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A CLASS OF LOCATION-INDEPENDENT VARIABILITY ORDERS, WITH APPLICATIONS

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Abstract

Li and Shaked (2007) introduced the family of generalized total time on test transform (TTT) stochastic orders, which is parameterized by a real function h that can be used to capture the preferences of a decision maker. It is natural to look for properties of these orders when there is an uncertainty in determining the appropriate function h. In this paper we study these orders when h is nondecreasing. We note that all these orders are location independent, and we characterize the dispersive order, and the location-independent riskier order, by means of the generalized TTT orders with nondecreasing h. Further properties, which strengthen known properties of the dispersive order, are given. A useful nontrivial closure property of the generalized TTT orders with nondecreasing h is obtained. Applications in poverty comparisons, risk management, and reliability theory are described.

Keywords: Dispersive order; excess wealth order; stochastic order; location-independent riskier order; increasing convex and concave orders; poverty measure; risk comparison; taxation policy; NBU; IFR

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1. Introduction

Recently, Li and Shaked (2007) introduced a family of stochastic orders parameterized by a real function h which captures the preferences of a decision maker. This family includes, for adequate choices of h, some well-known orders such as the usual stochastic order, the location-independent riskier order, and the total time on test transform order. Li and Shaked (2007) obtained some interesting relationships among various orders in this family, and provided some applications of this family of stochastic orders in actuarial science, reliability theory, and statistics.

Specifically, let X and Y be two random variables with distribution functions F and G, respectively, and let $F^{-1}(p) = \sup\{x\colon F(x) \le p\}$ and $G^{-1}(p) = \sup\{x\colon G(x) \le p\}$, $0 < p < 1$, denote the corresponding right-continuous quantile functions. Let h be a nonnegative real function defined on $[0, 1]$. According to Li and Shaked (2007), X is said to be smaller than Y in the *generalized total time on test transform order* with respect to h (denoted as

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$X \leq_{\mathrm{TTT}}^{(h)} Y$) if

$$\int_{-\infty}^{F^{-1}(p)} h(F(t))\,\mathrm{d}t \leq \int_{-\infty}^{G^{-1}(p)} h(G(t))\,\mathrm{d}t \quad \text{for all } p \in (0,1), \tag{1.1}$$

provided the integrals in (1.1) are well defined. The definition of the order '$\leq_{\mathrm{TTT}}^{(h)}$' is closely related to the comparison of one-sided distorted risk measures which were introduced in general form in Wang (1996).

Condition (1.1) leads, for certain choices of h, to some well-known stochastic orders. For example, if X and Y are random variables having the same finite left endpoint of support, and $h(p) = c$ for some constant $c > 0$, then (1.1) reduces to

$$F^{-1}(p) \leq G^{-1}(p) \quad \text{for all } p \in (0,1), \tag{1.2}$$

which means that X is smaller than Y in the usual stochastic order (denoted by $X \leq_{\mathrm{st}} Y$; see Section 1.A of Shaked and Shanthikumar (2007)). If $h(p) = p$ then (1.1) is equivalent to saying that X is smaller than Y in the location-independent riskier order (denoted by $X \leq_{\mathrm{lir}} Y$; see Jewitt (1989), Fagiuoli *et al.* (1999), and Kochar *et al.* (2002)). If X and Y are nonnegative random variables and $h(p) = 1 - p$ in (1.1), then (1.1) is equivalent to saying that X is smaller than Y in the total time on test transform order (denoted by $X \leq_{\mathrm{TTT}} Y$; see Kochar *et al.* (2002)).

In decision theory the role of the function h in (1.1) is to adjust the tails of the distributions F and G, before comparing the random variables, according to the preferences of the decision maker. A problem of natural interest then is to compare X and Y when there is an uncertainty in determining the appropriate function h. One possibility in this situation is to take care of this uncertainty by considering robust orderings based on the condition

$$X \leq_{\mathrm{TTT}}^{(h)} Y \quad \text{for all } h \in \Psi,$$

where Ψ is a large class of functions on $(0,1)$. In this paper we study the case when Ψ is the class of all nondecreasing functions $h\colon [0,1] \mapsto [0,1]$ that satisfy $h(0) = 0$ and $h(1) = 1$. Some specific applications of our study in economics and in reliability theory are described in Section 5 at the end of this paper.

In Section 2 we note that the orders that correspond to any $h \in \Psi$ are all location independent, and we characterize the dispersive order (see the definition in Section 2), and the location-independent riskier order, in terms of the orders '$\leq_{\mathrm{TTT}}^{(h)}$' for $h \in \Psi$. Further properties, which strengthen some known properties of the dispersive order, are given in Section 3. In Section 4 we obtain a nontrivial closure property of the orders '$\leq_{\mathrm{TTT}}^{(h)}$' for $h \in \Psi$, and, as a consequence, we correct some inaccuracies in the literature involving the excess wealth order (see the definition in Section 2). Finally, in Section 5 we illustrate the theory through applications in poverty comparisons, in risk management, and in reliability theory.

Throughout this paper, 'increasing' and 'decreasing' stand for 'nondecreasing' and 'nonincreasing', respectively. For any random variable X, we denote by l_X and u_X the respective left and right endpoints of its support. For any real number a, we denote by a_+ its positive part, that is, $a_+ = a$ if $a > 0$, and $a_+ = 0$ if $a \leq 0$. Also, for any distribution function F, we denote by $\bar{F} \equiv 1 - F$ the corresponding survival function.

2. Orders with an increasing h

Recall that, given two random variables X and Y with respective distribution functions F and G, X is said to be smaller than Y in the dispersive order (denoted by $X \le_{\text{disp}} Y$; see, e.g. Shaked and Shanthikumar (2007, Section 3.B)) if

$$F^{-1}(q) - F^{-1}(p) \le G^{-1}(q) - G^{-1}(p) \quad \text{for all } 0 < p < q < 1.$$

Recall also (see Section 1 above) that X is said to be smaller than Y in the location-independent riskier order ($X \le_{\text{lir}} Y$) if

$$\int_{-\infty}^{F^{-1}(p)} F(t)\,\mathrm{d}t \le \int_{-\infty}^{G^{-1}(p)} G(t)\,\mathrm{d}t \quad \text{for all } p \in (0,1),$$

provided the integrals above are well defined.

Note that both orders '$\le_{\text{disp}}$' and '$\le_{\text{lir}}$' are location independent in the sense that if $X \le_{\text{disp}} Y$ then $X + a \le_{\text{disp}} Y$ and if $X \le_{\text{lir}} Y$ then $X + a \le_{\text{lir}} Y$ for any $a \in \mathbb{R}$. The property of location independence is of importance and of use in the management of risky prospects; see Jewitt (1989). For example, if an order is location independent then it can be used to compare in variability random variables that need not have the same means or medians. Therefore, it is of interest to identify orders, other than '$\le_{\text{disp}}$' and '$\le_{\text{lir}}$', which are location independent. The following result, which characterizes the orders '$\le_{\text{disp}}$' and '$\le_{\text{lir}}$', leads later to an identification of a host of useful location-independent stochastic orders.

Recall from Section 1 that we denote by Ψ the collection of all increasing functions $h\colon [0,1] \mapsto [0,1]$ that satisfy $h(0) = 0$ and $h(1) = 1$. For the purpose of proving the next result, we also recall that, given two random variables X and Y with respective distribution functions F and G, X is said to be smaller than Y in the excess wealth order (denoted by $X \le_{\text{EW}} Y$; see, e.g. Shaked and Shanthikumar (2007, Section 3.C)) if

$$\int_{F^{-1}(p)}^{\infty} \bar{F}(t)\,\mathrm{d}t \le \int_{G^{-1}(p)}^{\infty} \bar{G}(t)\,\mathrm{d}t \quad \text{for all } p \in (0,1), \tag{2.1}$$

provided the integrals above are well defined. The excess wealth order is called the right-spread order in Fernandez-Ponce *et al.* (1998). This order is the same as saying that the expected shortfall risk measure (for the positive tail) is comparable; that is, $\mathrm{E}(X - F^{-1}(p))_+ \le \mathrm{E}(Y - G^{-1}(p))_+$ for all $p \in (0,1)$.

