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DERIVED RING ISOMORPHISMS OF VON NEUMANN ALGEBRAS

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1. Introduction. Let M be an associative *-algebra with complex scalar field. M may be turned into a Lie algebra by defining multiplication by $[A, B] = AB - BA$. A Lie *-subalgebra L of M is a *-linear subspace of M such that if $A, B \in L$ then $[A, B] \in L$. A Lie *-isomorphism ϕ between Lie *-subalgebras L_1 and L_2 of *-algebras M and N is a one-one, *-linear map of L_1 onto L_2 such that $\phi[A, B] = [\phi(A), \phi(B)]$ for all $A, B \in L_1$. We have previously shown [**5**] that if $L_1 = M$, $L_2 = N$ where M and N are von Neumann algebras with no central abelian summands, then, modulo a *-linear map from M into the center of N which annihilates brackets, ϕ is the direct sum of a *-isomorphism and the negative of a *-anti-isomorphism.

In this paper we show that if M and N are von Neumann algebras with no central abelian summands, and if $L_1 = [M, M]$ (= all finite linear combinations of elements $[A, B]$, $A, B \in M$) and $L_2 = [N, N]$ then ϕ can be extended to a mapping from M onto N which is the direct sum of a *-isomorphism and the negative of a *-anti-isomorphism. This result is analogous to that of R. A. Howland [**4**] who proved that if M and N are simple rings with M containing three non-zero orthogonal idempotents whose sum is the identity, then ϕ can be extended to an isomorphism of M onto N, or to the negative of an anti-isomorphism of M onto N. Although von Neumann algebras have an abundance of projections they will not, in general, be simple owing to the presence of central projections. It is known [**6**; **8**], that if M is an infinite von Neumann algebra then $[M, M] = M$, if M is of type I and finite then, modulo the center of M, $[M, M] = M$, and if M is of type II, then $[M, M]$ is uniformly dense in the set of operators with central trace zero.

In what follows we shall take Dixmier [**1**] as a general reference. A von Neumann algebra M is a weakly closed, self-adjoint algebra of operators on a Hilbert space H containing the identity operator. The set $Z_M = \{S \in M | ST = TS \text{ for all } T \in M\}$ is called the center of M. If P and Q are projections (= self-adjoint idempotents) in M then $M_P = \{PAP | A \in M\}$, $PMQ = \{PAQ | A \in M\}$ and

$$PMQMP = \left\{\sum_{i=1}^{n} PX_iQY_iP | X_i, Y_i \in M\right\}.$$

The central support $\bar{P}$ of a projection P is defined to be the smallest central projection larger than P, the central core $\tilde{P}$ of P is defined to be

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LUB$\{A \leqq P | A = A^* \in Z_M\}$. The central core of any self-adjoint element is defined analogously. If $\bar{P}\bar{Q} = 0$ we say P is parallel to Q, written $P \| Q$.

2. General results on Lie *-isomorphisms of $[M, M]$**.** The following results parallel those of [**5**, pp. 719–723]. In particular we first show that certain projections and relations between projections can be characterized in terms of bracket relations involving elements of $[M, M]$, and then that the image of these projections under a Lie *-isomorphism of $[M, M]$ onto $[N, N]$ can also be characterized.

LEMMA 1. *If A is a self-adjoint operator in the von Neumann algebra M and* $[[[[X, A], A], A], A] = 0$ *for all* $X \in M$, *then* $A - \tilde{A}$ *is a projection in* M.

Proof. An argument similar to [**5**, Theorem 1] with the polynomial $t^4 - t^2$ replacing $t^3 - t$ will show that the spectrum of $A - \tilde{A}$ consists of $\{0, 1\}$.

LEMMA 2. *If P and Q are commuting, core-free projections then* $[[P, Q], Q] = 0$ *for all* $X \in [M, M]$ *implies* $P \| Q$. *If* $P \| Q$ *then* $[[P, X], Q] = 0$ *for all* $X \in M$.

Proof. If $[[P, X], Q] = 0$ for all $X \in [M, M]$ then $[[[P, [X, P]], Q] = 0$ for all $X \in M$. Multiplying this out we have

$$2PXPQ - PXQ - XPQ - 2PQXP - PQX + QXP = 0.$$

Multiplying this relation on the left by PQ gives $PQX(I - P)(I - Q) = 0$ for all $X \in M$. Multiplying the relation on the left by P and on the right by Q gives $P(I - Q)XQ(I - P) = 0$ for all $X \in M$. The result now follows from the proof of [**5**, Lemma 2].

LEMMA 3. *Let P, Q be commuting projections in M such that* $[[[[X, P], Q], P], Q] + [[X, P], Q] = 0$ *for all* $X \in [M, M]$. *Then there exists a projection* $C \in Z_M$ *such that* $PQ(I - C) = 0$, $(I - P)(I - Q)C = 0$.

Proof. The bracket identity implies that

$$[[[[[X, Q], P], Q], P], Q] + [[[X, Q], P], Q] = 0$$

for all $X \in M$. Multiplying this relation on the left by PQ gives $PQX(I - P)(I - Q) = 0$ for all $X \in M$ which implies $PQ \| (I - P)(I - Q)$. Let $C = \bar{P}\bar{Q}$.

LEMMA 4. *Let* $\phi: [M, M] \rightarrow [N, N]$ *be a Lie *-isomorphism of* $[M, M]$ *onto* $[N, N]$ *where M and N are von Neumann algebras. Then* $\phi[Z_M \cap [M, M]) = Z_N \cap [N, N]$.

Proof. If $Z \in Z_M \cap [M, M]$ then $[[M, M], Z] = 0$. This implies $[[N, N], \phi(Z)] = 0$. By [**3**, Sublemma, p. 5] $\phi(Z) \in Z_N$. The reverse inclusion follows by applying the same argument to ϕ^{-1}.

LEMMA 5. *Let ϕ, M, and N be as in Lemma 4. If P is a projection in M such that* $P - Z \in [M, M]$ *for some* $Z = Z^* \in Z_M$ *then* $\phi(P - Z) = \theta(P) + \lambda(P - Z)$ *where θ is a core free projection and* $\lambda(P - Z) \in Z_N$. *This representation is* *unique. Also,* $\phi(P - Z) = -\theta'(P) + \lambda'(P - Z)$ *where* $\theta'(P)$ *is a core free projection and* $\lambda'(P - Z) \in Z_N$. *This representation is unique.*

Proof. Let $F = P - Z$. Then $[[[X, F], F], F] = [X, F]$ for all $X \in M$, and in particular for all $X \in [M, M]$. Thus, $[[[X, \phi(F)], \phi(F)], \phi(F)] = [X, \phi(F)]$ for all $X \in [N, N]$ since ϕ is onto. Let $X = [Y, \phi(F)]$. Then

$$[[[[Y, \phi(F)], \phi(F)], \phi(F)], \phi(F)] = [[Y, \phi(F)], \phi(F)].$$

By Lemma 1 this implies $\phi(F) - \phi(F)^{\sim}$ is a core-free projection, say $\theta(F)$. Suppose $P - Z' \in [M, M]$ for $Z' \in Z_M$. Then $Z - Z' = (P - Z') - (P - Z) \in [M, M]$ so that, by Lemma 4, $\phi(Z - Z') \in Z_N \cap [N, N]$. Also, $\phi(Z - Z') = \phi(P - Z') - \phi(P - Z) = \theta(P - Z') + \phi(P - Z')^{\sim} - \theta(P - Z) - \phi(P - Z)^{\sim}$ so that $\theta(P - Z') - \theta(P - Z) \in Z_N$. By [**5**, Lemma 1] this implies $\theta(P - Z') = \theta(P - Z)$. We call this common value $\theta(P)$. If $\phi(P - Z) = Q + Z'$ where Q is a core-free projection and $Z' \in Z_N$ then $\theta(P) - Q \in Z_N$ which would imply again by [**5**, Lemma 1] that $\theta(P) = Q$ and also that $\lambda(P - Z) = Z'$.

If we write $\theta'(P) = \overline{\theta(P)} - \theta(P)$ then $\theta'(P)$ is a core-free projection and $\phi(P - Z) = -\theta'(P) + \lambda'(P - Z)$. By an argument similar to the one above this representation is unique.

LEMMA 6. *If* $P - Z$, $Q - Z' \in [M, M]$, *for some self-adjoint* Z, $Z' \in Z_M$, *with* $[P, Q] = 0$ *then* $[\theta(P), \theta(Q)] = 0$.

Proof. $0 = [P, Q] = [P - Z, Q - Z']$. Hence, $0 = \phi(0) = \phi[P - Z, Q - Z'] = [\phi(P - Z), \phi(P - Z')] = [\theta(P), \theta(Q)]$.

LEMMA 7. *Let* Q *be a core-free projection in* N *such that* $Q - Z' \in [N, N]$ *for some* $Z' \in Z_N$. *There exists a core-free projection* $P \in M$ *and a self-adjoint* $Z \in Z_M$ *such that* $P - Z \in [M, M]$ *and* $\theta(P) = Q$.

