LIMITS OF LATTICES IN A COMPACTLY GENERATED GROUP

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1. Introduction. Let G be a locally compact and σ-compact[1] topological group and let H be a discrete subgroup of G.[2] We shall use G/H to denote the space of right cosets Hx of H with the usual topology (cf. (**8**, pp. 26–28)). Let μ be the left Haar measure in G. μ induces a measure in the space G/H;[3] this measure will, without ambiguity in this paper, also be denoted by μ. If $\mu(G/H)$ is finite, the group H is called a *lattice.* If the space G/H is compact, then H is certainly a lattice and is called a *bounded lattice.* These terms are an extension of the usage of the Geometry of Numbers, where G is the real n-dimensional vector space R^n. In this case any lattice is generated by n linearly independent vectors, all lattices are bounded, and the whole family of lattices is permuted transitively by the automorphisms of G (which are the non-singular linear transformations). The constant $\mu(G/H)$ is called the determinant of H in this case. The family of all lattices in Euclidean space forms a locally compact topological space. In (**7**) Mahler proved the following

SELECTION THEOREM. *Let $\{H_n\}$ be a sequence of lattices in R^n with the following properties*

(i) *There is a neighbourhood V of the zero-vector e such that, for all n, $H_n \cap V = \{e\}$,*

(ii) *$\mu(G/H_n)$ is bounded above.*

Then there exists a subsequence $\{H_{n'}\}$ of $\{H_n\}$ which converges to a lattice H.

Let now G, H, and μ be as in the beginning. Mahler's theorem suggests two definitions. [Notation: e is the unity of G, $\mathbf{N}$ the class of open sets containing e; $\bar{K}$, fr$K$ are closure and boundary of K; $\cup$, $-$, $\cap$ denote the set union, difference, intersection. We use φ: $G \rightarrow G/H$ for the natural mapping $\varphi(x) = Hx$.]

DEFINITION 1. A sequence $\{H_n\}$ of subgroups of G is called *uniformly discrete* if $H_n \cap V = \{e\}$ for a certain $V \in \mathbf{N}$ and all n.

DEFINITION 2. A sequence $\{H_n\}$ of subgroups of G *converges to a subgroup H* if, given any compact set C and any $V \in \mathbf{N}$,

$$H \cap C \subset H_n V \quad \text{and} \quad H_n \cap C \subset HV$$

holds for all but a finite number of n.

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[1]That is, G is a countable union of compact sets.

[2]As is well known, this implies that H is countable.

[3]We give a precise definition of the induced measure in § 3.

Chabauty **(2)** has generalized Mahler's theorem by showing that a uniformly discrete sequence $\{H_n\}$ of subgroups of G has a subsequence converging to a discrete subgroup H and moreover

$$\mu(G/H) \leqslant \liminf \mu(G/H_n) \tag{1}$$

so that H is a lattice if all the H_n are and $\mu(G/H_n)$ is bounded.

In the classical case $G = R^n$, it is of course easy to show that

$$\mu(G/H) = \lim \mu(G/H_n) \tag{2}$$

and Chabauty has shown that in certain circumstances this is true also for topological groups G. In this paper we make a further contribution to this problem by proving that, if H is a bounded lattice, then a necessary and sufficient condition for (2) to hold is that G should be compactly generated[4] or that H should be finitely generated. We shall give an example due to M. Kneser showing that the boundedness of H is essential. Thus it might seem better to consider bounded lattices only, particularly since in Geometry of Numbers all lattices are bounded. Unfortunately however, a lattice which is a limit of bounded lattices need not be bounded. In § 6 we shall give an example of such a lattice where G is a homomorphic image of the group of 2 by 2 matrices with determinant unity.

2. Fundamental domain. As in **(5)** and **(10)** a Borel set P will be called a packing if $P \cap hP = \phi$ for $e \neq h \in H$ and a Borel set C will be called a *covering* if $HC = G$. F is called a *fundamental domain* if it is both a packing and a covering. In cases of ambiguity we may refer to an H-packing, H-covering, or H-fundamental domain.

In this section we show, extending a result of Chabauty **(1)**, and Siegel **(9)**, that there is a fundamental domain F with $\mu(\mathrm{fr}F) = 0$, and also that if G/H is compact then there is such a fundamental domain with compact closure $\bar{F}$.