Theorem 2.1. *Let X and Y be two random variables with continuous distribution functions and interval supports. Then*

(i) *$X \le_{\text{disp}} Y$ if and only if $X \le_{\text{TTT}}^{(h)} Y$ for all $h \in \Psi$;*

(ii) *$X \le_{\text{lir}} Y$ if and only if $X \le_{\text{TTT}}^{(h)} Y$ for all concave $h \in \Psi$.*

Proof. Denote the distribution functions of X and Y by F and G, respectively. Let $h \in \Psi$. For each $p \in (0,1)$, we have

$$\int_{-\infty}^{F^{-1}(p)} h(F(t))\,\mathrm{d}t = \int_0^{\infty} h[F(-t + F^{-1}(p))]\,\mathrm{d}t. \tag{2.2}$$

Employing the continuity of F, it is seen that

$$\int_0^{\infty} h[F(-t + F^{-1}(p))]\,\mathrm{d}t = \int_0^{\infty} h[\bar{F}_{X,p}(t)]\,\mathrm{d}t, \tag{2.3}$$

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where $\bar{F}_{X,p}$ is the survival function of the random variable $(F^{-1}(p)-X)_+$. Using the fact that $h(0)=0$, we can write

$$\int_0^\infty h[\bar{F}_{X,p}(t)]\,\mathrm{d}t = \int_{t=0}^\infty \int_{u=0}^{\bar{F}_{X,p}(t)} \mathrm{d}h(u)\,\mathrm{d}t.$$

Changing the order of integration, and using the fact that F is strictly increasing on its support, we have

$$\int_{t=0}^\infty \int_{u=0}^{\bar{F}_{X,p}(t)} \mathrm{d}h(u)\,\mathrm{d}t = \int_{u=0}^{\bar{F}_{X,p}(0)} \int_{t=0}^{\bar{F}^{-1}_{X,p}(u)} \mathrm{d}t\,\mathrm{d}h(u) = \int_0^{\bar{F}_{X,p}(0)} \bar{F}^{-1}_{X,p}(u)\,\mathrm{d}h(u).$$

It is apparent that $\bar{F}_{X,p}(0)=p$. Hence, if we take into account the fact that $\bar{F}^{-1}_{X,p}(u)=0$ for all $u\geq p$, we obtain

$$\int_0^{\bar{F}_{X,p}(0)} \bar{F}^{-1}_{X,p}(u)\,\mathrm{d}h(u) = \int_0^p \bar{F}^{-1}_{X,p}(u)\,\mathrm{d}h(u) = \int_0^1 \bar{F}^{-1}_{X,p}(u)\,\mathrm{d}h(u). \tag{2.4}$$

(A reviewer pointed out that the equality of the right-hand sides of (2.3) and (2.4) follows from known results in the context of risk measures. For example, identity (4) of Wang (1996), or identity (2.3a) of Wang and Young (1998), imply the above equality. However, these identities are given without proofs in the above papers. The equivalent representation of distortion risk measures as spectral risk measures (see the equality $\mathcal{R}_\nu(X)=\mathcal{R}_\phi(X)$ in De Giorgi (2005, p. 916), or Corollary 4.66 of Föllmer and Schied (2004)) are even more general identities, at least when h is differentiable. Due to the extensive terminology that is needed in order to describe and use the results of De Giorgi (2005) or Föllmer and Schied (2004), we provide the reader here with a direct proof of the equality of the right-hand sides of (2.3) and (2.4).)

From the equality of the left-hand side of (2.2) and the right-hand side of (2.4), we see, given an $h\in\Psi$, that $X\leq^{(h)}_{\mathrm{TTT}} Y$ if and only if

$$\int_0^1 \bar{F}^{-1}_{X,p}(u)\,\mathrm{d}h(u) \leq \int_0^1 \bar{G}^{-1}_{Y,p}(u)\,\mathrm{d}h(u) \quad \text{for all } p\in(0,1);$$

here $\bar{G}_{Y,p}$ denotes the survival function of the random variable $(G^{-1}(p)-Y)_+$.

Now, in order to prove part (i), note that the condition

$$\int_0^1 \bar{F}^{-1}_{X,p}(u)\,\mathrm{d}h(u) \leq \int_0^1 \bar{G}^{-1}_{Y,p}(u)\,\mathrm{d}h(u) \quad \text{for all } p\in(0,1) \text{ and all } h\in\Psi$$

holds if and only if

$$\bar{F}^{-1}_{X,p}(u) \leq \bar{G}^{-1}_{Y,p}(u) \quad \text{for all } u\in(0,1) \text{ and all } p\in(0,1), \tag{2.5}$$

where we have used the fact that every $h\in\Psi$ satisfies $h(0)=0$ and $h(1)=1$. By (1.2), condition (2.5) is equivalent to

$$(F^{-1}(p)-X)_+ \leq_{\mathrm{st}} (G^{-1}(p)-Y)_+ \quad \text{for all } p\in(0,1),$$

which means that $X\leq_{\mathrm{disp}} Y$ (see Muñoz-Pérez (1990)).

In order to prove part (ii), we note that the condition

$$\int_0^1 \bar{F}_{X,p}^{-1}(u)\,\mathrm{d}h(u) \le \int_0^1 \bar{G}_{Y,p}^{-1}(u)\,\mathrm{d}h(u) \quad \text{for all } p \in (0,1) \text{ and all concave } h \in \Psi$$

is equivalent to the condition

$$\int_0^1 F_{X,p}^{-1}(u)\,\mathrm{d}\widehat{h}(u) \le \int_0^1 G_{Y,p}^{-1}(u)\,\mathrm{d}\widehat{h}(u) \quad \text{for all } p \in (0,1) \text{ and all convex } \widehat{h} \in \Psi, \tag{2.6}$$

where $F_{X,p}$ and $G_{Y,p}$ denote the respective distribution functions of $(F^{-1}(p) - X)_+$ and $(G^{-1}(p) - Y)_+$ (this follows by noting that h is a concave function in Ψ if and only if $\widehat{h}(u) = 1 - h(1-u)$ is a convex function in Ψ). Using Theorem 2.1 of Sordo and Ramos (2007), we see that (2.6) is equivalent to

$$(F^{-1}(p) - X)_+ \le_{\mathrm{icx}} (G^{-1}(p) - Y)_+ \quad \text{for all } p \in (0,1), \tag{2.7}$$

where '$\le_{\mathrm{icx}}$' denotes the increasing convex order (that is, in general, for any two random variables Z and W, we have $W \le_{\mathrm{icx}} Z$ if $\mathrm{E}\,\phi(W) \le \mathrm{E}\,\phi(Z)$ for all increasing convex functions ϕ for which the expectations exist). Now we argue that condition (2.7) is equivalent to $X \le_{\mathrm{lir}} Y$, and in order to do this, we use a result of Belzunce (1999) which is stated as Theorem 4.A.43 of Shaked and Shanthikumar (2007). That theorem shows that, for any two continuous random variables W and Z, with distribution functions F_W and F_Z, the condition

$$(W - F_W^{-1}(p))_+ \le_{\mathrm{icx}} (Z - F_Z^{-1}(p))_+ \quad \text{for all } p \in (0,1)$$

is equivalent to $W \le_{\mathrm{EW}} Z$. Using the fact that, for any two random variables W and Z, we have

$$W \le_{\mathrm{EW}} Z \iff -W \le_{\mathrm{lir}} -Z \tag{2.8}$$

(see Fagiuoli *et al.* (1999)), it is not hard now to verify, from Theorem 4.A.43 of Shaked and Shanthikumar (2007), that condition (2.7) is equivalent to $X \le_{\mathrm{lir}} Y$.

