeta^{2^n})$$

so that h is non-negative and not identically zero. Let $L = \{\zeta \in K : h(\zeta) = 0\}$ so that L is a compact proper subset of K. Let $\zeta \in L$. Then $g(\zeta^{2^n}) = 0$ for each n and so $h(\zeta^2) = 0$: that is, $\zeta^2 \in L$. Since K is minimal, it follows that L is empty and so h is invertible in $C(K)$. Thus, $1 \in hB \subseteq fB$ and so we can map f to 1. The proof is complete for the case of B. The proof adapts for the case of A, with the use at the last step of the above Wiener Lemma for $W(K)$. $\square$

For brevity we shall refer to minimal compact square-closed subsets of $\mathbb{T}$ as *μ-sets*. The existence of infinite μ-sets is guaranteed by Zorn's Lemma, but that approach gives us little insight. We have already met a countable family of finite μ-sets: the sets of the form $\{\alpha^{2^m} : m \in \mathbb{Z}_+\}$, where $\alpha^{2^k} = \alpha$ for some $k \in \mathbb{N}$. There are in fact uncountably many different infinite μ-sets (and each one is a Cantor set). Two such uncountable families are given in [**11**]. The union of all infinite μ-sets is dense in $\mathbb{T}$, and so the corresponding family of irreducible infinite-dimensional right representations again separates the points of A and B. This density property of μ-sets does not appear to be in the literature, but the authors give a proof in [**4**].

We conjecture that the only irreducible right representations of dimension greater than 1 for B are those given by minimal K as above. We do not know whether there exist any non-faithful irreducible left representations of A or B on infinite-dimensional spaces.

Appendix A. Some representation theory for the semigroup algebra $\mathbb{C}S$

We present here a different proof of Irving's result in [**8**] that $\mathbb{C}S$ is left-primitive but not right-primitive. The proof of Lemma A 5 is largely a translation of Irving's arguments with ideals to arguments with vectors. We prove further that the only irreducible right modules of $\mathbb{C}S$ are finite dimensional. As before, S is the semigroup with identity, generated by x and y, subject to y being invertible and $yx = xy^2$. Thus, the product in S is given by

$$x^m y^n x^p y^q = x^{m+p} y^{n2^p+q}.$$

There is a natural left module V_S for $\mathbb{C}S$ given by taking V_S to be the linear space $\mathbb{C}\mathbb{Z}$ and restricting to V_S the module action for V_A defined in § 2. To see that $\mathbb{C}S$ is left-primitive, we simply adapt the proof of Theorem 2.1. The use of Fourier analysis in proving faithfulness can be avoided; we leave the reader to give a direct algebraic proof that $aV_S \neq (0)$ whenever $a \neq 0$. We record the result.

Theorem A 1. *The linear algebra $\mathbb{C}S$ is left-primitive.*

The representation theory for $\mathbb{C}S$ is of course different from the representation theory for A since there are no continuity restrictions for $\mathbb{C}S$. For example, there are two (larger) families of one-dimensional irreducible representations of $\mathbb{C}S$ determined by multiplicative linear functionals χ. As before, these are determined by $\chi(x)$ and $\chi(y)$ subject to

$$\chi(y)\chi(x) = \chi(x)\chi(y)^2.$$

Since there is no boundedness constraint on χ, we have either $\chi(y) = 1$ and $\chi(x) \in \mathbb{C}$, or $\chi(x) = 0$ and $\chi(y) \in \mathbb{C}$ with $\chi(y) \neq 0$. Similarly we get a larger two-parameter family of finite-dimensional irreducible right representations of $\mathbb{C}S$. As before, let $\alpha \in \mathbb{C}$ with $\alpha^{2^k} = \alpha$ and the positive integer k minimal. Now let $\gamma \in \mathbb{C}$, $\gamma \neq 0$, and let X and Y be the matrices as in § 2. Let $\pi_{\alpha,\gamma}$ be the right representation of $\mathbb{C}S$ determined by

$$x\pi_{\alpha,\gamma} = \gamma X, \qquad y\pi_{\alpha,\gamma} = Y.$$

As for A and B in § 2, we now get the following result for $\mathbb{C}S$.

Theorem A 2. *The above finite-dimensional right representations $\pi_{\alpha,\gamma}$ of $\mathbb{C}S$ are irreducible and the family separates the points of $\mathbb{C}S$.*

Corollary A 3. *$\mathbb{C}S$ has the property of direct finiteness.*

Remark A 4. If we regard the matrices X and Y as acting on the left, then we get a corresponding family of irreducible finite-dimensional left representations of $\mathbb{C}S$ with the corresponding properties.

