

PART V

MINOR PLANETS

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1. SOME METHODS APPLIED TO CASES OF COMMENSURABLE MEAN MOTIONS

This paper consists of a review about work done on three-dimensional motion at commensurabilities of either the mean motions, or of secular periods, and of a report on the author's recent results on some special cases. Real and fictitious asteroidal orbits and the corresponding long-period effects are the main subject of interest. At first, methods are listed.

Let n and n_J be the mean motions of an asteroid and Jupiter (the index J relates to Jupiter) and let n/n_J be close to a rational number $(p+q)/p$, using the author's preferred notation (Schubart, 1964, 1968). The motion of the asteroid is described by osculating elements with respect to the sun, $a, e, i, \Omega, \dot{\omega} = \omega + \dot{\Omega}$, and by the mean longitude, ℓ . Periods of interest are given by the circulation or libration of $\omega, \tilde{\omega} - \tilde{\omega}_J, \Omega - \tilde{\Omega}_J$, and of a critical argument σ , where

$$q\sigma = (p + q) \ell_J - p \ell - q \tilde{\omega} \quad .$$

Numerical integration applied to the equations of motion of a suitable model for the acting forces of attraction has become a reliable means to study the changes in an orbit over an extended period. If the model fits well to the conditions of nature, the results will be applicable to real cases. The sun, the major planets, and an asteroid can be integrated on a computer as an n -body-system (Schubart and Stumpff, 1966), and ephemerides of high accuracy result. If, however, the interval of interest is very large, it will be necessary to reduce the number of attracting planets to two, or to one (sun-Jupiter-asteroid problem, $e_J \neq 0$ or $e_J = 0$).

Hagihara (1961) published an extensive listing of references of former work about commensurabilities. In many cases the work was based on analytical theories. A method by Poincaré (1902) suggests to drop the short-period terms in a development of the disturbing function of the sun-Jupiter-asteroid problem, but to retain terms depending on σ and

other slow arguments. To-day, it is no longer necessary to use a series development for this simplification, since numerical averaging of a Hamiltonian function (Schubart, 1964, 1966) or of a suitable set of differential equations (Schubart, 1968, 1978) allows an equivalent elimination of the short-period terms and a much wider range of applications. Numerical integration is used in the latter case. Care is necessary with respect to such an approximate method, since it is certainly not suitable for the study of orbits that allow close approaches to Jupiter (compare Schubart, 1970).

A further simplification of a corresponding model is possible, if the critical argument is eliminated during a suitable formal process of canonical transformation (Giacaglia, 1969). This method was applied to several special cases of commensurability.

2. METHODS GIVING EVIDENCE FOR SECULAR RESONANCES

If the orbital period of an asteroid is not close to a commensurability, methods are available to study the secular variations of e , $\tilde{\omega}$, i , Ω . Methods using numerical averaging allow an individual treatment of most asteroids (compare Smith, 1964). The usual analytical method (Brouwer and van Woerkom, 1950) is not applicable to large values of e or i , but it allows an explicit solution of the differential equations corresponding to an asteroid, in general. The constants of integration, called proper elements, gave rise to the discovery of families of asteroids by Hirayama. A basic period appearing in the solution depends on the element a . For certain values of a this period equals one of the fixed secular periods that are present in the differential equations, if the variations of e_j , i_j , and of analogous quantities are considered. Such a secular resonance needs a special treatment (Hagihara, 1972), since the difference of two respective frequencies causes a small divisor otherwise.

J.G. Williams (1969, 1971) proposed a new theory that allows to derive proper elements for most asteroids without a limitation in e or i . He starts with a zero-order approximation in neglecting the eccentricities and inclinations of the perturbing planets. These quantities are later considered by first-order terms, which introduce secular periods and can cause secular resonances again. Now the resonances are no longer restricted to specific values of a . They appear on surfaces in the space given by a and by proper elements corresponding to e and i . Comparatively few asteroids are close to these surfaces, but Williams (1969, 1973) performed a general study of such cases. There is evidence for a special type of libration of corresponding critical arguments. Strong variations of e or i can occur in case of a secular resonance, compare also the reviews by Peale (1976) and Greenberg (1977).