We shall overlap in places with Chabauty's results. We start with a lemma which shows that Chabauty's axiom (M) is always satisfied.

LEMMA 2.1. *If C is compact and U is open, $C \subset U$, then there is a Baire measurable open set V such that*

$$C \subset V \subset U, \quad \mu(\mathrm{fr}\,V) = 0.$$

In particular, taking $C = \{e\}$, there is a fundamental system of neighbourhoods of the identity each of which has a frontier of measure 0.

Proof. Since the measure μ is regular, and the measure of any compact set is finite **(4**, §§ 64 and 52**)**, we may assume, on replacing U by an open subset, if necessary, that $\mu(U) < \infty$. Since the group G is a completely regular space **(8**, p. 29**)**, a continuous function $f(x)$ exists such that $f(x) = 0$

[4]That is, have a compact set of generators.

for $x \in C, f(x) = 1$ for $x \notin U$. Let $E(r) = \{x : f(x) < r\}$. The function $\mu(E(r))$ is a monotonic function of the real variable r, and therefore has at most countably many discontinuities. Let r_0 be a value at which it is continuous. Then

$$\overline{E(r_0)} = \bigcap \{W : W \text{ open}, W \supset E(r_0)\} \subset \bigcap_{r > r_0} E(r).$$

Hence

$$\mu(E(r_0)) \leqslant \mu(\overline{E(r_0)}) \leqslant \lim_{r \to r_0} \mu(E(r)) = \mu(E(r_0)).$$

This completes the proof, since $V = E(r_0)$ is a Baire set.

The following two lemmas are easily verified:

LEMMA 2.2. *If A, B are packings and $C = (A - HB) \cup B$, then C is a packing and $HC = HA \cup HB$.*

LEMMA 2.3.

$$B \cap \mathrm{fr}A \subset \mathrm{fr}(A \cap B) \cup \mathrm{fr}B.$$

We begin now the construction of a fundamental domain. Our final result will be as follows.

THEOREM 1. *There is a fundamental domain F such that*
(i) $\mu(\mathrm{fr}F) = 0$;
(ii) *If G/H is compact, there exists a fundamental domain F satisfying* (i) *such that also $\bar{F}$ is compact.*

The proof of this Theorem is closely modelled on that of Siegel **(9)**.

Proof. Since G is locally compact and H is discrete, we can, by Lemma 2.1, choose $V \in \mathbf{N}$ so that $\mu(\mathrm{fr}\,V) = 0$, $\bar{V}$ is compact, and V is an H-packing. Since G is σ-compact, $G \subset \bigcup Vx_i$ for some sequence $\{x_i\} \subset G$. Define $F_1 = Vx_1$, $F_n = Vx_n - H(Vx_1 \cup \ldots \cup Vx_{n-1})$. Let $F = \bigcup F_n$. Then clearly F is an H-packing, since F_n is and since $F_m \cap hF_n = \phi$. Also $HF = G$, for if $g \in G$, there is a least integer n such that $g \in HVx_n$ and then $g \in HF_n \subset HF$. Thus F is a fundamental domain.

To show that $\mu(\mathrm{fr}F) = 0$, set $C_n = Vx_1 \cup \ldots \cup Vx_{n-1}$. Then $\mathrm{fr}C_n \subset \mathrm{fr}\,Vx_1 \cup \ldots \cup \mathrm{fr}\,Vx_{n-1}$, so $\mu(\mathrm{fr}C_n) = 0$. Also

$$F_n = Vx_n - HC_n = Vx_n - \bigcup_{h \in H} (hC_n \cap Vx_n).$$

If empty terms are dropped from the last union, only those h remain for which $h \in Vx_nC_n^{-1}$. Since $Vx_nC_n^{-1}$ is a bounded set, the number of h is finite, say $h_1, \ldots h_r$, and we have

$$F_n = Vx_n - (h_1C_n \cup \ldots \cup h_rC_n)$$

$$\mu(\mathrm{fr}F_n) \leqslant \mu(\mathrm{fr}\,Vx_n) + \sum_{i=1}^{r} \mu(\mathrm{fr}h_iC_n) = 0.$$

By Lemma 2.3, $Vx_n \cap \mathrm{fr}F \subset \mathrm{fr}(Vx_n \cap F) \cup \mathrm{fr}\,Vx_n = \mathrm{fr}F_n \cap \mathrm{fr}\,Vx_n$. Thus $\mu(\mathrm{fr}F) \leqslant \sum \mu(Vx_n \cap \mathrm{fr}F) = 0$.