As a corollary of Theorem 2.1(i), we obtain the following observation.

Corollary 2.1. *For any $h \in \Psi$, the order '$\le_{\mathrm{TTT}}^{(h)}$' is location independent when it applies to random variables with continuous distribution functions and interval supports.*

Proof. Let X and Y be random variables with continuous distribution functions and interval supports. Fix an $h \in \Psi$, and suppose that $X \le_{\mathrm{TTT}}^{(h)} Y$. Obviously, for any constant a, we have $X + a \le_{\mathrm{disp}} X$. Therefore, by Theorem 2.1(i) we have $X + a \le_{\mathrm{TTT}}^{(h)} X$, and by the transitivity property of '$\le_{\mathrm{TTT}}^{(h)}$' we obtain $X + a \le_{\mathrm{TTT}}^{(h)} Y$.

Some examples of interesting orders which are location independent are the ones determined by

$$h(u) = u^a, \qquad u \in [0,1],$$

for some $a > 0$, or by

$$h(u) = 1 - (1-u)^a, \qquad u \in [0,1],$$

for some $a > 0$. Note also that every reliability function of a coherent system (see Barlow and Proschan (1975, p. 21)) is a member in Ψ. Some applications in which some of the above hs are used will be studied in Section 5 below.

3. Further properties

Let $h \in \Psi$. From Theorem 2.1, it follows that

$$X \leq_{\mathrm{disp}} Y \quad \Longrightarrow \quad X \leq_{\mathrm{TTT}}^{(h)} Y.$$

In this section we obtain a few results involving the orders '$\leq_{\mathrm{TTT}}^{(h)}$' with $h \in \Psi$. Some of the results in this section strengthen some known properties of the dispersive order.

The following lemma, which generalizes Theorem 5(a) of Sordo (2009), will be needed below.

Lemma 3.1. *Let X and Y be two continuous random variables with respective distribution functions F and G having interval supports. Let $h \in \Psi$ be differentiable and strictly increasing on* $[0, 1]$*. Then, $X \leq_{\mathrm{TTT}}^{(h)} Y$ holds if and only if*

$$F^{-1}(p) - G^{-1}(p) \leq \frac{\int_0^u [F^{-1}(v) - G^{-1}(v)]\,\mathrm{d}h(v)}{h(u)} \quad \textit{whenever } 0 < u \leq p < 1. \tag{3.1}$$

Proof. From the proof of Theorem 2.1 we have

$$\int_{-\infty}^{F^{-1}(p)} h(F(t))\,\mathrm{d}t = \int_0^p \bar{F}_{X,p}^{-1}(u)\,\mathrm{d}h(u),$$

where $\bar{F}_{X,p}$ is the survival function of the random variable $(F^{-1}(p) - X)_+$ and $\bar{F}_{X,p}^{-1}$ is the corresponding inverse. Noting that

$$\bar{F}_{X,p}^{-1}(u) = \begin{cases} F^{-1}(p) - F^{-1}(u) & \text{if } 0 < u < p, \\ 0 & \text{if } p \leq u < 1, \end{cases} \tag{3.2}$$

we can write

$$\int_{-\infty}^{F^{-1}(p)} h(F(t))\,\mathrm{d}t = \int_0^p [F^{-1}(p) - F^{-1}(u)]\,\mathrm{d}h(u). \tag{3.3}$$

Therefore, $X \leq_{\mathrm{TTT}}^{(h)} Y$ holds if and only if

$$\int_0^p [F^{-1}(p) - F^{-1}(u)]\,\mathrm{d}h(u) \leq \int_0^p [G^{-1}(p) - G^{-1}(u)]\,\mathrm{d}h(u), \qquad 0 < p < 1,$$

or, equivalently, if and only if

$$F^{-1}(p) - G^{-1}(p) \leq \frac{1}{h(p)} \int_0^p [F^{-1}(v) - G^{-1}(v)]\,\mathrm{d}h(v), \qquad 0 < p < 1. \tag{3.4}$$

By differentiation, using the assumption that h is differentiable and strictly increasing, it is seen that (3.4) is the same as requiring that

$$\frac{1}{h(p)} \int_0^p [F^{-1}(v) - G^{-1}(v)]\,\mathrm{d}h(v) \text{ is decreasing in } p \in (0, 1). \tag{3.5}$$

Now, if $X \leq_{\mathrm{TTT}}^{(h)} Y$ then (3.1) follows from (3.4) and (3.5). Conversely, taking $u = p$ in (3.1), we obtain (3.4), which means that $X \leq_{\mathrm{TTT}}^{(h)} Y$.

Using characterization (3.1) of the order '$\le_{\text{TTT}}^{(h)}$', we obtain the following result which strengthens Theorem 3.B.13(a) of Shaked and Shanthikumar (2007) and a result of Sordo (2009) for the order '$\le_{\text{lir}}$'. The following result also indicates a potentially useful condition that implies the ordinary stochastic order '$\le_{\text{st}}$'.

Proposition 3.1. *Let X and Y be two continuous random variables with interval supports, and let $h \in \Psi$ be differentiable and strictly increasing on* $[0, 1]$. *If $X \le_{\text{TTT}}^{(h)} Y$ and $-\infty < l_X \le l_Y$, then $X \le_{\text{st}} Y$.*

Proof. Let F and G denote the distribution functions of X and Y, respectively. Letting $u \to 0+$ in (3.1) we obtain

$$F^{-1}(p) - G^{-1}(p) \le l_X - l_Y. \tag{3.6}$$

From (3.6) and the assumptions on l_X and l_Y, it follows that $F^{-1}(p) \le G^{-1}(p)$ for all $p \in (0, 1)$, and, by (1.2), this means that $X \le_{\text{st}} Y$.

It is known that if two random variables are ordered with respect to '$\le_{\text{disp}}$' and have the same finite support, then they must have identical distributions (see Theorem 3.B.14 of Shaked and Shanthikumar (2007)). A stronger result, which also generalizes a result of Sordo (2009) for the order '$\le_{\text{lir}}$', is the following. Below, '$=_{\text{st}}$' denotes equality in law.

Proposition 3.2. *Let X and Y be two continuous random variables with the same finite interval support, and let $h \in \Psi$ be differentiable and strictly increasing on* $[0, 1]$. *If $X \le_{\text{TTT}}^{(h)} Y$ then $X =_{\text{st}} Y$.*

Proof. Let F and G denote the distribution functions of X and Y, respectively. Suppose that $X \le_{\text{TTT}}^{(h)} Y$. Since $l_X = l_Y > -\infty$, it follows from Proposition 3.1 that $X \le_{\text{st}} Y$; that is, $F^{-1}(u) \le G^{-1}(u)$ for all $u \in (0, 1)$.

Next, letting $p \to 1-$ and then $u \to 1-$ in (3.1), taking into account the fact that $u_X = u_Y < \infty$, we see that

$$\int_0^1 [F^{-1}(v) - G^{-1}(v)]\, \mathrm{d}h(v) \ge u_X - u_Y \ge 0. \tag{3.7}$$

Since h is strictly increasing, it is easy to show that $X \le_{\text{st}} Y$ and (3.7), together, imply that X and Y have the same distribution.