Proof. Let $Q' = Q - Z'$. Then $[[[X, Q'], Q'], Q'] = [X, Q']$ for all $X \in [N, N]$. There exists a self-adjoint $P' \in [M, M]$ such that $\phi(P') = Q'$. This implies $[[[X, P'], P'], P'] = [X, P']$ for all $X \in [M, M]$. Hence $P' - \tilde{P}' = P$ is a core-free projection and $Q - Z' = Q' = \phi(P') = \phi(P + \tilde{P}') = \theta(P) + \lambda(P - (-\tilde{P}'))$. This implies $\theta(P) = Q$.

LEMMA 8. *Let* P *and* Q *be core-free projections in* M *with* $P - Z$, $Q - Z' \in [M, M]$ *for self-adjoint* $Z, Z' \in Z_M$. *Then* $P \| Q$ *if and only if* $\theta(P) \| \theta(Q)$ *and* $\bar{P} = \bar{Q}$ *if and only if* $\theta(P) = \overline{\theta(Q)}$.

Proof. If $P \| Q$ (P, Q need not be core-free here) then $0 = [[P, X], Q] = [[P - Z, X], Q - Z']$ for all $X \in [M, M]$. Thus $0 = \phi(0) = [[\phi(P - Z), X], \phi(Q - Z')] = [[\theta(P), X], \theta(P)]$ for all $X \in [N, N]$. This implies, by Lemma 2, since $\theta(P)$ and $\theta(Q)$ are core-free, that $\theta(P) \| \theta(Q)$. If $\theta(P) \| \theta(Q)$ then $0 = [[\theta(P), X], \theta(Q)] = [[\phi(P - Z), X], \phi(Q - Z')]$ for all $X \in [N, N]$. Thus $0 = \phi^{-1}(0) = [[P, X], Q]$ for all $X \in [M, M]$. By Lemma 2 we have $P \| Q$.

If $\bar{P} = \bar{Q}$ but $\overline{\theta(c)} \neq \overline{\theta(Q)}$ there exists a central projection $C \in Z_N$ such that $C\theta(Q) = 0$ but $C\theta(P) \neq 0$. By Lemma 7 there exists a core-free projection R in M and a self-adjoint $Z'' \in Z_M$ such that $R - Z'' \in [M, M]$ and $\theta(R) = C\theta(P)$. Hence $\theta(R)\|\theta(Q)$. By the preceding lemma $R\|Q$. Since $\bar{Q} = \bar{P}$ we have $R\|P$. This implies $\theta(R)\|\theta(P)$ a contradiction. Similarly $\overline{\theta(P)} = \overline{\theta(Q)}$ implies $\bar{P} = \bar{Q}$.

LEMMA 9. *Let $P_1, \ldots, P_n$ be parallel projections such that $P_i - Z_i \in [M, M]$ for self-adjoint $Z_i \in Z_M$. Then θ is additive on the P_i.*

Proof. Lemma 8 implies that the $\theta(P_i)$ and $\|$ core-free projections so that $\sum_{i=1}\theta(P_i)$ is a projection. It is core-free by parallelism. Then

$$0 = \phi\left(\sum_{i=1}^{n} P_i\right) - \sum_{i=1}^{n} \theta(P_i) + Z$$

where $Z \in Z_N$. Thus $\theta(\sum_{i=1}^{n} P_i) - \sum_{i=1}^{n}\theta(P_i) \in Z_N$ and are equal.

LEMMA 10. *Let C be a central projection in a von Neumann algebra M with no central abelian summands. There exists a core-free projection P in M, and a self-adjoint $Z \in Z_M$ such that $\bar{P} = C$ and $P - Z \in [M, M]$.*

Proof. Let $E + F + G = I$, the identity operator, where E, F, G are central projections, M_E is finite and discrete, M_F finite and continuous, and M_G infinite. CG is a central projection in M_G so there exists a core-free projection P_1 in M_G such that $\bar{P}_1 = CG$ by [**5**, Lemma 4]. Moreover, since M_G is infinite, $P_1 \in [M_G, M_G] = M_G$ by [**8**]. CE is central in M_E so there exists a core-free projection P_2 in M_E such that $\bar{P}_2 = CE$. By [**6**, Theorem 1] $P_2 - P_2^{\#} \in [M_E, M_E]$, where $P_2^{\#}$ is the center-valued trace of P_2. Finally, choose a projection $Q \backsim F - Q$ in M_F. Q is core-free, $\bar{Q} = F$, and if $VV^* = Q$, $V^*V = F - Q$, $\frac{1}{2}[V, V^*] = Q - \frac{1}{2}F \in [M_F, M_F]$. Thus $CQ - \frac{1}{2}CF \in [M_{CF}, M_{CF}] \subseteq [M_F, M_F]$. Let $P_3 = CQ$.

Set $P = P_1 + P_2 + P_3$. Then $\bar{P} = 0$ since the P_i are $\|$ and $P - (\frac{1}{2}CF + P_2^{\#}) \in [M, M] = [M_E, M_E] + [M_F, M_F] + [M_G, M_G]$. Moreover $\bar{P} = \bar{P}_1 + \bar{P}_2 + \bar{P}_3 = C$.

THEOREM 1. *Let $\phi: [M, M] \rightarrow [N, N]$ be a Lie *-isomorphism where M and N have no central abelian summands. There exists a *-isomorphism ψ of Z_M onto Z_N such that if P is a projection in M with $P - Z \in [M, M]$ for a self-adjoint $Z \in Z_M$, and if C is a central projection in M, then $\theta(CP) = \psi(C)\theta(P)$. Also $\theta'(PC) = \psi(C)\theta'(P)$.*

Proof. We first show that ϕ induces a projection orthoisomorphism of Z_M onto Z_N. Define ψ on a central projection C as follows: choose a core-free projection P in M such that $P - Z \in [M, M]$ for a self-adjoint $Z \in Z_M$ and $\bar{P} = C$. Define $\psi(C) = \overline{\theta(P)}$. If $\bar{Q} = C$ with $Q - Z' \in [M, M]$ then by Lemma 8, $\overline{\theta(P)} = \overline{\theta(Q)}$ so that the mapping is well defined. If D is a central projection in N, there exists a core-free projection $R \in N$ such that $R - Z' \in [N, N]$ for a self-adjoint $Z' \in Z_N$ and $\bar{R} = D$ by Lemma 10. There exists in M a core-free projection P with $P - Z \in [M, M]$ and $\theta(P) = R$. Hence $\psi(\bar{P}) = \overline{\theta(P)} = \bar{R} = D$ so that ψ is onto. If C, D are central projections in M with $CD = 0$ let $\bar{P} = C$, $\bar{Q} = D$ where $P - Z,\ Q - Z' \in [M, M]$ for self-adjoint central $Z, Z' \in Z_M$. Then

$$CD = 0 \Leftrightarrow \bar{P}\bar{Q} = 0 \Leftrightarrow P\|Q \Leftrightarrow \theta(P)\|\theta(Q) \Leftrightarrow \overline{\theta(P)}\,\overline{\theta(Q)} = 0$$
$$\Leftrightarrow \psi(C)\psi(D) = 0.$$

Thus ψ is a projection orthoisomorphism of Z_M onto Z_N and implements a *-isomorphism, also denoted ψ, of Z_M onto Z_N.

Let C be a central projection in M, P a projection such that $P - Z \in [M, M]$ for some self-adjoint $Z \in Z_M$. There exists a core-free projection Q in M such that $\bar{Q} = C(I - \bar{P})$ and $Q - Z' \in [M, M]$ for some self-adjoint $Z' \in Z_M$. $PC + Q$ has carrier C and $(P - Z)C + Q - Z' \in [M, M]$. (Note that if $X \in [M, M]$ and C is a central projection in M, $CX \in [M, M]$). Hence $\psi(C) = \overline{\theta(PC + Q)} = \overline{\theta(PC)} + \overline{\theta(Q)}$ since $PC\|Q$. Moreover, since PC and $P(I - C)$ are $\|$, $\theta(P) = \theta(PC + P(I - C)) = \theta(PC) + \theta(P(I - C))$. Both $\theta(PC)$ and $\theta(P(I - C))$ are $\|$ to $\theta(Q)$ since Q is $\|$ to P. Multiplying these relations we have $\psi(C)\,\theta(P) = \theta(P) = \theta(CP)$.

Definition. Let P, Q be projections in a von Neumann algebra M. If $PQ = 0$ we say P is orthogonal to Q written $P \perp Q$. If $(I - P)(I - Q) = 0$ we say P is co-orthogonal to Q, written P co $\perp$ Q.

LEMMA 11. *Let $P_1, \ldots, P_n$ be commuting core-free projections, each pair of which satisfy the identity of Lemma 3. Then there exists a central projection C such that the P_i are $\perp$ on C,* co $\perp$ *on $I - C$.*

Proof. This is essentially [**5**, Lemma 11].

