ON RAMANUJAN AND DIRICHLET SERIES WITH EULER PRODUCTS

by S. RAGHAVAN

(Received 27 January, 1983)

In his unpublished manuscripts (referred to by Birch [**1**] as Fragment V, pp. 247–249), Ramanujan [**3**] gave a whole list of assertions about various (transforms of) modular forms possessing naturally associated Euler products, in more or less the spirit of his extremely beautiful paper entitled "On certain arithmetical functions" (in Trans. Camb. Phil. Soc. 22 (1916)). It is simply amazing how Ramanujan could write down (with an ostensibly profound insight) a basis of eigenfunctions (of Hecke operators) whose associated Dirichlet series have Euler products, anticipating by two decades the famous work of Hecke and Petersson. That he had further realized, in the event of a modular form f not corresponding to an Euler product, the possibility of restoring the Euler product property to a suitable linear combination of modular forms of the same type as f, is evidently fantastic.

Ramanujan's assertions referred to at the beginning were recently upheld and elucidated in the light of the work of Hecke, Rankin and Serre by Rangachari [**4**]. Page 5 of [**4**] refers to "an additional list of Euler product developments found in [**3**] which is incomplete except for the first one" (already covered by one of Ramanujan's above-mentioned assertions): namely,

(1) If $\sum a(n)x^n = x^2\prod(1-x^{3n})^{16}$ and $\sum A(n)x^n = x\prod(1-x^{3n})^8\left(1+240\sum_1^\infty \frac{n^3x^{3n}}{1-x^{3n}}\right)$

then

$$\sum(A(n)+6a(n)\sqrt{10})\,.\,n^{-s} = (1-6\sqrt{10}\,.\,2^{-s}+2^{7-2s})^{-1}(1+96\sqrt{10}\,.\,5^{-s}+5^{7-2s})^{-1}$$
$$\times(1-260\,.\,7^{-s}+7^{7-2s})^{-1}(1+1920\sqrt{10}\,.\,11^{-s}+11^{7-2s})^{-1}\ldots \tag{1}$$

(2) If $\sum a(n)x^n = x^3\prod(1-x^{4n})^{18}$, then

$$156\sum a(n)n^{-s} = (1-78\,.\,3^{-s}+3^{8-2s})^{-1}(1+510\,.\,5^{-s}+5^{8-2s})^{-1}(1+1404\,.\,7^{-s}+7^{8-2s})^{-1}\ldots$$
$$-(1+78\,.\,3^{-s}+3^{8-2s})^{-1}(1+510\,.\,5^{-s}+5^{8-2s})^{-1}(1-1404\,.\,7^{-s}+7^{8-2s})^{-1}\ldots \tag{2}$$

Ramanujan writes further:

Presumably there are analogous results for $\sum a_n\,.\,n^{-s}$ where $\sum a_n\,.\,x^n$ is

$$x^5\prod(1-x^{12n})^{10},\quad x^7\prod(1-x^{12n})^{14},\quad x^5\prod(1-x^{6n})^{20}\quad\text{and}\quad x^{11}\prod(1-x^{12n})^{22} \tag{3}$$

In this brief note—an addendum to [**4**], our object is merely to point out that Ramanujan's relation (2) above does not seem to be correct and the "analogous

Glasgow Math. J. **25** (1984) 203–206.

results"for (3) appear to be covered by Ramanujan's assertions referred to in the first paragraph. Of course, as remarked in [**4**], Ramanujan's formula (1) above is taken care of already by the Euler product written down by him for $\eta^8 Q + 6\sqrt{10}\,\eta^{16}$ and verified in [**4**]; here, as usual, we have

$$\eta(x) = x^{1/24}\prod_1^\infty (1-x^n), \qquad E_4(x) = Q(x) = 1 + 240\sum_{n=1}^\infty \frac{n^3x^n}{1-x^n}$$

and

$$E_6(x) = R(x) = 1 - 504\sum_{n=1}^\infty \frac{n^5x^n}{1-x^n}.$$

Let us first consider Ramanujan's assertion (2). Writing $\prod_{m=1}^\infty (1-x^m)^r = \sum_{n=0}^\infty p_r(n)x^n$ for integral $r \geq 1$, after Newman [**2**], we have

$$\eta^{18}(x) = x^{3/4}\sum_{m=0}^\infty p_{18}(m)x^m = \sum_{n=3}^\infty a(n)x^{n/4}$$

where, as we can check from the tables in [**2**],

$$\begin{aligned} &a(3) = 1 \qquad a(7) = -18, \qquad a(11) = 135, \qquad a(15) = -510 \\ &a(27) = p_{18}(6) = -7038, \qquad a(39) = -27710 \\ &a(147) = p_{18}(36) = \sum_{j=0}^{36} p_{15}(j)p_3(36-j) = -767039 \end{aligned} \tag{4}$$

(as also confirmed recently by Prof. M. Newman).