The next result describes a property that we deem desirable for any location-independent variability order, and we show that the orders '$\le_{\text{TTT}}^{(h)}$', $h \in \Psi$, satisfy it. The following result strengthens Theorem 3.B.15 of Shaked and Shanthikumar (2007), as well as a result of Sordo (2009) for the order '$\le_{\text{lir}}$'.

Proposition 3.3. *Let X and Y be two continuous random variables whose supports are finite intervals, and let $h \in \Psi$ be differentiable and strictly increasing on* $[0, 1]$. *If $X \le_{\text{TTT}}^{(h)} Y$ then $m\{\text{supp}(X)\} \le m\{\text{supp}(Y)\}$, where m denotes the Lebesgue measure on $\mathbb{R}$.*

Proof. Let F and G denote the distribution functions of X and Y, respectively. Since the order '$\le_{\text{TTT}}^{(h)}$' is location independent, we can assume without loss of generality that $l_X = l_Y > -\infty$. As in the proof of Proposition 3.2, it can be shown that $X \le_{\text{TTT}}^{(h)} Y$ implies then that $F^{-1}(u) \le G^{-1}(u)$ for all $u \in (0, 1)$. Letting $u \to 1-$ we obtain $u_X \le u_Y$. Thus, $u_X - l_X \le u_Y - l_Y$.

Finally, Theorem 2.1(i) and the discussion in Shaked and Shanthikumar (2007, p. 149) yield the following result; its usefulness in economics is described in Subsection 5.1 below.

Proposition 3.4. *Let X be a random variable with finite interval support, and let $h \in \Psi$. Furthermore, let φ be bounded, differentiable, and strictly increasing on the support of X, such that its derivative φ' satisfies $\varphi'(x) \leq 1$ on the support of X. Then*

$$\varphi(X) \leq_{\mathrm{TTT}}^{(h)} X.$$

4. Closure under increasing concave transformations

Closure properties of a stochastic order under various monotone transformations are often useful in applications. For example, Li and Shaked (2007) proved, in their Theorem 2.11, that the order '$\leq_{\mathrm{TTT}}^{(h)}$', with a *decreasing* h, is preserved under increasing concave transformations. Some applications of this result were indicated in that paper. The main result in this section shows that, under minor conditions on the supports of the compared random variables, the order '$\leq_{\mathrm{TTT}}^{(h)}$', with an *increasing* h, is also preserved under such transformations. Our method of proof is a nontrivial modification of the proof of Theorem 4.1 of Kochar *et al.* (2002). While developing our proof we noticed a mistake in the proof of Theorem 4.1 of Kochar *et al.* (2002). In fact that theorem is not always true, but it is still correct under minor conditions on the supports of the compared random variables; details are given in Theorem 4.2 and Remark 4.3 at the end of this section.

In order to prove the main result below, we need a few lemmas. The first one is a special case of a theorem of Chong (1974). Let m denote the Lebesgue measure on $\mathbb{R}$, and, for a real function a on [0, 1], let us define

$$D_a(s) = m(\{u \in [0,1] \colon a(u) > s\}) \quad \text{for } s \in [-\infty, \infty].$$

The following result, which is an 'integration by parts' type of inequality, is a special case of Theorem 1.6 of Chong (1974).

Lemma 4.1. *Let $a \in L^1([0,1], m)$ and $b \in L^1([0,1], m)$ be decreasing functions. Then the following conditions are equivalent:*

(i) $\int_0^u a(v)\,\mathrm{d}v \leq \int_0^u b(v)\,\mathrm{d}v$ *for all* $u \in [0,1]$;

(ii) $\int_x^\infty D_a(t)\,\mathrm{d}t \leq \int_x^\infty D_b(t)\,\mathrm{d}t$ *for all* $x \in \mathbb{R}$.

Lemma 4.1 is used in the proof of the following technical result that will be needed below.

Lemma 4.2. *Let X and Y be two nonnegative random variables with respective survival functions $\bar{F}$ and $\bar{G}$. Let $h \in \Psi$ be differentiable and strictly increasing on* [0, 1] *such that the integrals below are well defined. Then the following conditions are equivalent:*

(i) $\int_x^\infty h[\bar{F}(t)]\,\mathrm{d}t \leq \int_x^\infty h[\bar{G}(t)]\,\mathrm{d}t$ *for all* $x \geq 0$;

(ii) $\int_0^u \bar{F}^{-1}(v)\,\mathrm{d}h(v) \leq \int_0^u \bar{G}^{-1}(v)\,\mathrm{d}h(v)$ *for all* $u \in [0,1]$.

Proof. Define $a(v) = \bar{F}^{-1} \circ h^{-1}(v)$ and $b(v) = \bar{G}^{-1} \circ h^{-1}(v)$ for all $v \in [0,1]$. Obviously, a and b are decreasing integrable functions. It is easy to see that

$$D_a(s) = m\{u \in [0,1] \colon a(u) > s\} = \begin{cases} h[\bar{F}(s)] & \text{if } s > 0, \\ 1 & \text{if } s \leq 0, \end{cases} \tag{4.1}$$

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and, analogously,

$$D_b(s) = \begin{cases} h[\bar{G}(s)] & \text{if } s > 0, \\ 1 & \text{if } s \le 0. \end{cases} \tag{4.2}$$

Now, from Lemma 4.1, it follows that

$$\int_0^u \bar{F}^{-1}[h^{-1}(v)]\,\mathrm{d}v \le \int_0^u \bar{G}^{-1}[h^{-1}(v)]\,\mathrm{d}v \quad \text{for all } u \in [0,1] \tag{4.3}$$

is equivalent to

$$\int_x^\infty D_a(t)\,\mathrm{d}t \le \int_x^\infty D_b(t)\,\mathrm{d}t \quad \text{for all } x \in \mathbb{R}. \tag{4.4}$$

From (4.1) and (4.2), we see that (4.4) is the same as

$$\int_x^\infty h[\bar{F}(s)]\,\mathrm{d}t \le \int_x^\infty h[\bar{G}(s)]\,\mathrm{d}t \quad \text{for all } x \ge 0.$$

On the other hand, a change of variable shows that (4.3) is the same as

$$\int_0^u \bar{F}^{-1}(v)\,\mathrm{d}h(v) \le \int_0^u \bar{G}^{-1}(v)\,\mathrm{d}h(v) \quad \text{for all } u \in [0,1],$$

and the stated result follows.

The following lemma plays a role, in our proof of the main result below, that parallels the role of Lemma A.1 in the proof of Theorem 4.1 of Kochar *et al.* (2002).

Lemma 4.3. *Let X and Y be two continuous random variables with distribution functions F and G, respectively. Let $h \in \Psi$ be differentiable and strictly increasing on* $[0, 1]$. *Then, $X \le_{\mathrm{TTT}}^{(h)} Y$ if and only if*

$$\int_x^\infty h[F(-t + F^{-1}(p))]\,\mathrm{d}t \le \int_x^\infty h[G(-t + G^{-1}(p))]\,\mathrm{d}t \quad \textit{for all } x \ge 0 \textit{ and } p \in (0,1). \tag{4.5}$$

Proof. From (2.4) in the proof of Theorem 2.1 we know that $X \le_{\mathrm{TTT}}^{(h)} Y$ holds if and only if

$$\int_0^p \bar{F}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \le \int_0^p \bar{G}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \quad \text{for all } p \in (0,1), \tag{4.6}$$

where $\bar{F}_{X,p}$ and $\bar{G}_{X,p}$ denote the survival functions of the random variables $(F^{-1}(p) - X)_+$ and $(G^{-1}(p) - Y)_+$, respectively. Now, by using the fact that $\bar{F}_{X,p}^{-1}(u) = \bar{G}_{Y,p}^{-1}(u) = 0$ for $u \ge p$ (see (3.2)), we can rewrite (4.6) as

$$\int_0^u \bar{F}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \le \int_0^u \bar{G}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \quad \text{for all } 0 < p \le u < 1. \tag{4.7}$$