To help simplify the notation in places, we shall write R for the algebra of Laurent polynomials in y; thus, R is simply a copy of $\mathbb{C}\mathbb{Z}$. We aim to show that any irreducible right module V for $\mathbb{C}S$ is finite dimensional, and it will follow immediately that $\mathbb{C}S$ is not right-primitive. The first (and most critical) step is to show that y must have an eigenvalue on V. We can then adapt the right averaging construction to our algebraic situation to prove that V must be finite dimensional. In the non-Banach algebra setting we need a proof that an irreducible representation of $\mathbb{C}\mathbb{Z}$ is one dimensional. We could use the *Nullstellensatz*, but in the course of Lemma A 5, below, we give instead a very short elementary argument for this.

Lemma A 5. *Let V be an irreducible right module for $\mathbb{C}S$. Then y has an eigenvalue on V.*

Proof. Suppose first that $Vx = (0)$. Let $v \in V$, $v \neq 0$. Consider $v(1+y)$. If $v(1+y) = 0$, then y has an eigenvalue. If $v(1+y) \neq 0$, then by irreducibility there is a polynomial ϕ in y and y^{-1} such that $v(1+y)\phi = v$. Multiply by a suitable power of y to get $v(1+y)\psi = vy^k$ for some polynomial ψ in y. Since $y^k - (1+y)\psi$ is a non-zero polynomial it follows, by factorizing into linear factors, that y has an eigenvalue.

Now suppose that $Vx \neq (0)$. Let $v \in V$ with $vx \neq 0$. Since V is irreducible, there exists $d \in \mathbb{C}S$ with $vxd = v$. This gives

$$0 = v(1 - xd) = v(1 + xd_1 + x^2 d_2 + \cdots)$$

for some $d_j \in R$. Suppose that $a = r_0 + xr_1 + \cdots + x^m r_m$ has m minimal such that $m \geqslant 0$, $va = 0$, $r_j \in R$ and $r_0 \neq 0$. Our aim is to show that $m = 0$. As above, this will prove that y has an eigenvalue on V.

From now on, we shall also regard the elements of R as functions on $\mathbb{T}$, via $y(\zeta) = \zeta$. We first choose an odd prime p so that for each zero $\mathrm{e}^{2\pi \mathrm{i}\beta_j}$ of r_m with $0 < \beta_j < 1$ we have $p\beta_j \notin \mathbb{Z}$. For this we need consider only those β_j which are rational, and we may take the prime p to be greater than any odd prime which appears in a denominator of any β_j in its lowest terms. Set $\alpha = \mathrm{e}^{2\pi\mathrm{i}/p}$. Then $\alpha^{2^k} \neq \mathrm{e}^{2\pi\mathrm{i}\beta_j}$ for any j or for any $k \in \mathbb{Z}_+$; for otherwise we should have $2^k 2\pi/p - 2\pi\beta_j \in 2\pi\mathbb{Z}$ and hence $p\beta_j \in \mathbb{Z}$. Also, $\alpha^{2^k} \neq 1$ since $2^k/p \notin \mathbb{Z}$. We thus have $r_m(\alpha^{2^k}) \neq 0$ for all $k \in \mathbb{Z}_+$.

Now put $q = y^p - 1$ and let $Q = Rq$. We have $qx = x(y^{2p} - 1) = xq(y^p + 1) \in xQ$. Similarly, $qx^2 \in x^2 Q$, $qx^3 \in x^3 Q$, and so on. If $vq = 0$, then y has an eigenvalue on V and we are done. Suppose instead that $vq \neq 0$ and so there is $d' \in \mathbb{C}S$ with $vqd' = v$. This gives $v(1 - qd') = 0$, where

$$1 - qd' = 1 + q_0 + xq_1 + x^2 q_2 + \cdots$$

and each $q_j \in Q$. By the choice of α we have $q(\alpha) = 0$ for every $q \in Q$. Thus, $(1+q_0)(\alpha) = 1$. Suppose now that

$$b = t_0 + xq_1 + \cdots + x^n q_n$$

has n minimal such that $n \geqslant 0$, $vb = 0$, $q_j \in Q$ and $t_0 \in R$ with $t_0(\alpha) \neq 0$. Since b satisfies the conditions given for a above, it follows that $n \geqslant m$. Put $n = m + k$ so that $k \geqslant 0$. We now suppose that $m > 0$ and derive a contradiction. This will give $m = 0$ and so complete the proof of the lemma.

For $r \in R$ put $\hat{r}(y) = r(y^{2^k})$. With a as above, we have

$$0 = vax^k = v(x^k \hat{r}_0 + \cdots + x^n \hat{r}_m)$$

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and hence $0 = vc$, where

$$c = b\hat{r}_m - ax^k q_n = T_0 + xQ_1 + \cdots + x^{n-1}Q_{n-1}$$

with $T_0 \in R$ and $Q_j \in Q$. Here we have $T_0 = t_0\hat{r}_m$ if $k > 0$, and $T_0 = t_0 r_m - r_0 q_n$ if $k = 0$. In either case we have

$$T_0(\alpha) = t_0(\alpha)\hat{r}_m(\alpha) = t_0(\alpha) r_m(\alpha^{2^k}) \neq 0.$$