LEMMA 12. *Let M and N be von Neumann algebras with no central abelian summands, and let $P_1, \ldots, P_n$ be mutually $\perp$ projections in M with $P_i - Z_i \in [M, M]$ for self-adjoint $Z_i \in Z_M$. There exists a projection $D \in Z_M$ such that the $\theta(P_iD)$ are mutually $\perp$, the $\theta'(P_i(I - D))$ are mutually $\perp$.*

Proof. The proof is similar to [**5**, Corollary to Lemma 11].

3. The I_2 case. Suppose now that M is of type I_2, N has no central abelian summands, and $\phi: [M, M] \to [N, N]$ is a Lie *-isomorphism onto. The I_2 case is isolated because the method of proof for the non-I_2 case requires the choice of three particular non-zero projections and this choice cannot be made if M is of type I_2. Let P_1, P_2 be $\perp$, equivalent, abelian projections such that $P_1 + P_2 = I$. By [**6**], $P_1 - P_1^{\#},\ P_2 - P_2^{\#} \in [M, M]$. We have $\tilde{P}_i = 0$, $\bar{P}_i = I$, and, by [**5**, Lemma 1], $\theta(P_1) \perp \theta(P_2)$ since $\theta(P_1) + \theta(P_2) \in Z_N$. Moreover $I = \psi(I) = \psi(\bar{P}_1) = \overline{\theta(P_1)} \leqq \theta(P_1) + \theta(P_2) \leqq I$. Hence $\theta(P_i) + \theta(P_2) = I$. For notation let $M_{ij} = P_iMP_j$, $N_{ij} = \theta(P_i)N\theta(P_j)$.

LEMMA 13. *N_{ii} $(i = 1, 2)$ is abelian.*

Proof. We first show that $N_{11} \cap [N, N]$ is abelian. Suppose $Y \in N_{11} \cap [N,N]$ and let $X \in [M, M]$ be such that $\phi(X) = Y$. We have $0 = [Y, \theta(P_2)] = [\phi(X), \phi(P_2 - P_2^{\#})]$. Applying ϕ^{-1} we have $0 = [X, P_2]$. This implies $X \in M_{11} + M_{22}$. If $Y, Y' \in N_{11} \cap [N, N]$ and X, X' are such that $\phi(X) = Y$, $\phi(X') = Y'$ then, by the above, $X, X' \in M_{11} + M_{22}$ which is abelian. Hence $0 = [X, X']$ implies $0 = [Y, Y']$.

$N_{11} \cap [N, N]$ is an abelian Lie *-ideal in N_{11} and so by [**5**, Lemma 36], $N_{11} \cap [N, N] \subseteq Z_{N_{11}}$, the center of N_{11}. This implies that

$$[N_{11}, N_{11}] \subseteq N_{11} \cap [N, N] \subseteq Z_{N_{11}}.$$

By [**5**, Lemma 6], $[N_{11}, N_{11}] = 0$.

COROLLARY. *N has no continuous part.*

Proof. Let C be a non-zero central projection in N such that N_C is continuous. $C = C\theta(P_1) + C\theta(P_2)$ and one of $C\theta(P_1)$, $C\theta(P_2)$ is nonzero. We also have that $N_{C\theta(P_i)} \subseteq N_{11}$, $N_{C\theta(P_2)} \subseteq N_{22}$. Thus $N_{C\theta(P_1)}$ and $N_{C\theta(P_2)}$ are abelian and one is nonzero. But C can have no discrete projections contained in it [**1**, p. 125, Proposition 4] a contradiction.

THEOREM 2. *Let $\phi:[M, M] \to [N, N]$ be a Lie *-isomorphism where M is of type I_2 and N has no abelian summands. There exists an extension σ of ϕ to a *-isomorphism of M onto N.*

Proof. We first extend ϕ to $\bar{\phi}$, a near-isomorphism of M to N (see [**5**, p. 722]). If $A \in M$, then by [**6**, Theorem 1] there exists a unique central element, namely $A^{\#}$, such that $A - A^{\#} \in [M, M]$. Define $\bar{\phi}(A) = \phi(A - A^{\#}) + \psi(A^{\#})$. Since $A^{\#}$ is unique the mapping is well defined.

Obviously $\bar{\phi}$ is a *-linear map from M into $[N, N] + Z_N$. If $X \in [M, M]$ then $X^{\#} = 0$ so that $\bar{\phi}(X) = \phi(X)$.

$$\bar{\phi}[A, B] = \phi[A, B] = \phi[A - A^{\#}, B - B^{\#}] = [\phi(A - A^{\#}), \phi(B - B^{\#})] = [\phi(A - A^{\#}) + \psi(A^{\#}), \phi(B - B^{\#}) + \psi(B^{\#})] = [\bar{\phi}(A), \bar{\phi}(B)]$$

so that $\bar{\phi}$ preserves brackets. If $\bar{\phi}(A) = \bar{\phi}(B)$ then $\phi(A - B - (A^{\#} - B^{\#})) \in Z_N \cap [N, N]$ so that $A - B - (A^{\#} - B^{\#}) \in Z_M$. This shows $A - B \in Z_M$. If $B + Z' \in [N, N] + Z_N$ there exists $A \in [N, N]$, $Z \in Z_M$ with $\phi(A) = B$, $\psi(Z) = Z'$. Then $\bar{\phi}(A + Z) = \phi(A) + \psi(Z) = B + Z'$ so that $\bar{\phi}$ is onto $[N, N] + Z_N$. By the Corollary to Lemma 13, N has no continuous part so that by [**6**], and [**8**], $[N, N] + Z_N = N$.

Applying [**5**, Theorem 2] to the near isomorphism $\bar{\phi}: M \to N$ we have $\bar{\phi} = \sigma + \tau$ where σ is an associative *-isomorphism of M onto N and τ is a *-linear map which annihilates $[M, M]$. If $A \in [M, M]$, $\phi(A) = \bar{\phi}(A) = \sigma(A) + \tau(A) = \sigma(A)$.

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4. The non-I_2 case. Let $\phi : [M, M] — [N, N]$ be a Lie *-isomorphism where M and N have no abelian summands and M is not of type I_2 (M may have a type I_2 summand). We wish to employ techniques of [**4**], but in order to do this we must make a particular choice of three projections.

LEMMA 14. *There exist projections P_1, P_2, P_3 in M such that $\sum P_i = I$, $\bar{P}_1 = \bar{P}_2 = I$, $P_1 \backsim P_2$, $I - \bar{P}_3$ is the I_2-summand, $I - \bar{P}_3 \leqq P_1 + P_2$, $P_1(I - \bar{P}_3)$ and $P_2(I - \bar{P}_3)$ are the equivalent, $\perp$, abelian projections comprising $I - \bar{P}_3$, and there exist central self-adjoint elements Z_i, $i = 1, 2, 3$, such that $P_i - Z_i \in [M, M]$. Moreover we have $\bar{P}_3 P_i M P_j = \bar{P}_3 P_i M P_k M P_j$ for $i, j, k \in \{1, 2, 3\}$.*

Proof. Let $C_n^{(1)}$ be the I_n part of M ($n \geqq 2$), $C^{(2)}$ the II_1 part, and $C^{(3)}$ the infinite part. $C_n^{(1)}$ is the sum of n equivalent (abelian) projections $P_1^{(n)}, \ldots, P_n^{(n)}$. If n is even ($n \geqq 4$) let

$$Q_1^{(n)} = \sum_{i=1}^{(n-2)/2} P_i^{(n)}, \qquad Q_2^{(n)} = \sum_{i=n/2}^{n-2} P_i^{(n)}, \qquad Q_3^{(n)} = \sum_{i=n-1}^{n} P_i^{(n)}.$$

If n is odd let

$$Q_1^{(n)} = \sum_{i=1}^{(n-1)/2} P_i^{(n)}, \qquad Q_2^{(n)} = \sum_{i=(n+1)/2}^{n-1} P_i^{(n)}, \qquad Q_3^{(n)} = P_n^{(n)}.$$

Moreover, by [**6**], there exist central self-adjoint elements T_1, T_2, T_3 in $M_{C^{(1)}}$ where $C^{(1)} = \sum_{n=1}^{\infty} C_n^{(1)}$ such that

$$\sum_{n=1}^{\infty} Q_1^{(n)} - T_1, \qquad \sum_{n=1}^{\infty} Q_2^{(n)} - T_2, \qquad \sum_{n=1}^{\infty} Q_3^{(n)} - T_3 \in [M, M].$$

$C^{(2)} = \sum_{i=1}^{4} D_i$ where $D_i \backsim D_j$. If $VV^* = D_1 + D_2$, $V^*V = D_3 + D_4$ then $[V, V^*] = C^{(2)} - (D_3 + D_4) \in [M_{C^{(2)}}, M_{C^{(2)}}] \subseteq [M, M]$. Since $D_3 \sim D_4$, $D_3 - D_4 \in [M, M]$ which implies $D_4 - \frac{1}{2}C^{(2)} \in [M, M]$. The same argument holds for D_1, D_2, D_3. Similarly $C^{(3)} = \sum_{i=1}^{4} E_i$ with $E_i \backsim E_j$ and $E_i \in [M, M]$ by [**8**].