On the other hand, if we take into account the facts that $\bar{F}_{X,p}^{-1}(v) = F^{-1}(p) - F^{-1}(v)$ for $v < p$ and $\bar{G}_{X,p}^{-1}(v) = G^{-1}(p) - F^{-1}(v)$ for $v < p$ (see (3.2) again), we can rewrite (3.1) as

$$\int_0^u \bar{F}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \le \int_0^u \bar{G}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \quad \text{for all } 0 < u \le p < 1. \tag{4.8}$$

Combining (4.7) and (4.8), we see that $X \leq_{\mathrm{TTT}}^{(h)} Y$ holds if and only if

$$\int_0^u \bar{F}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \leq \int_0^u \bar{G}_{X,p}^{-1}(v)\,\mathrm{d}h(v) \quad \text{for all } u \in (0,1) \text{ and } p \in (0,1). \tag{4.9}$$

Now, by using Lemma 4.2 (note that $(F^{-1}(p) - X)_+$ and $(G^{-1}(p) - Y)_+$ are nonnegative random variables) we see that (4.9) is equivalent to

$$\int_x^\infty h[\bar{F}_{X,p}(t)]\,\mathrm{d}t \leq \int_x^\infty h[\bar{G}_{Y,p}(t)]\,\mathrm{d}t \quad \text{for all } x \geq 0 \text{ and } p \in (0,1),$$

which is the same as (4.5).

We now have the tools needed to prove the main result.

Theorem 4.1. *Let X and Y be two continuous random variables with interval supports, such that $u_Y \leq u_X < \infty$. Let $h \in \Psi$ be differentiable and strictly increasing on* $[0,1]$. *Then, for any increasing concave function φ, we have*

$$X \leq_{\mathrm{TTT}}^{(h)} Y \implies \varphi(X) \leq_{\mathrm{TTT}}^{(h)} \varphi(Y). \tag{4.10}$$

Proof. Let F and G denote the distribution functions of X and Y, respectively. Let φ be an increasing concave function; for simplicity, we assume that φ is strictly increasing and differentiable. Note that $\varphi(X) \leq_{\mathrm{TTT}}^{(h)} \varphi(Y)$ holds if and only if

$$\int_{-\infty}^{\varphi(F^{-1}(p))} h(F(\varphi^{-1}(x)))\,\mathrm{d}x \leq \int_{-\infty}^{\varphi(G^{-1}(p))} h(G(\varphi^{-1}(x)))\,\mathrm{d}x, \qquad p \in (0,1).$$

Using the substitution $x' = \varphi^{-1}(x)$ (then $x = \varphi(x')$ and $\mathrm{d}x = \varphi'(x')\,\mathrm{d}x'$), we see that $\varphi(X) \leq_{\mathrm{TTT}}^{(h)} \varphi(Y)$ is equivalent to

$$\int_{-\infty}^{F^{-1}(p)} h(F(x))\varphi'(x)\,\mathrm{d}x \leq \int_{-\infty}^{G^{-1}(p)} h(G(x))\varphi'(x)\,\mathrm{d}x, \qquad p \in (0,1), \tag{4.11}$$

which is equivalent to

$$\int_0^\infty h[F(-t + F^{-1}(p))]\varphi'[-t + F^{-1}(p)]\,\mathrm{d}t \\ \leq \int_0^\infty h[G(-t + G^{-1}(p))]\varphi'[-t + G^{-1}(p)]\,\mathrm{d}t, \qquad p \in (0,1). \tag{4.12}$$

First we show that (4.12) holds for all $p \in (0,1)$ such that $G^{-1}(p) \leq F^{-1}(p)$. For such a p, using the decreasingness of φ', it is seen that

$$\int_0^\infty \{h[G(-t + G^{-1}(p))]\varphi'[-t + G^{-1}(p)] - h[F(-t + F^{-1}(p))]\varphi'[-t + F^{-1}(p)]\}\,\mathrm{d}t \\ \geq \int_0^\infty \{h[G(-t + G^{-1}(p))] - h[F(-t + F^{-1}(p))]\}\varphi'[-t + F^{-1}(p)]\,\mathrm{d}t. \tag{4.13}$$

On the other hand, from Lemma 4.3 we have

$$\int_x^\infty \{h[G(-t + G^{-1}(p))] - h[F(-t + F^{-1}(p))]\}\,\mathrm{d}t \geq 0 \quad \text{for all } x \geq 0.$$

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Since φ' is nonnegative and decreasing, it follows that $\varphi'[-t + F^{-1}(p)]$ is nonnegative and increasing in t. Therefore, from Lemma 7.1(a) of Barlow and Proschan (1975, p. 120) we obtain

$$\int_0^\infty \{h[G(-t + G^{-1}(p))] - h[F(-t + F^{-1}(p))]\}\varphi'[-t + F^{-1}(p)]\,\mathrm{d}t \ge 0.$$

This inequality, applied to (4.13), yields (4.12) (and, hence, (4.11)) for all $p \in (0, 1)$ such that $G^{-1}(p) \le F^{-1}(p)$. Note that, by the continuity assumption on F and G, we see that (4.11) also holds for $p = 1$.

Now we are going to show that (4.12) also holds for $p \in (0, 1)$ such that $G^{-1}(p) > F^{-1}(p)$. Since $u_Y \le u_X < \infty$, we see that

$$1 \in \{u > p : G^{-1}(u) \le F^{-1}(u)\}.$$

Consider the point

$$p_0 = \inf\{u > p : G^{-1}(u) \le F^{-1}(u)\},$$

and define $t_0 = F^{-1}(p_0)$, as in Figure 1. It is apparent that $t_0 = G^{-1}(p_0)$ by the continuity of F and G (note, in particular, that if $p_0 = 1$ then $u_X = u_Y$). Since $G^{-1}(p) > F^{-1}(p)$, we have $F(t) \ge G(t)$ for all $t \in [G^{-1}(p), G^{-1}(p_0)]$, which implies that $h[F(t)] \ge h[G(t)]$ for all $t \in [G^{-1}(p), G^{-1}(p_0)]$. Therefore, we have

$$\begin{aligned}\int_{-\infty}^{G^{-1}(p)} h(G(t))\varphi'(t)\,\mathrm{d}t &= \int_{-\infty}^{G^{-1}(p_0)} h(G(t))\varphi'(t)\,\mathrm{d}t - \int_{G^{-1}(p)}^{G^{-1}(p_0)} h(G(t))\varphi'(t)\,\mathrm{d}t \\ &\ge \int_{-\infty}^{G^{-1}(p_0)} h(G(t))\varphi'(t)\,\mathrm{d}t - \int_{F^{-1}(p)}^{F^{-1}(p_0)} h(F(t))\varphi'(t)\,\mathrm{d}t \\ &\ge \int_{-\infty}^{F^{-1}(p_0)} h(F(t))\varphi'(t)\,\mathrm{d}t - \int_{F^{-1}(p)}^{F^{-1}(p_0)} h(F(t))\varphi'(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{F^{-1}(p)} h(F(t))\varphi'(t)\,\mathrm{d}t,\end{aligned}$$

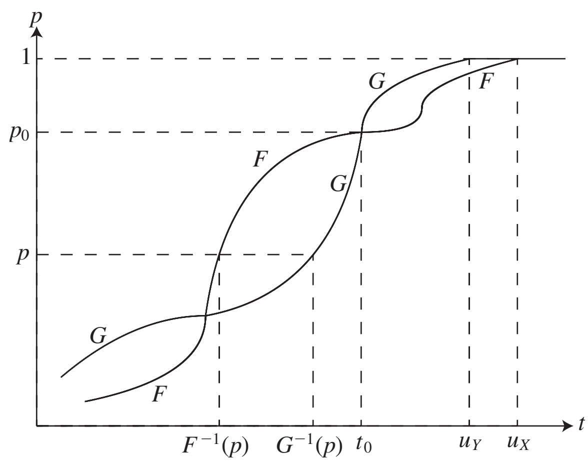

FIGURE 1: Typical crossing points of the distributions F and G (of X and Y, respectively) when $X \le_{\mathrm{TTT}}^{(h)} Y$.