3. FORMER WORK ON SPECIAL ASTEROIDS WITH COMMENSURABLE MEAN MOTIONS

As demonstrated by Peale's (1976) review, asteroid-Jupiter commensurabilities contribute many interesting cases to the wider field of orbital resonances in the solar system. At first commensurable asteroids were studied with methods based on a development of the disturbing function, but later, especially in Japan, astronomers started to replace the approximate analytical theories by computations of orbital elements with numerical integration (Hagihara, 1961). This was most important for asteroids with n/n_J close to $3/2$, or $4/3$, compare the papers by Akiyama (1962) and Takenouchi (1962), who studied the motion of (153) Hilda and (279) Thule more recently. They found evidence for libration of σ in both cases. Belyaev and Chebotarev (1968), and Chebotarev et al. (1974) applied numerical integration over an interval of 400 yr to many interesting asteroidal cases of resonance, especially to Trojan asteroids (1/1-case). For some other cases compare Froeschlé and Scholl (1978). Schweizer (1969) used integration for the asteroids corresponding to the vicinity of the 2/1-case. This type of resonance was studied in detail by Franklin et al. (1975), after Marsden (1970) had treated this and other types as given by minor planets and comets. The 2/1-case corresponds to a typical Kirkwood gap (Hagihara, 1961), but (1362) Griqua and two other numbered asteroids show an oscillation of the mean motion about the exact value of resonance with respect to Jupiter. These three numbered objects have a large eccentricity, but there is some indication for the existence of Griqua-type objects with lower e as well (Franklin et al., 1975).

Numerical integration of averaged differential equations was mainly applied to the asteroids of the Hilda group (3/2-case). Planar models gave first results and demonstrated, that almost all asteroids of the group show libration of σ about 0° (Schubart, 1968; Ip, 1976), compare the next chapter. Direct numerical integration gave evidence about resonances and librations of Apollo and Amor asteroids with respect to earth and Venus (Janiczek et al., 1972; Ip and Mehra, 1973). The librations with respect to inner planets are temporary in most, and perhaps in all cases.

4. RECENT WORK ON GRIQUA- AND HILDA-TYPE ASTEROIDS

The author's results presented in this and the following chapter depend on three models for the acting forces. Model 1 is a generalization to three dimensions of the former computer program (Schubart, 1968), that allowed the elimination of short-period terms from the differential equations of the planar, elliptic restricted three-body problem by averaging, and then the numerical integration of these equations (compare Schubart, 1978). Model 2 is the corresponding rigorous elliptic sun-Jupiter-asteroid problem. Model 3 approximates the real variations of e_J and $\tilde{\omega}_J$, since Saturn is added to the bodies of the second model on an appropriate orbit that is turned into the orbital plane of Jupiter, so that this fixed plane can be used as the plane of reference

in all three models ($i_J = 0$). The masses of Jupiter and Saturn correspond to the IAU (1976) System of Constants, while all other perturbing masses are entirely neglected. The two last models are realized by an n -body program (Schubart and Stumpff, 1966). Further definitions are $a_J = 1$, $e_J = 0.048$, $\tilde{\omega}_J = 0^\circ$; they are related to a moment in 1975 August in case of model 3, together with $\ell_J = 0^\circ$. The time is counted from this moment, or otherwise from the date of osculation of the asteroidal orbit to be studied. The elements of this orbit are transformed to the new scale, plane of reference, and zero-direction of longitude.

The three numbered Griqua-type objects ($p = q = 1$) and ten members of the Hilda group ($p = 2$, $q = 1$) have been studied over periods that range from 20 000 yr to 140 000 yr in total and cover both past and future motion in most cases. Results by model 1 are available in all cases. They depend on orbital elements that were averaged over the interval of an appropriate short-period integration corresponding to model 2, before the long-term integration by model 1 started. This numerical integration allowed a step-length of a half to one year for twelve of the asteroids, but 0.25 yr was needed for (1921) Pala, a special case of 2/1-libration. From former experience, and from recent comparisons it is known that model 1 gives a good approximation to the conditions of model 2 except in special cases, compare chapter 1. Model 2 is expected to give a qualitatively correct picture of the real motion, but frequencies of very long periods and related amplitudes will differ from values found by more accurate models (Froeschlé and Scholl, 1978). Model 3 is important for tests about this, and for special cases with a strong influence of the motion of $\tilde{\omega}_J$ and of the long-term oscillation of e_J between 0.03 and 0.06 (compare Cohen et al., 1973). A comparison of results by model 1 for (1748) Mauderli with n -body results by Froeschlé and Scholl (1978), that correspond to model 3, appears satisfactory.