Let

$$P_1 = P_1^{(2)} + \sum_{n=3}^{\infty} Q_1^{(n)} + D_1 + E_1,$$

$$P_2 = P_2^{(2)} + \sum_{n=3}^{\infty} Q_2^{(n)} + D_2 + E_2,$$

$$P_3 = \sum_{n=3}^{\infty} Q_3^{(n)} + D_3 + D_4 + E_3 + E_4.$$

All assertions except the last are clear. If $P \backsim Q \backsim R$ with $VV^* = Q$ $V^*V = R$ then $PXQ = PXVRV^*Q$ so that $PMQ = PMRMQ$. We apply this technique to each I_n summand ($n \geqq 3$), to the II_1 summand, and to the infinite summand. For example, examine $C_4^{(1)}$. $C_4^{(1)} = Q_1^{(4)} + Q_2^{(4)} + Q_3^{(4)}$ where $Q_3^{(4)} =$

$P_3^{(4)} + P_4^{(4)}$ and $Q_1^{(4)} \backsim Q_2^{(4)} \backsim P_3^{(4)} \backsim P_4^{(4)}$. We prove a few representative cases:

(i) $Q_1^{(4)}MQ_2^{(4)} = Q_1^{(4)}MQ_3^{(4)}MQ_2^{(4)}$. For,
$Q_1^{(4)}XQ_2^{(4)} = Q_1^{(4)}XVP_3^{(4)}Q_3^{(4)}V^*Q_2^{(4)}$ where $V^*V = P_3^{(4)}$, $VV^* = Q_2^{(4)}$.

(ii) $Q_1^{(4)}MQ_3^{(4)} = Q_1^{(4)}MQ_2^{(4)}MQ_3^{(4)}$. For,

$$Q_1^{(4)}XQ_3^{(4)} = Q_1^{(4)}XP_3^{(4)} + Q_1^{(4)}XP_4^{(4)} = Q_1^{(4)}XVP_2^{(4)}V^*P_3^{(4)}Q_3^{(4)} + Q_1^{(4)}XWP_2^{(4)}W^*P_4^{(4)}Q_3^{(4)}$$

where

$$V^*V = P_2^{(4)},\ VV^* = P_3^{(4)},\ W^*W = P_2^{(4)},\ WW^* = P_4^{(4)}.$$

(iii) $Q_3^{(4)}MQ_3^{(4)} = Q_3^{(4)}MQ_1^{(4)}MQ_3^{(4)}$. For,

$$Q_3^{(4)}XQ_3^{(4)} = P_3^{(4)}XP_3^{(4)} + P_3^{(4)}XP_4^{(4)} + P_4^{(4)}XP_3^{(4)} + P_4^{(4)}XP_4^{(4)} =$$

$$Q_3^{(4)}P_3^{(4)}XVP_1^{(4)}V^*P_3^{(4)}Q_3^{(4)} + Q_3^{(4)}P_3^{(4)}XWP_1^{(4)}W^*P_4^{(4)}Q_3^{(4)} + Q_4^{(4)}P_4^{(4)}XVP_1^{(4)}V^*P_3^{(4)}Q_3^{(4)} + P_4^{(4)}XWP_1^{(4)}W^*P_4^{(4)}Q_3^{(4)}$$

where

$$V^*V = P_1^{(4)},\ VV^* = P_3^{(4)},\ W^*W = P_1^{(4)},\ WW^* = P_4^{(4)}.$$

Similar arguments work in the other cases.

Let P_i, $i = 1, 2, 3$, be as in Lemma 14 and let $Q_1 = P_1(I - \bar{P}_3)$, $Q_2 = P_2(I - \bar{P}_3)$, $Q_3 = P_1\bar{P}_3$, $Q_4 = P_2\bar{P}_3$, $Q_5 = P_3$. By Lemma 12 there exists a central projection $D \in M$ such that the $\theta(Q_iD)$ are $\perp$ and the $\theta'(Q_i(I - D)$ are $\perp$ for $i = 3, 4, 5$. (Note that $Q_iD - Z_iD \in [M_D, M_D] \subseteq [M, M]$.)

LEMMA 15. *$\theta(Q_1) \perp \theta(Q_2)$ and $\theta(Q_1) + \theta(Q_2) = \psi(I - \bar{P}_3)$.*

Proof. $Q_1 - Z_1(I - \bar{P}_3)$, $Q_2 - Z_2(I - \bar{P}_3) \in [M, M]$ and $Q_1 + Q_2 = I - \bar{P}_3$. Hence $Q_1 - Z_1(I - \bar{P}_3) + Q_2 - Z_2(I - \bar{P}_3) \in Z_N \cap [M, M]$. This implies $\phi(Q_1 - Z_1(I - \bar{P}_3) + Q_2 - Z_2(I - \bar{P}_3)) \in Z_N$. Hence $\theta(Q_1) + \theta(Q_2) \in Z_N$. As before this implies $\theta(Q_1) \perp \theta(Q_2)$ since they are core-free.

$\theta(Q_i) = \theta(P_i)\psi(I - \bar{P}_3)$ so that $\theta(Q_i) \leqq \psi(I - \bar{P}_3)$, $i = 1, 2$. $\psi(I - \bar{P}_3) = \psi(\bar{Q}_1) = \overline{\theta(Q_1)} \leqq \theta(Q_1) + Q(Q_2) \leqq \psi(I - \bar{P}_3)$.

COROLLARY. $\theta'(Q_1) = \theta(Q_2)$.

Proof. $\theta'(Q_1) = \overline{\theta(Q_1)} - \theta(Q_1) = \psi(I - \bar{P}_3) - \theta(Q_1) = \theta(Q_2)$.

For notation let $M_{ij} = Q_iMQ_j$, $N_{ij} = \theta(Q_i)M\theta(Q_j)$ for $i, j \in \{1, 2\}$, and let $M_{ij} = Q_iDMQ_jD$, $\tilde{M}_{ij} = Q_i(I - D)MQ_j(I - D)$, $N_{ij} = \theta(Q_iD)N\theta(\theta_jD)$, and $\tilde{N}_{ij} = \theta'(Q_i(I - D))N\theta'(Q_j(I - D))$ for $i, j \in \{3, 4, 5\}$. Notice that if $X_{ij} \in M_{ij}$ $(i \neq j)$ then $X_{ij} = [X_{ij}, Q_j] \in [M, M]$.

LEMMA 16. $\phi^{-1}((\sum_{i=1}^5 N_{ii} + \sum_{i=3}^5 \tilde{N}_{ii}) \cap [N, N]) = (\sum_{i=1}^5 M_{ii} + \sum_{i=3}^5 \tilde{M}_{ii}) \cap [M, M]$.

Proof. See [**5**, Lemma 26]. Note that $Z_M \subseteq \sum_{i=1}^5 M_{ii} + \sum_{i=3}^5 \tilde{M}_{ii}$.

LEMMA 17. $\phi^{-1}(N_{ij}) = M_{ij}$, $\phi^{-1}(\tilde{N}_{ij}) = \tilde{M}_{ij}$ *if* $i \neq j$.

Proof. See [**5**, Lemma 27].

LEMMA 18. $\sum_{i=3}^{5}\theta(Q_iD) = \psi(D\bar{P}_3)$, $\sum_{i=3}^{5}\theta'(Q_i(I - D)) = \psi((I - D)\bar{P}_3)$.

Proof. In [**5**, Lemma 13] replace D by $D\bar{P}_3$, and the result follows.

LEMMA 19. $\phi((Z_{M_{11}} + Z_M) \cap [M, M]) \subseteq (N_{11} + Z_N) \cap [N, N]$.

Proof. If $A \in (Z_{M_{11}} + Z_M) \cap [M, M]$ then $[A, X] = 0$ for all X in

$$\sum_{i\neq j;\, i,j\geqq 2} M_{ij} + \sum_{i=1}^{5} M_{ii} + \sum_{i\neq j;\, i,j\geqq 3} \tilde{M}_{ij} + \sum_{i\geqq 3} \tilde{M}_{ii}.$$

Hence $[\phi(A), X] = 0$ for all X in

$$\sum_{i\neq j;\, i,j\geqq 2} N_{ij} + \left(\sum_{i=2}^{5} N_{ii} + \sum_{i=3}^{5} \tilde{N}_{ii}\right) \cap [N, N] + \sum_{i\neq j;\, i,j\geqq 3} \tilde{N}_{ij}$$

$$= \left(\sum_{i\neq j;\, i,j\geqq 2} N_{ij} + \left(\sum_{i=2}^{5} N_{ii} + \sum_{i=3}^{5} \tilde{N}_{ii}\right) + \sum_{i\neq j;\, i,j\geqq 3} \tilde{N}_{ij}\right) \cap [N, N]$$

$$= \left\{STS | T \in N,\, S = \theta(Q_2) + \sum_{i=3}^{5} \theta(Q_iD) + \sum_{i=3}^{5} \theta'(Q_i(I - D))\right\} \cap [N, N]$$

$$= N_S \cap [N, N].$$

(Note that by Lemmas 15 and 18, $S = I - \theta(Q_1)$.) In particular $[\phi(A), X] = 0$ for all X in $N_S \cap [N_S, N_S] = [N_S, N_S]$. Since $A \in \sum_{i=1}^{5} M_{ii} + \sum_{i=3}^{5}\tilde{M}_{ii}$, $\phi(A) = B_1 + C$ where $B_1 \in N_{11}$ and $C \in N_S$ by Lemma 16. Thus $0 = [\phi(A), X] = [B_1 + C, X] = [C, X]$ for all X in $[N_S, N_S]$. By [**3**, Sublemma, p. 5] this implies $[C, X] = 0$ for all X in N_S, or that $C \in Z_{N_S} = Z_S$. Since $S = I - \theta(Q_1)$ we have $C = Z(I - \theta(Q_1))$. Finally, $\phi(A) = B_1 + C = B_1 - \theta(Q_1)Z + Z \in (N_{11} + Z_N) \cap [N, N]$.