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where the second inequality follows from the validity of (4.11) for p_0 proven earlier. This proves that (4.11) also holds for $p \in (0, 1)$ such that $G^{-1}(p) > F^{-1}(p)$, and the proof of the theorem is complete.

As a corollary of Theorem 4.1, we obtain the following interesting result (which will also be useful in the sequel).

Corollary 4.1. *Let X and Y be two continuous random variables with interval supports, such that $u_Y \leq u_X < \infty$. Then, for any increasing concave function φ, we have*

$$X \leq_{\text{lir}} Y \quad \Longrightarrow \quad \varphi(X) \leq_{\text{lir}} \varphi(Y). \tag{4.14}$$

Proof. The function $h(p) = p,\ p \in [0, 1]$, satisfies the assumptions of Theorem 4.1, and the order '$\leq_{\text{lir}}$' is equivalent to the order '$\leq_{\text{TTT}}^{(h)}$' with this function.

Surprisingly, the implications in (4.10) and (4.14) need not hold without the assumption that u_X, and, hence, also u_Y, are finite. In order to see this, we need some definitions, and we also need to correct some related results in the literature.

Recall that the random variable X is said to be smaller in the convex order than the random variable Y (denoted $X \leq_{\text{cx}} Y$) if $\mathrm{E}\,\phi(X) \leq \mathrm{E}\,\phi(Y)$ for all convex functions ϕ for which the expectations exist. If X and Y have finite means then X is said to be smaller in the dilation order than the random variable Y (denoted $X \leq_{\text{dil}} Y$) if $X - \mathrm{E}\,X \leq_{\text{cx}} Y - \mathrm{E}\,Y$. The following proposition is a corrected version of Theorem 4.A.30 of Shaked and Shanthikumar (2007).

Proposition 4.1. *Let X and Y be two random variables with finite means. If $-\infty < l_X \leq l_Y$ then $X \leq_{\text{dil}} Y$ implies that $X \leq_{\text{icx}} Y$.*

The proof of the above proposition is the same as the proof of Theorem 4.A.30 of Shaked and Shanthikumar (2007), or, equivalently, as the proof of Theorem 2.1 of Belzunce *et al.* (1997), except that in the above references it is implicitly assumed that l_X is finite, although this assumption was not stated there. We note that, without assuming the finiteness of l_X, the conclusion of Proposition 4.1 need not hold; see Remark 4.1, below.

Now recall from (2.1) the order '$\leq_{\text{EW}}$'. The following proposition is a corrected version of Corollary 4.A.32 of Shaked and Shanthikumar (2007). Its proof follows from Proposition 4.1 above, and implication (3.C.7) of Shaked and Shanthikumar (2007).

Proposition 4.2. *Let X and Y be two random variables with finite means. If $-\infty < l_X \leq l_Y$ then $X \leq_{\text{EW}} Y$ implies that $X \leq_{\text{icx}} Y$.*

Remark 4.1. It is worthwhile to mention that the conclusion of Proposition 4.2, and, hence, also of Proposition 4.1, need not hold if l_X is not assumed to be finite. In order to see this, let X be a normal random variable with mean μ_X and variance σ_X^2, and let Y be a normal random variable with mean μ_Y and variance σ_Y^2. Suppose that $\sigma_X^2 \leq \sigma_Y^2$. Then $X \leq_{\text{disp}} Y$ (this easily follows from Theorem 3.B.4 of Shaked and Shanthikumar (2007)), and, hence, by (3.C.9) of Shaked and Shanthikumar (2007), we have $X \leq_{\text{EW}} Y$. However, if $\mu_X > \mu_Y$ then it is not true that $X \leq_{\text{icx}} Y$.

Recall that the random variable X is said to be smaller in the increasing concave order than the random variable Y (denoted $X \leq_{\text{icv}} Y$) if $\mathrm{E}\,\phi(X) \leq \mathrm{E}\,\phi(Y)$ for all increasing concave functions ϕ for which the expectations exist. By applying Proposition 4.2 to the negative of its random variables, and using (2.8), we obtain the following result.

Proposition 4.3. *Let X and Y be two random variables with finite means. If $u_X \le u_Y < \infty$ then $X \le_{\text{lir}} Y$ implies that $X \ge_{\text{icv}} Y$.*

We now have the ingredients needed to show, in the following remark, the necessity of the finiteness of u_X in Theorem 4.1 and in Corollary 4.1.

Remark 4.2. Let X be a normal random variable with parameters $(\mu_X, \sigma_X^2) = (-1, 1)$, and let Y be a normal random variable with parameters $(\mu_Y, \sigma_Y^2) = (0, 2)$. Note that both X and Y have an infinite right endpoint of support. Since $\sigma_X^2 \le \sigma_Y^2$, we have $X \le_{\text{disp}} Y$ (see Remark 4.1), and, hence, by Theorem 2.1(i) in Section 2 above, we have

$$X \le_{\text{lir}} Y.$$

Consider the increasing concave function $\varphi(t) = -\mathrm{e}^{-t}$, $t \in \mathbb{R}$. Then $-\varphi(X) = \mathrm{e}^{-X}$ and $-\varphi(Y) = \mathrm{e}^{-Y}$ are lognormal random variables, and from the well-known expression of the mean of a lognormal distribution, it holds that

$$\mathrm{E}(\varphi(X)) = -\mathrm{E}(\mathrm{e}^{-X}) = -\mathrm{e}^{-\mu_X+\sigma_X^2/2} = -\mathrm{e}^{1+1/2} < -\mathrm{e}^{0+2/2} = -\mathrm{e}^{-\mu_Y+\sigma_Y^2/2} = \mathrm{E}(\varphi(Y)). \tag{4.15}$$

Now suppose that $\varphi(X) \le_{\text{lir}} \varphi(Y)$. Note that $\varphi(X)$ and $\varphi(Y)$ have the same finite right endpoint of support. So, by Proposition 4.3 we would have $\varphi(X) \ge_{\text{icv}} \varphi(Y)$. This would imply that $\mathrm{E}(\varphi(X)) \ge \mathrm{E}(\varphi(Y))$, which contradicts (4.15). So, the conclusion of Corollary 4.1, and, hence, also the conclusion of Theorem 4.1, do not hold for the example above.

We end this section by providing a corrected version of Theorem 4.1 of Kochar *et al.* (2002).

Theorem 4.2. *Let X and Y be two continuous random variables with interval supports, such that $-\infty < l_X \le l_Y$. Then, for any increasing convex function φ, we have*

$$X \le_{\text{EW}} Y \quad \Longrightarrow \quad \varphi(X) \le_{\text{EW}} \varphi(Y).$$

The above theorem can be derived from Corollary 4.1 by applying the latter to the negative of its random variables, and using (2.8). We omit the straightforward details.