For z in the complex upper half-plane and $q = e^{2\pi i z}$, let $\eta^6(q)\,E_6(q) = \sum_{n=1}^\infty b(n)e^{2\pi i n z/4}$ so that $b(1) = 1$, $b(5) = -510$, $b(13) = -27710$, $b(21) = 504\times 720$. Then (as functions of z), η^{18}, $\eta^6 E_6$ are two cusp forms generating a 2-dimensional subspace S of the space of cusp forms in $M_9(12)$ of weight 9, divisor 3 and character ε in the sense of Hecke, where for $(n, 12) = 1$, $\varepsilon(n) = \sin^9(\pi n/2)$; actually in the notation of Rankin, $S = M_9^{(0)} \oplus M_9^{(6)}$ and further S is invariant under the Hecke operators T_n for $(n, 12) = 1$ (See [**5**]). In fact, for primes $p \equiv 1, 5 \pmod{12}$,

$$T_p(\eta^{18}) = a(3p)\eta^{18}, \qquad T_p(\eta^6E_6) = b(p)\eta^6E_6.$$

For primes $r \equiv 7, 11 \pmod{12}$, T_r permutes η^{18} and η^6E_6 up to scalar multiples and indeed

$$T_7(\eta^6E_6) = b(21)\eta^{18} = (504\times720)\eta^{18}, \qquad T_7(\eta^{18}) = a(7)\eta^6E_6 = -18\eta^6E_6.$$

The eigenvalues of the matrix $A = \begin{pmatrix} 0 & -18 \\ 504\times720 & 0\end{pmatrix}$ are $\pm 432\sqrt{-35}$, and if $\lambda = 432\, i\sqrt{35}$ and $P = \begin{pmatrix} -\lambda/18 & 1 \\ \lambda/18 & 1\end{pmatrix}$, we have $PAP^{-1} = \begin{pmatrix}\lambda & 0 \\ 0 & -\lambda\end{pmatrix}$. Thus $\eta^6E_6 \pm (\lambda/18)\eta^{18}$ are normalized

https://doi.org/10.1017/S0017089500005620 Published online by Cambridge University Press

eigenfunctions of the Hecke algebra and as a consequence, the associated Dirichlet series have Euler product expansions. (Incidentally, we have, from above, a justification, although indirect, for the application—untenable on the face of it—of formula (1) in page 6 of [**4**] *even for the prime* $p=3$ *with* $t=3$, in order to find out the value of λ_3 in line 3 from the bottom of page 9 of [**4**].) One now concludes that for $48\, i\sqrt{35}\,\eta^{18}$, the associated Dirichlet series is the difference of two Euler products. To this extent, therefore, Ramanujan's assertion (2) is true.

However, *if assertion* (2) *were true exactly as stated, we would have*

$$\begin{aligned}156\, a(147)\,.\,(147)^{-s} &= 78\,.\,3^{-s}\,.\,7^{-2s}((1404)^2-7^8)-(-78\,.\,3^{-s}\,.\,7^{-2s}((1404)^2-7^8))\\ &= -156\,.\,(147)^{-s}\times 3793585\end{aligned}$$

which gives a *contradiction* to $a(147)=-767039$ from (4).

Now the Hecke algebra acting on the 2-dimensional space S contains the representation matrices $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, A and further, being commutative and semisimple over $\mathbb{Q}$ of rank ≤ 2, is necessarily isomorphic to $\mathbb{Q}(\sqrt{-35})$. Since the representation matrix corresponding to the Hecke operator T_p on S for primes $p\equiv 1, 5 \pmod{12}$ has the form $\begin{pmatrix}a(3p) & 0\\ 0 & b(p)\end{pmatrix}$, it follows necessarily that $a(3p)=b(p)$ for all such p. But then we have

$$\sum_{n=0}^{\infty} a(3p^n)p^{-ns}=\sum_{n=0}^{\infty} b(p^n)p^{-ns}=(1-a(3p)p^{-s}+p^{8-2s})^{-1}$$

and these are precisely the Eulerian factors occurring in the above two Euler products corresponding to primes $p\equiv 1,5 \pmod{12}$. The Eulerian factors corresponding to primes $r\equiv 7, 11 \pmod{12}$ have the form $(1-\lambda(r)\,.\,r^{-s}+\varepsilon(r)r^{8-2s})^{-1}$. In the light of the different structure of the Eulerian factors in (2) for primes $r\equiv 5, 7 \pmod{12}$, perhaps Ramanujan had in mind some other cusp form g of weight 9 and level 12 such that $g+78\,\eta^{18}$ is an eigenfunction of the Hecke algebra.