COROLLARY. $\phi((M_{11} + Z_M) \cap [M, M]) \subseteq (N_{11} + Z_N) \cap [N, N]$.

Proof. M_{11} is abelian since Q_1 is an abelian projection. Hence $M_{11} \subseteq Z_{M_{11}}$.

We now extend $\phi | [M_{I-\bar{P}_3}, M_{I-\bar{P}_3}]$ to a Lie *-isomorphism of ϕ of $\sum_{1\leqq i,j\leqq 2} M_{ij}$ into N, and then analyze ϕ. We cannot proceed exactly as in Theorem 2 because of a lack of information about the image of $\sum_{1\leqq i,j\leqq 2} M_{ij}$ under ϕ.

If $A \in \sum_{1\leqq i,j\leqq 2} M_{ij}$ define $\bar{\phi}(A) = \phi(A - A^{\#}) + \psi(A^{\#})$. This is well defined since $M_{(I-\bar{P}_3)}$ is finite. If $A \in M_{ij}$, $(i, j) = (1, 2)$ or $(2, 1)$ then $A^{\#} = 0$ and $\bar{\phi}(A) = \phi(A)$. If $A \in M_{ii}$ then $A - A^{\#} \in (M_{ii} + Z_M) \cap [M, M]$ by [**6**, Theorem 1] and by Lemma 19, $\bar{\phi}(A) = \phi(A - A^{\#}) + \psi(A^{\#}) \in N_{11} + Z_M$. $\bar{\phi}$ is obviously *-linear. If $\bar{\phi}(A) = 0$ then $\phi(A - A^{\#}) \in Z_N \cap [N, N]$ so that $A - A^{\#} \in Z_M \cap [M, M]$. Thus $A \in Z_M$ so that $A = A^{\#}$ and $0 = \bar{\phi}(A) = \phi(A - A^{\#}) + \psi(A^{\#}) = 0 + \psi(A^{\#})$. Hence $A^{\#} = 0$. This shows $\bar{\phi}$ is 1-1. $\bar{\phi}$ preserves brackets as in Theorem 2.

Defining mappings σ_0 and λ_0 as follows: if $A \in M_{ij}$, $(i, j) = (1, 2)$ or $(2, 1)$ let $\sigma_0(A) = \bar{\phi}(A) = \phi(A)$. If $A \in M_{ii}$ $(i = 1, 2)$ then $\bar{\phi}(A) = \sigma_0(A) + \lambda_0(A)$

where $\sigma_0(A) \in N_{ii}$, $\lambda_0(A) \in Z_N$. σ_0 and λ_0 are well defined for if $\sigma_0(A) + \lambda_0(A) = \sigma_0(B) + \lambda_0(B)$ then $\sigma_0(A) - \sigma_0(B) \in N_{ii} \cap Z_N = \{0\}$. σ_0 and λ_0 can be shown to be *-linear maps with $\sigma_0(AB) = \sigma_0(A)\,\sigma_0(B)$ for all $A, B \in M_{I-P_3}$ as in [**5**, Lemmas 18-22].

LEMMA 20. *σ_0 extends $\phi|[M_{I-\bar{P}_3}, M_{I-\bar{P}_3}]$ to a *-homomorphism of $M_{I-\bar{P}_3}$ into N.*

Proof. We show that λ_0 annihilates brackets of elements in $M_{(Q_1+Q_2)}$. $\lambda_0[A, B] = \bar{\phi}[A, B] - \sigma_0[A, B] = [\bar{\phi}(A), \bar{\phi}(B)] - [\sigma_0(A), \sigma_0(B)] = [\sigma_0(A) + \lambda_0(A), \sigma_0(B) + \lambda_0(B)] - [\sigma_0(A), \sigma_0(B)] = 0$ since $\lambda_0(A) \in Z_N$. Hence $\phi[A, B] = \bar{\phi}[A, B] = \sigma_0[A, B] + \lambda[A, B] = \sigma_0[A, B]$.

We turn our attention to $M_{\bar{P}_3}$. By Lemma 14, $Q_iMQ_i = Q_iMQ_kMQ_j$ for $i, j, k \in \{3, 4, 5\}$ so that we also have $Q_iDMQ_jD = Q_iDMQ_kDMQ_jD$ for $i, j, k \in \{3, 4, 5\}$. A similar relation will hold with D replaced by $I - D$.

LEMMA 21. *Let (i, j, k) be any permutation of $(3, 4, 5)$. If $X_{ij} \in M_{ij}$, $X_{jk} \in M_{jk}$ then $\phi(X_{ij}X_{jk}) = \phi(X_{ij})\,\phi(X_{jk})$. If $X_{ij} \in \tilde{M}_{ij}$, $X_{jk} \in \tilde{M}_{jk}$ then $\phi(X_{ij}X_{jk}) = -\phi(X_{jk})\,\phi(X_{ij})$.*

Proof. If $i \neq j$ and $X_{ij} \in M_{ij}$ then $\phi(X_{ij}) \in N_{ij}$ by Lemma 17. Hence $\phi(X_{ij}X_{jk}) = \phi[X_{ij}, X_{jk}] = [\phi(X_{ij}), \phi(X_{jk})] = \phi(X_{ij})\,\phi(X_{jk})$. If $i \neq j$ and $X_{ij} \in \tilde{M}_{ij}$ then $\phi(X_{ij}) \in \tilde{N}_{ji}$. $\phi(X_{ij}X_{jk}) = \phi[X_{ij}, X_{jk}] = [\phi(X_{ij}), \phi(X_{jk})] = -\phi(X_{jk})\,\phi(X_{ij})$.

LEMMA 22. *ϕ is a homomorphism from the algebra generated algebraically by $M_{ij} + M_{jk} + M_{ik}$ into the one generated algebraically by $N_{ij} + N_{jk} + N_{ik}$, and the negative of an anti-homomorphism of the algebra generated algebraically by $\tilde{M}_{ij} + \tilde{M}_{jk} + \tilde{M}_{ik}$ into the one generated algebraically by $\tilde{N}_{ji} + \tilde{N}_{kj} + \tilde{N}_{ki}$, where (i, j, k) is a permutation of $(3, 4, 5)$.*

Proof. It suffices to let $(i, j, k) = (3, 4, 5)$. If $X_{34} \in M_{34}$, $X_{45} \in M_{45}$, then by Lemma 21, $\phi(X_{34}X_{45}) = \phi(X_{34})\,\phi(X_{45})$. In all other cases $0 = \phi(0) = \phi(X_{ij}X_{kl}) = \phi(X_{ij})\,\phi(X_{kl})$ by Lemma 17.

For the other part, if $\tilde{X}_{34} \in \tilde{M}_{34}$, $\tilde{X}_{45} \in \tilde{M}_{45}$ then $\phi(\tilde{X}_{34}\tilde{X}_{45}) = -\phi(\tilde{X}_{45})\,\phi(\tilde{X}_{34})$ by Lemma 21. In all other cases $0 = \phi(0) = \phi(\tilde{X}_{ij}\tilde{X}_{kl}) = -\phi(\tilde{X}_{kl})\,\phi(\tilde{X}_{ij})$ by Lemma 17.

LEMMA 23. *A von Neumann algebra M is generated algebraically by $[M, M]$ if and only if M has no abelian summands.*

Proof. By [**6**], $[M, M]$ is the set of all finite sums of niloptent operators of index two. By [**2**], M is algebraically generated by nilpotents of index two if and only if M has no abelian summands.