Orbital elements from Ephemerides of Minor Planets for 1978 are the basis of study for twelve of the asteroids under consideration. In case of (153) Hilda, the starting values used before (Schubart, 1978) were retained. Table 1 shows results obtained by model 1 for all 13 objects. The results on (1746), (1748), and (1921) refer only to future motion, although a backward computation is available for (1921), see below. During the intervals related to Table 1, all 13 asteroids show stable libration of σ about a mean value of 0° with a mean period T_L . The periods of retrograde revolution of perihelion (T_P) and node (T_N) are comparatively large. They are given in millennia and represent mean values in many cases. If (1921) Pala is excluded, the variations of the orbits appear to be quasi-periodic, and it is expected that this will show up in more extended computations by model 1 as well. It appears from more detailed studies of four objects, that mean frequencies corresponding to T_L , T_P , and T_N , and combinations of them, as the mean angular velocity of $\omega = \dot{\omega} - \Omega$, determine the changes in the elements of the orbits. Since it turned out that the former planar model (Schubart, 1968) gave a good description of basic effects for Hilda (Schubart 1978), the present study includes mainly objects with larger inclination and the more recently numbered Hilda-asteroids (compare Marsden, 1970). Table 1 shows for the

Table 1. Periods and extreme values for three 2/1-librators followed by ten Hilda-type asteroids (3/2-case). Results from model 1.

No.	Name	T_L	T_P	T_N	σ_M	e_M	i_M	Δ_m
1362	Griqua	396	10.6	36	122 ^o	0.37	23 ^o .6	2.11
1921	Pala	307	29	22	140	.56	20.2	1.97
1922	Zulu	422	26	132	98	.53	39.5	2.65
153	Hilda	275	2.7	22	43	0.24	9.3	1.89
361	Bononia	270	4.1	26	66	.29	12.6	1.94
1345	Potomac	273	3.8	27	50	.28	11.2	1.98
1746	Brouwer	278	1.8	18	54	.21	9.6	1.76
1748	Mauderli	253	2.7	17	87	.26	2.3	1.73
1754	Cunningham	269	3.5	24	70	.27	11.7	1.88
1877	Marsden	283	4.2	30	60	.29	18.2	1.98
1902	Shaposhnikov	277	3.3	26	35	.26	11.7	1.96
1911	Schubart	267	3.1	23	47	.26	3.1	1.93
1941	Wild	258	4.2	24	68	.30	3.2	1.95

Notes. T_L (yr): mean period of σ -libration; T_P, T_N (10^3 yr): mean or approximate periods of retrograde revolution of $\tilde{\omega}$ and Ω ; The subscript M indicates the maximum-value of $|\sigma| \leq 180^\circ$, e, or i found ($i_J = 0$); Δ_m is the smallest distance Jupiter-asteroid in AU. For (1921) Pala the backward computation is not considered.

ten objects of this type a sufficiently large minimum of distance to Jupiter, Δ_m , and a similar amount in each of the three periods. Differences in amount of T_P or T_N are correlated with Δ_m . The upper limit for e is about 0.3. (1877) Marsden has the largest inclination of the group.

The three 2/1-librators reach much larger values in e or i, and the amplitude of libration in σ is comparatively large, as found before by Franklin et al. (1975). Δ_m is large, since a is smaller than in the Hilda-case, and due to the libration. T_P and T_N are rather large, especially for (1922). They have a comparable amount in case of (1921). Although this turns out to hold only for a limited interval, it gives rise to slow variations of ω during this interval, and ω can remain between 0° and 90° for more than 20 000 yr. This seems to cause an exchange between the mean amounts of i and e. A backward computation leads to small values of e, large amplitudes in σ , and indicates the possibility of a temporary circulation of σ . A computation on (1921) Pala by model 2 gives analogous results. They are shown in Fig. 1 and Fig. 2, left curve. All these curves are smoothed with respect to effects by T_L . In Fig. 2 the solid part of the left curve extends backward in time for 30 000 yr. It shows temporary libration of $\tilde{\omega}$. A little open dot close to the curve indicates a time 14 500 yr ago, where one single revolution of σ is observed. Libration of σ takes place at earlier and later times. More typical examples of change between libration and circulation of σ were