Thus, c has the same form as b but is of lesser degree, which is impossible. Hence, $m = 0$, which was needed to complete the proof. □

Theorem A 6. *Let V be any irreducible right module for $\mathbb{C}S$. Then V is finite dimensional, and the representation is one dimensional (as listed above) or is equivalent to one of the form $\pi_{\alpha,\gamma}$. Thus, $\mathbb{C}S$ is not right-primitive.*

Proof. By Lemma A 5 we have $v \in V$ and $\alpha \in \mathbb{C}$ such that $vy = \alpha v$. Since y is invertible, $\alpha \neq 0$. Let ρ be a linear functional on V with $\rho(v) = 1$. Consider first the case $|\alpha| \geqslant 1$. We note that $\alpha^{-n}\rho(vy^n) = 1$, $n \in \mathbb{N}$. Also

$$\alpha^{-2m}\rho(vxy^{2m}) = \alpha^{-2m}\rho(vy^m x) = \alpha^{-m}\rho(vx)$$

and similarly $\alpha^{-2m-1}\rho(vxy^{2m+1}) = \alpha^{-m-1}\rho(vxy)$. Thus, $\{\alpha^{-n}\rho(vxy^n) : n \in \mathbb{N}\}$ is a bounded sequence and hence we may apply a Banach limit to define

$$f(x) = \operatorname*{LIM}_{n\to\infty} \alpha^{-n}\rho(vxy^n).$$

Similarly, we may define

$$f(x^j) = \operatorname*{LIM}_{n\to\infty} \alpha^{-n}\rho(vx^j y^n), \quad j \in \mathbb{N},$$

and then we can extend to a linear functional f on $\mathbb{C}S$ given by

$$f(a) = \operatorname*{LIM}_{n\to\infty} \alpha^{-n}\rho(vay^n).$$

If $|\alpha| < 1$, we instead define $f(a) = \mathrm{LIM}_{n\to\infty}\, \alpha^n \rho(vay^{-n})$. In each case, as before, $f(1) = 1$ and $f(ay) = \alpha f(a)$. We can now proceed exactly as in § 2 to get our given right representation equivalent to the right representation on the space $G = \{g_a : a \in \mathbb{C}S\}$ with the actions $(g_a x)(m) = g_a(m+1)$ and $(g_a y)(m) = \alpha^{2^m} g_a(m)$ for $m \in \mathbb{Z}_+$.

We have $h \in G$, $h \neq 0$, with $hy = \alpha h$. If, for all $m \in \mathbb{N}$, $\alpha^{2^m} \neq \alpha$, then $h(m) = 0$ for all $m \in \mathbb{N}$. Then $hx = 0$. In this case $G = \mathbb{C}h$ and the representation is one dimensional.

Finally, consider the case $\alpha^{2^k} = \alpha$ for some $k \in \mathbb{N}$. We may assume that k is minimal. For simplicity we shall now consider the case when $k = 2$; the general case is just an elaboration of the argument. Thus, we have $\alpha^4 = \alpha \neq \alpha^2$. We write X and Y for the operators on G corresponding to x and y, respectively. It is immediate from the action of y given above that the algebra generated by Y consists of all diagonal operators with entries of the form $(\gamma, \delta, \gamma, \delta, \dots)$. It is clear that our eigenvector h for y must be

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of the form $h = (h_0, 0, h_2, 0, h_4, 0, \dots)$. If $hX^2 = 0$, then $h = (h_0, 0, 0, \dots)$, and again our representation is one dimensional. Suppose that $hX^2 \neq 0$. By irreducibility we get $hX^2 \sum_j X^j p_j(Y) = h$ and, since the support of hX^{2m+1} is contained in $\{1, 3, 5, \dots\}$, we can assume that only even j appear in the sum. But each $p_{2m}(Y)$ acts as a multiple of I on hX^{2+2m}. It follows that $hQ(X) = 0$ for some non-zero complex polynomial Q. The usual argument shows that X has an eigenvector $g' \in G$. Suppose that $g'X = \beta g'$ for some $\beta \in \mathbb{C}$. Then we have $g' = (1, \beta, \beta^2, \beta^3, \dots)$. If $\beta = 0$, the representation is one dimensional. Suppose $\beta \neq 0$. Let G_1 be the two-dimensional subspace of G given by

$$G_1 = \{g \in G : g(j+2) = \beta^2 g(j),\ j \in \mathbb{Z}_+\}.$$

A calculation shows that G_1 is invariant under X and Y. Since $G_1 \neq (0)$, it follows by irreducibility that $G_1 = G$ and hence $\dim G = 2$.

Verification that our representation is equivalent to the representation $\pi_{\alpha,\beta}$ is routine. □

Remark A 7. The use of LIM can be avoided in § 2 by using numerical range results from [**10**]; in this appendix the Banach limits can be calculated explicitly, if desired.

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