LEMMA 24. *$[M_{\bar{P}_3}, M_{\bar{P}_3}]$ is linearly generated by M_{ij}, $\tilde{M}_{ij}$, $[M_{ij}, M_{ji}]$, and $[\tilde{M}_{ij}, \tilde{M}_{ji}]$ for $i \neq j$, $i, j \in \{3, 4, 5\}$. $[M_{I-\bar{P}_3}, M_{I-\bar{P}_3}]$ is linearly generated by M_{ij} and $[M_{ij}, M_{ji}]$, $i \neq j$, $i, j \in \{1, 2\}$.*

Proof. $[M_{\bar{P}_3}, M_{\bar{P}_3}] = [M_{\bar{P}_3 D}, M_{\bar{P}_3 D}] + [M_{\bar{P}_3(I-D)}, M_{\bar{P}_3(I-D)}]$.

$$[M_{\bar{P}_3 D}, M_{\bar{P}_3 D}] = \left[\sum_{3 \leqq i,j \leqq 5} M_{ij}, \sum_{3 \leqq i,j \leqq 5} M_{ij}\right]$$

$$= \sum_{i \neq j; 3 \leqq i,j \leqq 5} M_{ij} + \sum_{i \neq j; 3 \leqq i,j \leqq 5} [M_{ij}, M_{ji}] + \sum_{i=3}^{5} [M_{ii}, M_{ii}].$$

It suffices to show that $[M_{33}, M_{33}] \subseteq [M_{34}, M_{43}]$. $M_{33} = Q_3 D M Q_3 D = Q_3 D M Q_4 D M Q_3 D$. If $A, B \in M_{33}$ then $A = Q_3 D A Q_3 D = Q_3 D A V O_4 D V^* Q_3 D$ and $B = Q_3 D B Q_3 D$. Thus $[A, B] = [Q_3 D X Q_4 D Y Q_3 D, Q_3 D B Q_3 D]$ (for appropriate X, Y) =
$[Q_3 D X Q_4 D, Q_4 D Y Q_3 D B Q_3 D] - [Q_3 D B Q_3 D X Q_4 D, Q_4 D Y Q_3 D] \in [M_{34}, M_{43}]$.
The other parts of the lemma are proved similarly.

COROLLARY. *$[N, N]$ is linearly generated by N_{ij}, $\tilde{N}_{ij}$, $[N_{ij}, N_{ji}]$, and $[\tilde{N}_{ij}, \tilde{N}_{ji}]$ for $i \neq j$.*

LEMMA 25. *If $X_{ij}, Y_{ij} \in M_{ij}$, $X_{ji} \in M_{ji}$ then $\phi(X_{ij} X_{ji} Y_{ij}) = \phi(X_{ij}) \phi(X_{ji}) \phi(Y_{ij})$ for $i \neq j$, $i, j \in \{3, 4, 5\}$. If $X_{ij}, Y_{ij} \in \tilde{M}_{ij}$, $X_{ji} \in \tilde{M}_{ji}$ then $\phi(X_{ij} X_{ji} Y_{ij}) = \phi(Y_{ij}) \phi(X_{ji}) \phi(X_{ij})$ for $i \neq j$, $i, j \in \{3, 4, 5\}$.*

Proof. Let $X_{34}, Y_{34} \in M_{34}$, $X_{43} \in M_{43}$. We will show that $[\phi(X_{34} X_{43} Y_{34}) - \phi(X_{34}) \phi(X_{43}) \phi(Y_{34})] [N, N] = 0$. This will imply the result by Lemma 23. By the Corollary to Lemma 24 it suffices to show that

(1) $[\phi(X_{34} X_{43} Y_{34}) - \phi(X_{34}) \phi(X_{43}) \phi(Y_{34})] \phi(X_{ij}) = 0$ for $i \neq j$ and $X_{ij} \in M_{ij}$ or $\tilde{M}_{ij}$. Since, by Lemma 17, both $\phi(X_{34} X_{43} Y_{34})$ and $\phi(X_{34}) \phi(X_{43}) \phi(Y_{34})$ are in N_{34}, (1) will be true if $i \neq 4$ and $X_{ij} \in M_{ij}$ or if $X_{ij} \in \tilde{M}_{ij}$ for $i \neq j$. We need only check X_{43} and X_{45}. (Note that $X_{41} = 0$ since $Q_4 \leqq \bar{P}_3$, $Q_1 \leqq I - \bar{P}_3$).

$$\text{(2)} \qquad \phi(X_{34} X_{43} Y_{34}) \phi(X_{45}) - \phi(X_{34}) \phi(X_{43}) \phi(Y_{34}) \phi(X_{45})$$
$$= \text{(by Lemma 21)}$$
$$\phi(X_{34} X_{43} Y_{34} X_{45}) - \phi(X_{34}) \phi(X_{43}) \phi(Y_{34} X_{45})$$
$$= \text{(by Lemma 21)}$$
$$\phi(X_{34} X_{43} Y_{34} X_{45}) - \phi(X_{34}) \phi(X_{43} Y_{34} X_{45})$$
$$= \text{(by Lemma 21)}$$
$$\phi(X_{34} X_{43} Y_{34} X_{45}) - \phi(X_{34} X_{43} Y_{34} X_{45}) = 0.$$

As for X_{43}, we can write $X_{43} = \sum_{i=1}^{n} X_{45}{}^{(i)} X_{53}{}^{(i)}$ by Lemma 14. We have

$$\phi(X_{43}) = \sum_{i=1}^{n} \phi(X_{45}{}^{(i)}) \phi(X_{53}{}^{(i)})$$

by Lemma 21. By the preceding argument we have (1) if $(i, j) = (4, 3)$.

The second statement is proved similarly. For example if $\tilde{X}_{34}$, $\tilde{Y}_{34} \in \tilde{M}_{34}$, $\tilde{X}_{43} \in \tilde{M}_{43}$ and $\tilde{X}_{53} \in \tilde{M}_{53}$ then

$$\begin{aligned} &\phi(\tilde{X}_{34}\tilde{X}_{43}\tilde{Y}_{34})\,\phi(\tilde{X}_{53}) - \phi(\tilde{Y}_{34})\,\phi(\tilde{X}_{43})\,\phi(\tilde{X}_{34})\,\phi(\tilde{X}_{53}) \\ &= -\,\phi(\tilde{X}_{53}\tilde{X}_{34}\tilde{X}_{43}\tilde{Y}_{34}) + \phi(\tilde{Y}_{34})\,\phi(X_{43})\,\phi(\tilde{X}_{53}\tilde{X}_{34}) \\ &= -\,\phi(\tilde{X}_{53}\tilde{X}_{34}\tilde{X}_{43}\tilde{Y}_{34}) - \phi(\tilde{Y}_{34})\,\phi(\tilde{X}_{53}\tilde{X}_{34}\tilde{X}_{43}) \\ &= -\,\phi(\tilde{X}_{53}\tilde{X}_{34}\tilde{X}_{43}\tilde{Y}_{34}) + \phi(\tilde{X}_{53}\tilde{X}_{34}\tilde{X}_{43}\tilde{Y}_{34}) = 0. \end{aligned}$$

LEMMA 26. *Let* (i, j, k) *be any permutation of* $(3, 4, 5)$. *If*

$$\sum_{s=1}^{n} X_{ij}^{(s)}X_{ji}^{(s)} = \sum_{t=1}^{m} X_{ik}^{(t)}X_{ki}^{(t)}$$

where $X_{ij} \in M_{ij}$ *then*

$$\sum_{s=1}^{n} \phi(X_{ij}^{(s)})\phi(X_{ji}^{(s)}) = \sum_{t=1}^{m} \phi(X_{ik}^{(t)})\phi(X_{ki}^{(t)}).$$

If

$$\sum_{s=1}^{n} \tilde{X}_{ij}^{(s)}\tilde{X}_{ji}^{(s)} = \sum_{t=1}^{m} \tilde{X}_{ik}^{(t)}\tilde{X}_{ki}^{(t)}$$

where $\tilde{X}_{ij} \in \tilde{M}_{ij}$ *then*

$$\sum_{s=1}^{n} \phi(\tilde{X}_{ji}^{(s)})\phi(\tilde{X}_{ij}^{(s)}) = \sum_{t=1}^{m} \phi(\tilde{X}_{ki}^{(t)})\phi(\tilde{X}_{ik}^{(t)}).$$

Proof. We prove the second statement. The proof of the first is similar. Let $(i, j, k) = (3, 4, 5)$. We show that

$$\left(\sum_{s=1}^{n} \phi(\tilde{X}_{43}^{(s)})\phi(\tilde{X}_{34}^{(s)}) - \sum_{t=1}^{m} \phi(\tilde{X}_{53}^{(t)})\phi({}_{35}\tilde{X}^{(5)})\right)[N, N] = 0. \tag{1}$$

As before, we check elements of $[N, N]$ of the form $\phi(Y_{ij})$, $i \neq j$ where $Y_{ij} \in M_{ij}$ or $\tilde{M}_{ij}$. Since $\tilde{X}_{34}^{(s)} \in \tilde{M}_{34}$, $\phi(\tilde{X}_{34}^{(s)}) \in \tilde{N}_{43}$ and similarly $\phi(\tilde{X}_{35}^{(t)}) \in \tilde{N}_{53}$, (1) will hold if $Y_{ij} \in M_{ij}$ $i \neq j$ or if $Y_{ij} \in \tilde{M}_{ij}$ with $j \neq 3$. We need only check the cases $\tilde{Y}_{ij} \in \tilde{M}_{ij}$ for $(i, j) = (4, 3)$ or $(5, 3)$.