described by Froeschlé and Scholl (1977). The right curve in Fig. 2, smoothed like the others, represents a solution of the planar elliptic restricted problem, that is not periodic. It has the quality of simultaneous libration of $\tilde{\omega}$ and σ (2/1-case), with large amplitudes about 0° in both cases. The two respective mean periods are about 18 000 and 340 yr. The amplitude and period for σ vary along the curve shown. Simultaneous small-amplitude oscillation of $\tilde{\omega}$ and σ may occur near stable equilibrium points, given by a Hamiltonian that corresponds to model 1. Libration of $\tilde{\omega}$ can also occur in nearly resonant orbits, if e is small, but in some cases studied, σ circulates, if $\tilde{\omega}$ librates. The plane orbit of Fig. 2 corresponds to starting values $a = 0.63$, $e = 0.37$, $\sigma = 133^\circ$ ($p = q = 1$), $\tilde{\omega} = \ell_J = 0^\circ$. Model 2 was used with $i = 0^\circ$.

The solid part of the left curve in Fig. 2 resembles the right one. The approximate symmetry of these curves with respect to the direction $\tilde{\omega}_J = 0^\circ$ suggests, that the amount of e_J is important for the effects of libration in $\tilde{\omega}$. Probably due to the comparatively small amount of e_J in the respective interval, a backward computation for (1921) Pala by model 3 did not confirm the temporary libration of $\tilde{\omega}$ and circulation of σ . However, a corresponding forward computation agrees to the prediction of a slowly changing ω , and of big changes in e and i , and it may be expected, that libration of $\tilde{\omega}$ and circulation of σ will occur at more remote times, when the phase of oscillation of e_J is more favorable for this.

5. HILDA- AND THULE - TYPE ORBITS OF HIGH INCLINATION

The known asteroids of Hilda- and Thule-type have a small or moderate inclination. However, there is theoretical evidence for the possibility of long-term motion on highly inclined orbits at the 3/2 and 4/3 commensurabilities as well. This appears from the author's recent studies (compare Schubart, 1978) by means of models 1-3. Orbits were found that give evidence for a permanent simultaneous libration of ω about 90° and of σ about 0° , with different main periods of libration, and with a sufficiently large minimum distance Δ_m . The following two sets of starting values refer to typical orbits of this kind with small-amplitude libration.

$$p = 2, q = 1, a = 0.7630, e = 0.54, i = 41^\circ, \omega = 90^\circ;$$

$$p = 3, q = 1, a = 0.8277, e = 0.38, i = 30^\circ, \omega = 90^\circ.$$

These values were used together with $\sigma = \tilde{\omega} = \ell_J = 0^\circ$. Studies by both models 2 and 3 cover more than 30 000 yr in each case. The observed amplitudes of ω -libration do not exceed 10° and 22° in the two cases, respectively. The amplitudes in σ are small too. Since high-inclination orbits occur in other regions of the asteroid belt, there may be small unknown objects on orbits of the above types as well.