We next come to the "analogous" Euler products connected with $\eta^{10}(12z)$, $\eta^{14}(12z)$, $\eta^{20}(6z)$ and $\eta^{22}(12z)$ in (3) and hinted at by Ramanujan. From [**3**] and [**4**], we know, in fact, that for $\rho, \delta=\pm 1$, $\rho\,.\,48^{10}+\eta^2E_4$, $\rho\,.\,360i\sqrt{3}\,\eta^{14}+\eta^2E_6$, $\rho\,.\,288\sqrt{70}\,\eta^{20}+\eta^4E_4^2$, $\delta\,.\,103680\, i\sqrt{7}\,\eta^{22}+\eta^2E_4E_6+\rho\,.\,96\sqrt{1045}\,\eta^{10}E_6+\delta\rho\,.\,216\, i\sqrt{7315}\,\eta^{14}E_4$ are eigenfunctions of the Hecke algebra and the associated Dirichlet series have respectively the Euler product expansions

$$\begin{aligned}\Pi_\rho(5)&=(1-\rho\,.\,48\cdot 5^{-s}+5^{4-2s})^{-1}(1-238\,.\,13^{-s}+13^{4-2s})^{-1}\ldots\\ \Pi_\rho(7)&=(1-\rho\,.\,360\, i\sqrt{3}\,.\,7^{-s}-7^{6-2s})^{-1}(1+506\,.\,13^{-s}+13^{6-2s})^{-1}\ldots\\ \Pi_\rho(10)&=(1-\rho\,.\,288\sqrt{70}\,.\,5^{-s}+5^{9-2s})^{-1}(1-476\,.\,7^{-s}-7^{9-2s})^{-1}\ldots \qquad (5)\\ \Pi_{\rho,\delta}(11)&=(1-\rho\,.\,96\sqrt{1045}\,.\,5^{-s}+5^{10-2s})^{-1}(1-\delta\rho\,.\,216\, i\sqrt{7315}\,.\,7^{-s}-7^{10-2s})^{-1}\\ &\quad\times(1-\delta\,.\,103680\, i\sqrt{7}\,.\,11^{-s}-11^{10-2s})^{-1}\ldots\end{aligned}$$

and we have by appropriate elimination, Ramanujan's "analogous results":

https://doi.org/10.1017/S0017089500005620 Published online by Cambridge University Press

THEOREM (Ramanujan). *If* $\sum_n a(n)x^n$ *is the expansion for* $96\,\eta^{10}(x^{12})$, $720\,i\sqrt{3}\,\eta^{14}(x^{12})$, $576\sqrt{70}\,\eta^{20}(x^6)$, $414720\,i\sqrt{7}\,\eta^{22}(x^{12})$, *respectively, then* $\sum_n a(n)\,.\,n^{-s}$ *is equal respectively to*

$$\Pi_1(5)-\Pi_{-1}(5), \qquad \Pi_1(7)-\Pi_{-1}(7), \qquad \Pi_1(10)-\Pi_{-1}(10),$$
$$\Pi_{1,1}(11)-\Pi_{1,-1}(11)+\Pi_{-1,1}(11)-\Pi_{-1,-1}(11)$$

for the Euler products $\Pi_*(*)$ *given by* (5).

The author is grateful to Prof. R. A. Rankin for his critical reading of the manuscript and several helpful comments.

REFERENCES

1. B. J. Birch, A lookback at Ramanujan's notebooks, *Math. Proc. Camb. Phil. Soc.* **78** (1975), 73–79.

2. M. Newman, A table of coefficients of powers of $\eta(\tau)$, *Indag. Math.* **18** (1956), 204–216.

3. S. Ramanujan, Unpublished manuscripts.

4. S. S. Rangachari, Ramanujan and Dirichlet series with Euler products, *Proc. Indian Acad. Sci.*, **91** (1981), 1–15.

5. R. A. Rankin, Hecke operators on congruence subgroups of the modular group, *Math. Ann.* **168** (1967), 40–58.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH
BOMBAY 400 005
INDIA