$$\begin{aligned} &\sum_{s=1}^{n} \phi(\tilde{X}_{43}^{(s)})\phi(\tilde{X}_{34}^{(s)})\phi(\tilde{Y}_{43}) - \sum_{t=1}^{m} \phi(\tilde{X}_{53}^{(t)})\phi(\tilde{X}_{35}^{(t)})\phi(\tilde{Y}_{43}) \\ &\quad= \sum_{s=1}^{n} \phi(\tilde{Y}_{43}\tilde{X}_{34}^{(s)}\tilde{X}_{43}^{(s)}) + \sum_{t=1}^{m} \phi(\tilde{X}_{53}^{(t)})\phi(\tilde{Y}_{43}\tilde{X}_{35}^{(t)}), \text{ by Lemmas 21, 25} \\ &\quad= \sum_{s=1}^{n} \phi(\tilde{Y}_{43}\tilde{X}_{34}^{(s)}\tilde{X}_{43}^{(s)}) - \sum_{t=1}^{m} \phi(\tilde{Y}_{43}\tilde{X}_{35}^{(t)}\tilde{X}_{53}^{(t)}), \text{ by Lemma 21} \\ &\quad= \phi\left(\sum_{s=1}^{n} \tilde{Y}_{43}\tilde{X}_{34}^{(s)}\tilde{X}_{43}^{(s)} - \sum_{t=1}^{m} \tilde{Y}_{43}\tilde{X}_{35}^{(t)}\tilde{X}_{53}^{(t)}\right) = \phi(0) = 0. \end{aligned}$$

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A similar computation shows that

$$(2) \quad \sum_{s=1}^{n} \phi(\tilde{X}_{43}{}^{(s)})\phi(\tilde{X}_{34}{}^{(s)})\phi(\tilde{Y}_{53}) - \sum_{t=1}^{m} \phi(\tilde{X}_{53}{}^{(t)})\phi(\tilde{X}_{35}{}^{(t)})\phi(\tilde{Y}_{53}) = 0.$$

We are now in a position to define the extension of ϕ on $[M_{\bar{P}_3}, M_{\bar{P}_3}]$.

Definition. Let σ_1 and σ' be mappings of $M_{D\bar{P}_3}$ and $M_{(I-D)\bar{P}_3}$ into $N_{\psi(D\bar{P}_3)}$ and $N_{\psi((I-D)\bar{P}_3)}$, respectively, defined in the following manner:

(1) if $X \in M_{ij}$ $(i \neq j)$, $\sigma_1(X) = \phi(X) \in N_{ij}$ for $i, j \in \{3, 4, 5\}$;

(2) if $X \in M_{ii}$ and $X = \sum_{i=1}^{n} X_{ij}{}^{(t)} X_{ji}{}^{(t)} = \sum_{s=1}^{m} X_{ik}{}^{(s)} X_{ki}{}^{(s)}$ for i, j, $k \in \{3, 4, 5\}$ then

$$\sigma_1(X) = \sum_{t=1}^{n} \phi(X_{ij}{}^{(t)})\phi(X_{ji}{}^{(t)}) = \sum_{s=1}^{m} \phi(X_{ik}{}^{(s)})\phi(X_{ki}{}^{(s)});$$

(3) if $\tilde{X} \in \tilde{M}_{ij}$ $(i \neq j)$, $\sigma'(\tilde{X}) = \sigma(\tilde{X}) \in \tilde{N}_{ji}$ for $i, j \in \{3, 4, 5\}$;

(4) if $\tilde{X} \in \tilde{M}_{ii}$ and $\tilde{X} = \sum_{i=1}^{n} \tilde{X}_{ij}{}^{(t)} \tilde{X}_{ji}{}^{(t)} = \sum_{s=1}^{m} \tilde{X}_{ik}{}^{(s)} \tilde{X}_{ki}{}^{(s)}$ then

$$\sigma'(\tilde{X}) = -\sum_{t=1}^{n} \phi(\tilde{X}_{ji}{}^{(t)})\phi(\tilde{X}_{ji}{}^{(t)}) = -\sum_{s=1}^{m} \phi(\tilde{X}_{ki}{}^{(s)})\phi(\tilde{X}_{ik}{}^{(s)}).$$

Extend σ_1 (respectively σ') to all of $M_{D\bar{P}_3}$ (respectively $M_{(I-D)\bar{P}_3}$) by linearity. These maps are well defined by Lemma 26. It is a straightforward computation to check that σ_1 and σ' are *-linear.

LEMMA 27. *σ_1 is an extension of $\phi|[M_{D\bar{P}_3}, M_{D\bar{P}_3}]$ to $M_{D\bar{P}_3}$, and σ' is an extension of $\phi|[M_{(I-D)\bar{P}_3}, M_{(I-D)\bar{P}_3}]$ to $M_{(I-D)\bar{P}_3}$.*

Proof. $M_{D\bar{P}_3}$ is linearly generated by X_{ij} and $[X_{ij}, X_{ji}]$ where $i \neq j$ and $X_{ij} \in M_{ij}$, $i, j \in \{3, 4, 5\}$. By definition, $\sigma_1 = \phi$ on M_{ij}. $\sigma[X_{ij}, X_{ji}] = \sigma(X_{ij}X_{ji} - X_{ji}X_{ij}) = \sigma(X_{ij}X_{ji}) - \sigma(X_{ji}X_{ij}) = \phi(X_{ij})\,\phi(X_{ji}) - \phi(X_{ji})\,\phi(X_{ij}) = \phi[X_{ij}, X_{ji}]$ from the definition of σ_1 on M_{ii}.

Similarly $M_{(I-D)\bar{P}_3}$ is generated by $\tilde{X}_{ij}$ and $[\tilde{X}_{ij}, \tilde{X}_{ji}]$ where $i \neq j$ and $\tilde{X}_{ij} \in \tilde{M}_{ij}$, $i, j \in \{3, 4, 5\}$. Again $\sigma' = \phi$ on $\tilde{M}_{ij}$. $\sigma'[\tilde{X}_{ij}, \tilde{X}_{ji}] = \sigma'(\tilde{X}_{ij}\tilde{X}_{ji}) - \sigma'(\tilde{X}_{ji}\tilde{X}_{ij}) = -\phi(\tilde{X}_{ji})\,\phi(\tilde{X}_{ij}) + \phi(\tilde{X}_{ij})\,\phi(\tilde{X}_{ji}) = \phi[\tilde{X}_{ij}, \tilde{X}_{ji}]$.

LEMMA 28. *σ_1 is a homomorphism of $M_{D\bar{P}_3}$ into $N_{\psi(D\bar{P}_3)}$, and σ' is the negative of an anti-homomorphism of $M_{(I-D)\bar{P}_3}$ into $N_{\psi((I-D)\bar{P}_3)}$.*

Proof. We show the anti-homomorphism part. The homomorphism proof is analogous. We must show that $\sigma'(\tilde{X}_{ij}\tilde{X}_{kl}) = -\sigma'(\tilde{X}_{kl})\,\sigma'(\tilde{X}_{ij})$ for i, j, k, $l \in \{3, 4, 5\}$.

(1) $i \neq j$, $k \neq l$, $j \neq k$. In this case $\tilde{X}_{ij}\tilde{X}_{kl} = 0$ so $\sigma'(\tilde{X}_{ij}\tilde{X}_{kl}) = 0$. $\sigma'(\tilde{X}_{ij}) \in \tilde{N}_{ji}$ and $\sigma'(\tilde{X}_{kl}) \in \tilde{N}_{lk}$ so that $\sigma'(\tilde{X}_{kl})\,\sigma'(\tilde{X}_{ij}) = 0$.

(2) $i \neq j$, $k \neq l$, $j = k$. If $i = l$, $\sigma'(\tilde{X}_{ij}\tilde{X}_{jl}) = -\phi(\tilde{X}_{jl})\ \phi(\tilde{X}_{ij}) = -\sigma'\,(\tilde{X}_{jl})\,\sigma'(\tilde{X}_{ij})$ since $\sigma'(\tilde{X}_{ij}) = \phi(\tilde{X}_{ij})$ for $i \neq j$. If $i \neq l$ then $\sigma'(\tilde{X}_{ij}\tilde{X}_{jl}) = \phi(\tilde{X}_{ij}\tilde{X}_{jl}) = -\phi(\tilde{X}_{jl})\,\phi(\tilde{X}_{ij}) = -\sigma'(\tilde{X}_{jl})\,\sigma'(\tilde{X}_{ij})$.

(3) $i = j$, $k \neq l$, $i \neq k$. We can assume, in this case, that $\tilde{X}_{ii} = \tilde{X}_{ik}\tilde{X}_{ki}$. Then $\sigma'(X_{ii}X_{kl}) = 0$. Also $-\sigma'(\tilde{X}_{kl})\,\sigma'(\tilde{X}_{ik}\tilde{X}_{ki}) = \phi(\tilde{X}_{kl})\,\phi(\tilde{X}_{ki})\,\phi(\tilde{X}_{ik}) = 0$ since $\phi(\tilde{X}_{kl}) \in \tilde{N}_{lk}$, $\phi(\tilde{X}_{ki}) \in \tilde{N}_{ik}$ and $i \neq k$.