Remark 4.3. The conclusion of Theorem 4.2 may not hold if l_X is not finite. In order to see this, let X be a normal random variable with parameters $(\mu_X, \sigma_X^2) = (1, 1)$, and let Y be a normal random variable with parameters $(\mu_Y, \sigma_Y^2) = (0, 2)$. Note that both X and Y have an infinite left endpoint of support. As in Remark 4.2, we have $X \le_{\text{disp}} Y$, and from (3.C.9) of Shaked and Shanthikumar (2007), it follows that

$$X \le_{\text{EW}} Y.$$

Considering the function $\varphi(t) = \mathrm{e}^t$, $t \in \mathbb{R}$, it can be shown, with a calculation similar to the one in Remark 4.2, that, for these random variables, we have

$$\mathrm{E}(\varphi(X)) > \mathrm{E}(\varphi(Y)). \tag{4.16}$$

On the other hand, if we suppose that $\mathrm{e}^X \le_{\text{EW}} \mathrm{e}^Y$ then, from Proposition 4.2 above, it would follow that $\mathrm{e}^X \le_{\text{icx}} \mathrm{e}^Y$, which would imply that $\mathrm{E}(\mathrm{e}^X) \le \mathrm{E}(\mathrm{e}^Y)$. The latter inequality contradicts (4.16), and, therefore, the conclusion of Theorem 4.2 does not hold for this example.

5. Applications

5.1. Poverty comparisons

Various indices of income poverty have been proposed by several authors, each of them providing a particular view on the nature of poverty. These indices can be used to make poverty comparisons of different societies.

Let X be an income random variable with a continuous distribution function F. Let z be the poverty line, that is, the income which separates the population into poor and nonpoor. Let $X^* = \min\{X, z\}$ be the random variable X, censored at z, with a distribution function F_z. Its corresponding quantile function is F_z^{-1}, where

$$F_z^{-1}(v) = \begin{cases} F^{-1}(v) & \text{if } 0 \le v < F(z), \\ z & \text{if } F(z) \le v \le 1. \end{cases}$$

Censored quantiles are, therefore, just the incomes $F^{-1}(v)$ for those in poverty (below z) and z for those whose income exceeds the poverty line. Note that usually the income random variable X is taken to be nonnegative; however, we need not assume this below, and, therefore, our analysis also applies to situations where incomes can be negative.

The poverty gap associated with income $F^{-1}(v)$ is defined as $z - F_z^{-1}(v)$. Many poverty indices can be expressed in terms of poverty gaps (see Jenkins and Lambert (1997)). Following Duclos and Grégoire (2002, p. 478) and Duclos and Araar (2006, Equations (3.5) and (10.15)) we focus on a class of poverty measures given by the following functional form:

$$\int_0^1 (z - F_z^{-1}(v))\, \mathrm{d}h(v), \tag{5.1}$$

where the poverty gaps are weighted with a continuous probability distribution h with support in $[0, 1]$. This class includes some well-known poverty indices, including the 'per-capita income gap' P_1 proposed in Foster *et al.* (1984) (obtained when h is the uniform distribution on $(0, 1)$), the poverty indices proposed in Shorrocks (1995, p. 1228) and Foster and Shorrocks (1988, p. 174) (obtained when $h(v) = 1 - (1 - v)^2$), and the general class of poverty indices proposed in Thon (1983, p. 61–62), obtained from (5.1) by choosing

$$h(v) = \frac{c^2}{4(c-1)} - \frac{1}{c-1}\left(\frac{c}{2} - v\right)^2 \quad \text{for some } c > 2.$$

Duclos and Grégoire (2002, pp. 477–478) analyzed the poverty measures corresponding to the hs in (5.1) given by

$$h(v) = 1 - (1 - v)^a, \qquad v \in [0, 1],$$

for some $v \ge 1$. Hagenaars (1987, p. 595) studied further indices of the form (5.1).

Sordo *et al.* (2007) characterized the comparisons of classes of indices of the form (5.1) by means of stochastic orderings when the poverty line z is defined as a level income. In this subsection, following Hagenaars and van Praag (1985) and Zheng (2001), we suppose that the poverty line z is defined as a quantile of the income distribution, $z = z_p = F^{-1}(p)$ for some $p \in (0, 1)$. This means that a person is poor if his/her income is below $F^{-1}(p)$. Such an index is called *relative* rather than *absolute*; see Hagenaars and van Praag (1985) and Zheng (2001).

So, for any $p \in (0, 1)$ and $h \in \Psi$, let us consider the poverty index $I_X(h, p)$ given by

$$I_X(h, p) = \int_0^1 (z_p - F_{z_p}^{-1}(v))\, \mathrm{d}h(v) = \int_0^p (F^{-1}(p) - F^{-1}(v))\, \mathrm{d}h(v). \tag{5.2}$$

From (3.3), it follows that

$$I_X(h, p) = \int_0^{F^{-1}(p)} h(F(v))\,\mathrm{d}v,$$

and the following fact is apparent from (1.1).

Observation 5.1. *Let X and Y be two random variables with continuous distribution functions and interval supports, and let* $h \in \Psi$. *Then*

$$X \le_{\mathrm{TTT}}^{(h)} Y \iff I_X(h, p) \le I_Y(h, p) \quad \textit{for all } p \in (0, 1).$$

From the above observation and Theorem 2.1(i), it is seen that

$$X \le_{\mathrm{disp}} Y \implies I_X(h, p) \le I_Y(h, p) \quad \text{for all } h \in \Psi \text{ and all } p \in (0, 1).$$

This result is particularly useful when the choice of h and p in (5.2) is arbitrary.

By replacing the assumption $X \le_{\mathrm{disp}} Y$ above by the weaker assumption $X \le_{\mathrm{lir}} Y$, we can still get comparisons of a lot of poverty measures. That is, from Observation 5.1 and Theorem 2.1(ii), it follows that

$$X \le_{\mathrm{lir}} Y \implies I_X(h, p) \le I_Y(h, p) \quad \text{for all concave } h \in \Psi \text{ and all } p \in (0, 1).$$

We next interpret Theorem 4.1 in terms of poverty measures. As is well known, there are frequent attempts to modify income distributions by means of intervention in the economic process (taxation and welfare programs are some examples). Then, the original income distribution X is replaced by some function $g(X)$. If we assume that X and Y are two income distributions satisfying the assumptions of Theorem 4.1 and φ is an increasing and concave function, then, for each $h \in \Psi$ and $p \in (0, 1)$, we have

$$I_X(h, p) \le I_Y(h, p) \implies I_{\varphi(X)}(h, p) \le I_{\varphi(Y)}(h, p);$$

that is, φ preserves poverty comparisons for all measures of the form (5.2).

Finally, we point out that Proposition 3.4 has an interesting interpretation in terms of taxation policies and their influence on the relative poverty indices defined in (5.2). A taxation policy transforms the income distribution X into another income distribution $\varphi(X)$. If φ is strictly increasing, and its derivative φ' satisfies $\varphi' \le 1$, then the taxation policy may be called *progressive*. It is thus seen from Proposition 3.4 that a progressive taxation policy φ reduces the corresponding relative poverty measure; that is, for all $h \in \Psi$ and $p \in (0, 1)$, we have

$$I_{\varphi(X)}(h, p) \le I_X(h, p).$$

5.2. Risk comparisons

In the context of risk management, outcomes with values smaller than the target value are viewed as risky. In this case, the concern is the downside risk and the interest relies on the left tail of the distributions. Drawing on earlier joint work, Artzner (1999) discussed the properties that a risk measure should satisfy. Aebi *et al.* (1992) suggested using probability metrics for the purpose of measuring the downside risk. One of the most popular probability metrics is the Kantorovich or, equivalently, the Mallows metric. This yields a commonly used class of downside risk measures (see Jones and Zitikis (2003, p. 50)) that is given by

$$I_r(X) = \int_{-\infty}^{\infty} [(F(t))^r - F(t)]\,\mathrm{d}t \quad \text{for } \tfrac{1}{2} \le r < 1, \tag{5.3}$$