In the case when G/H is compact, G/H can be covered by a finite union $\varphi(Vx_1) \cup \ldots \cup \varphi(Vx_n)$, so $F = F_1 \cup \ldots \cup F_n$ will be a fundamental domain. Since F is then contained in the bounded set C_{n+1}, it is itself bounded. This completes the proof of Theorem 1.

We conclude this section with a slightly more precise form of the statement of Theorem 1. This is required for a later application.

LEMMA 2.4. *If S is any covering, then there is a fundamental domain contained in S.*

Proof. Let F be any fundamental domain. We have $F \subset HS$. Thus F is a union of h-translates of subsets of S and therefore F is also a disjoint union of h-translates of subsets of S, say $F = h_1 S_1 \cup h_2 S_2 \cup \ldots$. It is obvious that $F_0 = S_1 \cup S_2 \cup \ldots$ is a fundamental domain contained in S.

3. The induced measure in G/H. Since we regard the group H as a group of permutations acting on G by left translation, it follows that each H-orbit is a *right* coset Hx. This is why we use G/H for the space of right cosets, instead of the more usual homogeneous space of left cosets. On the space G/H the group G acts transitively by *right* translation. If $\Delta(x)$ is the real-valued function defined on G by the relation $\mu(Ex) = \Delta(x) \cdot \mu(E)$, then it follows from the criterion in (**11**, p. 45) that there is a measure $\tilde{\mu}$ on Borel subsets $\tilde{E}$ of G/H such that $\tilde{\mu}(\tilde{E}x) = \tilde{\mu}(\tilde{E}) \cdot \Delta(x)$. For our purposes it is more convenient to define μ directly from the natural mapping $\varphi: G \rightarrow G/H$, ($\varphi(x) = Hx$), as follows: If F is any fundamental domain, define

$$\tilde{\mu}(\tilde{E}) = \mu(\varphi^{-1}(E) \cap F).$$

It follows from (**5**, Theorem 1, Corollary), applied to the measure space $\varphi^{-1}(E)$ and the group H of transformations of this space, that this expression does not depend on the particular fundamental domain chosen. We shall, for $S \subset G$, use S/H to denote $\varphi(S)$ and we shall write μ for $\tilde{\mu}$.

We conclude this section with three lemmas which will be useful later.

Before stating the first lemma, we note that if G_1 is any open subgroup of G, the same measure μ, but with its domain of definition restricted to G_1 will serve as a Haar measure on G_1.

LEMMA 3.1. *If G_1 is an open subgroup of G and $H_1 = H_1 \cap H$, then $\mu(G_1/H_1) = \mu(G_1/H)$.*

Proof. Let F_1 be a fundamental domain for H_1 in G_1. Then $hF_1 \cap F_1 \neq \phi$, $h \in H$, implies $h \in F_1 F_1^{-1} \subset G_1$, so $h \in H_1$ and $h = e$. Thus F_1 is an H-packing. If F is a fundamental domain for H in G, then, so is $F^* = F_1 \cup (F - HF_1)$, by Lemma 2.2. By our definition of induced measure,

$$\mu(G_1/H) = \mu(F^* \cap G_1) = \mu(F_1) = \mu(G_1/H_1).$$

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LEMMA 3.2. *If $H_1 \subset H$ and $H{:}H_1$ denotes the index of H_1 in H, then we have*

$$\mu(G/H_1) = (H{:}\, H_1)\mu(G/H).$$

Proof. Let F be an H-fundamental domain and let X be a complete system of representatives of left cosets of H_1 in H. One checks that XF is an H_1-fundamental domain and our result follows then since $\overline{\overline{X}} = H{:}\, H_1$.