(4) $i = j$, $k \neq l$, $i = k$. We can assume, in this case, that $\tilde{X}_{ii} = \tilde{X}_{il}\,\tilde{X}_{li}$. Then $\sigma'(\tilde{X}_{ii}\tilde{Y}_{il}) = \sigma'(\tilde{X}_{il}\,\tilde{X}_{li}\,\tilde{Y}_{il}) = \phi(\tilde{X}_{il}\tilde{X}_{li}\tilde{Y}_{il}) = \phi(\tilde{Y}_{il})\,\phi(\tilde{X}_{li})\,\phi(\tilde{X}_{il}) = \sigma'(\tilde{Y}_{il})\,\sigma'(\tilde{X}_{li})\,\sigma'(\tilde{X}_{il}) = -\sigma'(\tilde{Y}_{il})\,\sigma'(\tilde{X}_{il}\tilde{X}_{li}) = -\sigma'(\tilde{Y}_{il})\,\sigma'(\tilde{X}_{ii})$.

(5) $i \neq j$, $k = l$. This case is proved in a manner similar to (3) and (4).

(6) $i = j$, $k = l$, $i \neq k$. We can assume, in this case, that $\tilde{X}_{ii} = \tilde{X}_{ik}\tilde{X}_{ki}$ and $\tilde{X}_{kk} = \tilde{Y}_{ki}\tilde{Y}_{ik}$. $\tilde{X}_{ii}\tilde{X}_{kk} = 0$ so that $\sigma'(\tilde{X}_{ii}\tilde{X}_{kk}) = 0$. $\sigma'(\tilde{X}_{ik}\tilde{X}_{ki})\,\sigma'(\tilde{Y}_{ki}\tilde{Y}_{ik}) = \phi(\tilde{X}_{ki})\,\phi(\tilde{X}_{ik})\,\phi(\tilde{Y}_{ik})\,\phi(\tilde{Y}_{ki}) = 0$, since $\phi(\tilde{X}_{ik}) \in \tilde{N}_{ki}$, and $\phi(\tilde{Y}_{ik}) \in \tilde{N}_{ki}$.

(7) $i = j$, $k = l$, $i = k$. We can assume $\tilde{X}_{ii} = \tilde{X}_{ip}\tilde{X}_{pi}$, $X_{kk} = \tilde{Y}_{ip}\tilde{Y}_{pi}$ $(i \neq p)$. $\sigma'(\tilde{X}_{ii}\tilde{X}_{kk}) = \sigma'(\tilde{X}_{ip}\tilde{X}_{pi}\tilde{Y}_{ip}\tilde{Y}_{pi}) = -\sigma(\tilde{Y}_{pi})\,\phi(\tilde{X}_{ip}\tilde{X}_{pi}\tilde{Y}_{ip}) = -\phi(\tilde{Y}_{pi})\,\phi(\tilde{Y}_{ip})\,\phi(\tilde{X}_{pi})\,\phi(\tilde{X}_{ip}) = -\sigma'(\tilde{Y}_{ip}\tilde{Y}_{pi})\,\sigma'(\tilde{X}_{ip}\tilde{X}_{pi}) = -\sigma'(\tilde{X}_{kk})\,\sigma'(\tilde{X}_{ii})$.

THEOREM 3. *Let $\phi:[M, M] \to [N, N]$ be a Lie *-isomorphism of $[M, M]$ onto $[N, N]$ where M and N are von Neumann algebras with no central abelian summands. There exists a map $\Pi: M \to N$ which extends ϕ and such that $\Pi = \sigma + \sigma'$ where σ is a *-isomorphism of M_C onto $N_{\psi(C)}$ and σ' is the negative of a *-anti-isomorphism of M_{I-C} onto $N_{\psi(I-C)}$ for an appropriate central projection $C \in M$.*

Proof. By Theorem 2 it suffices to assume M is not of type I_2. Let D, P_1, P_2, P_3 be as above, let $C = I - \bar{P}_3 + D\bar{P}_3$, and let $\sigma = \sigma_0 + \sigma_1$.

In general if $\phi:[M, M] \to N$ is a Lie *-isomorphism where M is a von Neumann algebra with no central summands, and N is a *-algebra, and if Π is an extension of ϕ to an associative *-homomorphism or *-anti-homomorphism of M, then Π is 1-1. For, suppose $A = A^*$ and $\Pi(A) = 0$. Then $\Pi([A, B], B]) = 0$ for all self-adjoint B in M. This implies that $\phi([[A, B], B]) = 0$ (since $\phi = \Pi$ on $[M, M]$) and thus $[[A, B], B] = 0$. By [**7**] this implies $A \in Z_M$, or ker $\Pi \subseteq Z_M$. But ker Π is a two-sided *-ideal of M and cannot be contained in Z_M unless it is zero. The proof of this claim goes as follows:

Let $\mathscr{I}$ be a two-sided, *-ideal of M contained in Z_M, and let $A = A^* \in \mathscr{I}$ with $\|A\| \leqq 1$. If P is a core-free projection of M then $PA = AP \in \mathscr{I} \subseteq Z_M$ and $PA \leqq P$. Thus PA is central, self-adjoint, and so is equal to 0 since P is core-free. Now choose a core-free P with $\bar{P} = I$. Then $\bar{P} - P = I - P$ is core-free so that $0 = A(I - P) = A - AP = A$.

Applying the above to σ_0, σ_1, and σ' we see that each of these is 1-1.

Π itself is an extension of ϕ to M since σ_0 extends $\phi|[M_{I-\bar{P}_3}, M_{I-\bar{P}_3}]$ to $M_{I-\bar{P}_3}$, σ extends $\phi|[M_{D\bar{P}_3}, M_{D\bar{P}_3}]$ to $M_{D\bar{P}_3}$, and σ' extends $\phi|[M_{(I-D)\bar{P}_3}, M_{(I-D)\bar{P}_3}]$ to $M_{(I-D)\bar{P}_3}$ and so $[N, N] \subseteq$ Range Π. Moreover, since the image of M under Π is a *-subalgebra of N, the *-algebra generated by $[N, N]$ is contained in Range Π. But this algebra is just N by Lemma 23. Thus Π onto. This implies that each of σ_0, σ_1, and σ' is onto.

COROLLARY. *If $\phi: M \to N$ is a Lie *-isomorphism of M onto N where M and N have no central abelian summands, there exists a central projection C in M such* *that* $\phi = \sigma + \sigma' + \lambda$ *where* σ *is a *-isomorphism of* M_C *onto* $N_{\psi(C)}$, σ' *is the negative of a *-anti-isomorphism of* M_{I-C} *onto* $N_{\psi(I-C)}$ *and* λ *is a *-linear map of* M *into* Z_N *which annihilates brackets.*

Proof. $\eta = \phi|_{[M,M]}$ is a Lie *-isomorphism of $[M, M]$ onto $[N, N]$. Let C, σ, σ' be as in Theorem 3, and set $\lambda = \phi - (\sigma + \sigma')$. λ is *-linear since both ϕ and $\sigma + \sigma'$ are, and λ annihilates brackets since $\phi = \sigma + \sigma'$ on brackets.

We need to show that $\lambda(A) \in Z_N$ for $A \in M$. Since the ring generated by $[N, N]$ is N and since ϕ maps $[M, M]$ on $[N, N]$, it suffices to show that $[\lambda(A), \phi(X)] = 0$ for all X in $[M, M]$. $[\lambda(A), \phi(X)] = [\phi(A) - (\sigma + \sigma')(A), \phi(X)] = [\phi(A), \phi(X)] - [(\sigma + \sigma')(A), \phi(X)] = \phi[A, X] - [(\sigma + \sigma')(A), (\sigma + \sigma')(X)]$ (since $\phi = \sigma + \sigma'$ on $[M, M]$) $= \phi[A, X] - (\sigma + \sigma')[A, X] = \phi[A, X] - \phi[A, X] = 0$.

References

1. J. Dixmier, *Les algebres d'operateurs dans l'espace Hilbertien*, Cahiers Scientifiques, Fas. XXV (Gauthier-Villars, Paris, 1969).

2. R. G. Douglas and D. M. Topping, *Operators whose squares are zero*, Rev. Roumaine Math. Pures Appl. *12* (1967), 647–652.

3. I. N. Herstein, *Topics in ring theory* (University of Chicago Press, Chicago, 1969).

4. R. A. Howland, *Lie isomorphisms of derived rings of simple rings*, Trans. Amer. Math. Soc. *145* (1969), 383–396.

5. C. R. Miers, *Lie homomorphisms of operator algebras*, Pacific J. Math. *38* (1971), 717–737.

6. C. Pearcy and D. Topping, *Commutators and certain II-factors*, J. Functional Analysis *3* (1969), 69–78.

7. I. M. Singer, *Uniformly continuous representations of Lie Groups*, Ann. of Math. *56* (1952), 242–247.

8. H. Sunouchi, *Infinite Lie rings*, Tôhoku Math. J. *8* (1956), 291–307.

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