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provided the above integral exists. For example, the Wang's left-tail deviation (see Wang (1998, p. 101)) is given by (5.3) with $r = \frac{1}{2}$. From the table of results in Jones and Zitikis (2003, p. 50), it follows that an alternative representation of $I_r(X)$ is

$$I_r(X) = \int_0^1 F^{-1}(u)\,\mathrm{d}u - \int_0^1 F^{-1}(u)\,\mathrm{d}u^r = \mathrm{E}\,X - \int_0^1 F^{-1}(u)\,\mathrm{d}u^r$$

for $\frac{1}{2} \le r < 1$. More generally, we can consider measures of the form

$$I(X, h) = \mathrm{E}\,X - \mathrm{E}_h(X),$$

where h is an increasing concave function on [0, 1], such that $h(0) = 0$, $h(1) = 1$, and

$$\mathrm{E}_h(X) = \int_0^1 F^{-1}(u)\,\mathrm{d}h(u).$$

Note that $I(X, h) \ge 0$; this can be verified by noting from the hypotheses on h that $h(u) \ge u$ for $u \in [0, 1]$, and, therefore,

$$I(X, h) = \int_0^1 F^{-1}(u)\,\mathrm{d}(u - h(u)) = \int_0^1 (h(u) - u)\,\mathrm{d}F^{-1}(u) \ge 0.$$

In this context of risk comparisons, we have the following result.

Proposition 5.1. *Let X and Y be two continuous random variables with interval supports. Let $h \in \Psi$ be differentiable, strictly increasing, and concave. If $X \le_{\mathrm{TTT}}^{(h)} Y$ then $I(X, h) \le I(Y, h)$.*

Proof. Let F and G denote the distribution functions of X and Y, respectively. Suppose that $X \le_{\mathrm{TTT}}^{(h)} Y$, which is equivalent to assuming that (3.5) holds. From (3.5), it follows that

$$\frac{1}{h(p)} \int_0^p [F^{-1}(u) - G^{-1}(u)]\,\mathrm{d}h(u) \ge \int_0^1 [F^{-1}(u) - G^{-1}(u)]\,\mathrm{d}h(u), \qquad p \in (0, 1).$$

Rearranging the above inequality we obtain

$$\int_0^p (F^{-1}(u) - \mathrm{E}_h(X))\,\mathrm{d}h(u) \ge \int_0^p (G^{-1}(u) - \mathrm{E}_h(Y))\,\mathrm{d}h(u), \qquad p \in (0, 1). \tag{5.4}$$

Since

$$\int_0^p (F^{-1}(u) - \mathrm{E}_h(X))\,\mathrm{d}h(u) + \int_p^1 (F^{-1}(u) - \mathrm{E}_h(X))\,\mathrm{d}h(u) = 0,$$

it follows that (5.4) is equivalent to

$$\int_p^1 (F^{-1}(u) - \mathrm{E}_h(X))\,\mathrm{d}h(u) \le \int_p^1 (G^{-1}(u) - \mathrm{E}_h(Y))\,\mathrm{d}h(u), \qquad p \in (0, 1),$$

or, equivalently, to

$$\int_p^1 [G^{-1}(u) - F^{-1}(u) - \mathrm{E}_h(Y) + \mathrm{E}_h(X)]h'(u)\,\mathrm{d}u \ge 0, \qquad p \in (0, 1). \tag{5.5}$$

Since $h(u)$ is strictly increasing and concave on [0, 1], it is seen that $1/[h'(u)]$ is increasing on [0, 1]. It follows from (5.5), using Lemma 7.1(a) of Barlow and Proschan (1975, p. 120), that

$$\int_0^1 [G^{-1}(u) - F^{-1}(u) - \mathrm{E}_h(Y) + \mathrm{E}_h(X)]\,\mathrm{d}u \geq 0,$$

that is,

$$\mathrm{E}\,X - \mathrm{E}_h(X) \leq \mathrm{E}\,Y - \mathrm{E}_h(Y),$$

which is the same as $I(X, h) \leq I(Y, h)$.

In particular, from Proposition 5.1 and Theorem 2.1(ii), we obtain the following corollary.

Corollary 5.1. *Let X and Y be two continuous random variables with interval supports. Then, $X \leq_{\mathrm{lir}} Y$ implies that $I(X, h) \leq I(Y, h)$ for all differentiable, strictly increasing, concave functions $h \in \Psi$. Specifically, $X \leq_{\mathrm{lir}} Y$ implies that $I_r(X) \leq I_r(Y)$ for all $\frac{1}{2} \leq r \leq 1$.*

The corollary shows, in particular, that the Wang's left-tail deviation is consistent with the order '$\leq_{\mathrm{lir}}$'.

5.3. Relationships among NBU notions

In reliability theory, stochastic orders are often used to define or characterize ageing notions (see, for instance, Barlow and Proschan (1975) or Shaked and Shanthikumar (2007)). For example, if the nonnegative random variable X describes the lifetime of an item and X_t is the residual life of the item at time t, defined by $X_t =_{\mathrm{st}} [X - t \mid X > t]$ for $t > 0$, we have the following characterization of the NBU (new better than used) ageing notion (see Barlow and Proschan (1975, Chapter 6)):

$$X \text{ is NBU} \quad \Longleftrightarrow \quad X_t \leq_{\mathrm{st}} X \quad \text{for all } t > 0.$$

Similarly, for a continuous random variable X with support $[0, \infty)$, the IFR (increasing failure rate) ageing notion can be characterized (see Pellerey and Shaked (1997)) by

$$X \text{ is IFR} \quad \Longleftrightarrow \quad X_t \leq_{\mathrm{disp}} X \quad \text{for all } t > 0. \tag{5.6}$$

Recently, Kayid (2007) introduced a general family of ageing notions based on the orders '$\leq_{\mathrm{TTT}}^{(h)}$'. He defined a nonnegative random variable X as $\mathrm{NBU}_{(h)}$ (new better than used with respect to h) if

$$X_t \leq_{\mathrm{TTT}}^{(h)} X \quad \text{for all } t > 0,$$

and he gave some applications of it in actuarial science and reliability. Here, as a consequence of our results in Section 2, we first characterize the IFR ageing notion in terms of $\mathrm{NBU}_{(h)}$ ageing notions. Recall that we denote by Ψ the collection of all increasing functions $h\colon [0, 1] \mapsto [0, 1]$ that satisfy $h(0) = 0$ and $h(1) = 1$.

Corollary 5.2. *Let X be a continuous random variable with support* $[0, \infty)$. *Then*

$$X \textit{ is IFR} \quad \Longleftrightarrow \quad X \textit{ is } \mathit{NBU}_{(h)} \quad \textit{for all } h \in \Psi.$$

Proof. From (5.6), it follows that X is IFR if and only if $X_t \leq_{\mathrm{disp}} X$ for all $t > 0$. By Theorem 2.1(i), it is seen that the latter condition is equivalent to the condition that $X_t \leq_{\mathrm{TTT}}^{(h)} X$ for all $t > 0$ and $h \in \Psi$, and the stated result follows.

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From Corollary 5.2 and Theorem 2.2(b) of Kayid (2007), we see that, when $h \in \Psi$, the $\mathrm{NBU}_{(h)}$ ageing notion occupies an intermediate position between the IFR and the NBU notions. This is formally stated in the next result.

Corollary 5.3. *Let X be a continuous random variable with support $(0, \infty]$, and let $h \in \Psi$. Then*

$$X \text{ is IFR} \quad \Longrightarrow \quad X \text{ is } \mathrm{NBU}_{(h)} \quad \Longrightarrow \quad X \text{ is NBU}.$$

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