LEMMA 3.3. *If $H \subset G_1$, where G_1 is an open subgroup of G, then*

$$\mu(G/H) = (G{:}\, G_1)\mu(G_1/H).$$

Proof. If G_1 is not unimodular, neither is G and $\mu(G/H) = \mu(G_1/H) = \infty$ (**9**, Lemma 5). Suppose next that G_1 is unimodular, but not G. Then, since G_1 is open, μ is also the Haar measure for G_1 and we have $\Delta(x) = 1$ for $x \in G_1$. However, if $\Delta(x) \neq 1$, where $x \in G$, then $\Delta(x^n) \neq 1$ for each natural n. All the elements x^n must then belong to different left cosets of G_1 and hence $G{:}\, G_1 = \infty$. Again both sides are infinite.

The remaining case to consider is when G is unimodular. Then, if F is an H-fundamental domain for G_1 and X is a complete system of representatives of right cosets of G_1, we verify that FX is an H-fundamental domain for G. Since G is unimodular our result follows from $\overline{\overline{X}} = G{:}\, G_1$.

LEMMA 3.4. *If G_1 is an open subgroup of G, then*

$$\mu(G/H) \leqslant (G{:}\, G_1)\mu(G_1/H).$$

Proof. Let $H_1 = G_1 \cap H$. By Lemmas 3.1, 3.2, and 3.3 we have

$$\mu(G/H_1) = (G{:}\, G_1)\mu(G_1/H_1) = (G{:}\, G_1)\mu(G_1/H),$$
$$\mu(G/H_1) = (H{:}\, H_1)\mu(G/H) \geqslant \mu(G/H).$$

This proves the lemma.

LEMMA 3.5. *If K is an open subgroup of G and HK is also a subgroup, then $\mu(HK/H) = \mu(K/K \cap H)$.*

Proof. Let F be a $(K \cap H)$-fundamental domain for the group K. One checks that F is an H-fundamental domain for HK.

4. Limits of discrete subgroups. In this section we assume G/H compact. We consider the following two closely related questions:

I. In what groups G does the relation $\lim H_n = H$ imply $\lim \mu(G/H_n) = \mu(G/H)$ for any uniformly discrete sequence of subgroups $\{H_n\}$?

II. Under what circumstances does $\lim H_n = H$ imply $\lim \mu(G/H_n) = \mu(G/H)$ if $\{H_n\}$ is restricted to be a uniformly discrete sequence of *lattices*?

Our answer to I is complete, given by the theorem below. As to question II we give a little extra information in Theorem 3. Another kind of answer was found by Chabauty and we present in § 5 an alternative proof of his result (our Theorem 4).

THEOREM 2. *The following four statements are equivalent:*
(i) *G is compactly generated.*
(ii) *H is finitely generated.*
(iii) *If $\{H_n\}$ is a sequence of discrete subgroups,* $\lim H_n = H$, *then*
$$\limsup \mu(G/H_n) \leqslant \mu(G/H).$$
(iv) *If $\{H_n\}$ is a uniformly discrete sequence of subgroups,* $\lim H_n = H$, *then*
$$\lim \mu(G/H_n) = \mu(G/H).$$

Proof. We have proved in a recent paper that (i) implies (ii) (see **6**). Suppose (ii) holds. It follows from Theorem 1 (ii) that there exists an H-fundamental domain F with compact closure $\bar{F}$. If Γ is the finite set of generators of H, then the compact set $\Gamma \cup \bar{F}$ is obviously a set of generators of G. Hence (ii) implies (i) and so (i) and (ii) are equivalent. By Chabauty's inequality (1), (iii) implies (iv). Thus it remains to prove that (iv) implies (ii) and that (i) implies (iii).

Proof that (iv) *implies* (ii). Suppose that (ii) is false. Then H being countable let its elements be enumerated $h_1, h_2, \ldots$, and let H_n be the subgroup generated by the elements $h_1, \ldots, h_n$. If C is a compact set, $C \cap H$ is finite and if n_0 is the largest value of r for which h_r lies in C, we have $C \cap H = C \cap H_n$ for $n > n_0$. Thus $\lim H_n = H$. However, the index $H{:}\,H_n$ is infinite, otherwise H would have a finite system of generators given by $h_1, \ldots, h_n$ together with a complete system of representatives of the H_n-cosets. It follows from Lemma 3.2 that $\mu(G/H_n) = \infty$ for all n. But $\mu(G/H) < \infty$, so (iv) is false.