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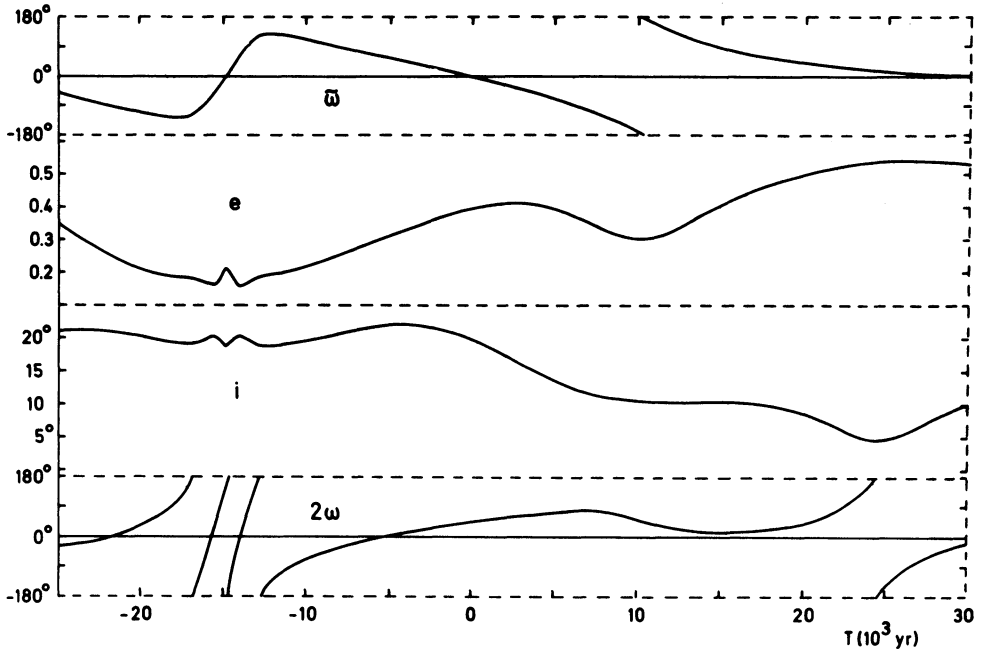


Fig. 1. Smoothed curves representing a solution of the elliptic restricted problem (model 2) at the 2/1-resonance. $\tilde{\omega}$, e , i , 2ω are plotted against time. The starting values correspond to (1921) Pala.

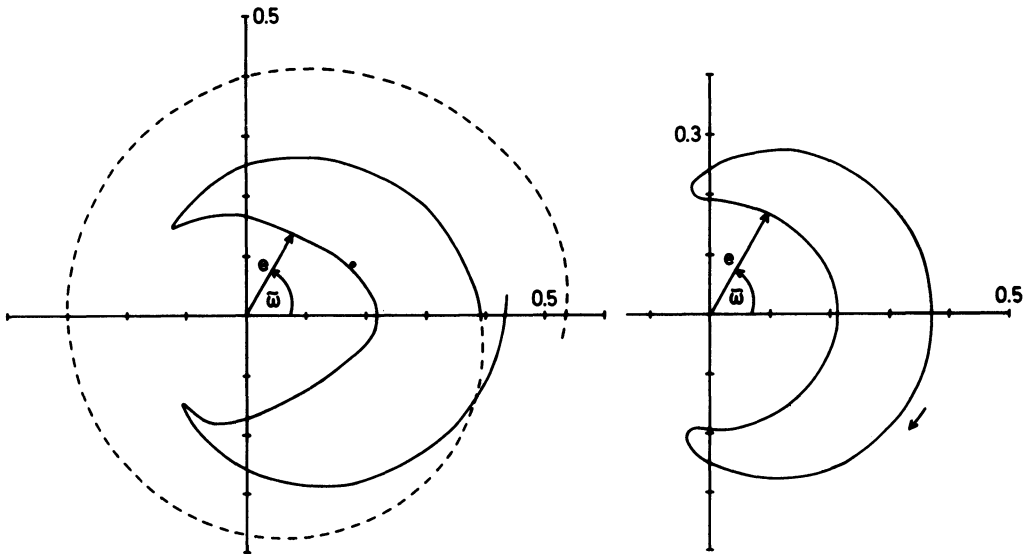


Fig. 2. Smoothed curves showing e and $\tilde{\omega}$ in polar coordinates. The left curve corresponds to the upper part of Fig. 1 (dashed line gives future motion), the right one to a case of planar motion (see text).

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DISCUSSION

Garfinkel: What is the number of secondary resonances that you mentioned in your talk?

Schubart: I have only mentioned cases of simultaneous libration. I did not treat resonances of σ with respect to short-period arguments.

Garfinkel: How do you treat the critical divisors?

Schubart: I have done numerical integration, where such divisors do not arise.

Marsden: In the model giving circulation of σ for (1921) Pala, what was the minimum approach distance to Jupiter?

Schubart: In the backward computation by model 2 the minimum approach distance was about 1.3 AU.