Proof that (i) *implies* (iii). Let F be an H-fundamental domain with compact closure $\bar{F}$, such that $\mu(\mathrm{fr}F) = 0$. We have $\mu(G/H) = \mu(F) = \mu(\bar{F})$. Let $\epsilon > 0$. We have to show that, for sufficiently large n, $\mu(G/H_n) < \mu(G/H) + \epsilon$. Choose $V \in \mathbf{N}$, $\bar{V}$ compact, so that

$$\mu(VF) < \mu(F) + \epsilon. \tag{3}$$

Let D be a compact system of generators of G. Replacing D by $D \cup D^{-1}$, if necessary, we may assume that

$$\bigcup_1^\infty D^k = G. \tag{4}$$

The set $\overline{VFDF^{-1}}$ is compact, so there is a finite number of elements $h_1, \ldots, h_r$ of H in it. We have $VFD \subset HF$; but $hF \cap VFD = \phi$ unless $h \in VFDF^{-1}$, that is, unless h is one of the elements $h_1, \ldots, h_r$. It follows that

$$VFD \subset h_1F \cup \ldots \cup h_rF. \tag{5}$$

Since $\lim H_n = H$, there is a number n_0, such that, for $n > n_0$, $H_n V$ contains each of the elements $h_1, \ldots, h_r$, and hence from (5), $VFD \subset H_n VF$. But H_n is a subgroup, $H_n = H_n{}^k$ for each integer k, and thus

$$VFD^k \subset H_n VFD^{k-1} \subset \ldots \subset H_n^{k-1} VFD \subset H_n^k VF = H_n VF.$$

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Thus $G = H_n VF$ by (4). Hence VF is an H_n-covering and by the theorem on packings and coverings in **(5)** it follows from (3) that

$$\mu(G/H_n) \leqslant \mu(VF) \leqslant \mu(G/H) + \epsilon.$$

To state our next theorem briefly, it is convenient to have another definition. A pair (G, H) consisting of a locally compact σ-compact group G and a discrete subgroup H with G/H compact will be called a *tractable pair* if the following condition holds. Given any uniformly discrete sequence $\{H_n\}$ of lattices in G such that $\lim H_n = H$, then $\lim \mu(G/H_n) = \mu(G/H)$.

THEOREM 3. *If G contains an open compactly generated subgroup K such that for $h \in H$*

$$hKh^{-1} = K \tag{6}$$

then (G, H) is tractable if and only if (H, H) is tractable.

Proof. It is quite clear that if (H, H) is not tractable, then (G, H) is not tractable. For there will be a sequence $\{Q_n\}$ of subgroups of H of finite index such that $\lim Q_n = H$, but $H{:}\, Q_n > 1$ for infinitely many n. By Lemma 3.2, $\mu(G/Q_n) = (H{:}\, Q_n)\mu(G/H) \geqslant 2\mu(G/H)$ for infinitely many n. Thus (G, H) is not tractable.

We now assume therefore, that (H, H) is tractable and our aim is to prove that (G, H) is tractable. We shall show that if $\{H_n\}$ is a sequence of lattices in G and $\lim H_n = H$, then

$$\limsup \mu(G/H_n) \leqslant \mu(G/H). \tag{7}$$

Hence for a uniformly discrete sequence $\{H_n\}$ of lattices we have by (1), $\lim \mu(G/H_n) = \mu(G/H)$, that is, (G, H) is tractable.

Since the topology in H is discrete, our assumption that (H, H) is tractable means that, if $\{Q_n\}$ is a sequence of subgroups of H with the following properties:

$$\text{(8)}\quad \text{(i)}\qquad H = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} Q_n,$$

$$\text{(ii)}\qquad H{:}\, Q_n < \infty,$$

then there is a number n_0 such that $H = Q_n$ for $n > n_0$.

Suppose now that $\{H_n\}$ is a sequence of lattices in G such that $\lim H_n = H$. To show (7) we shall associate with the sequence $\{H_n\}$ a sequence $\{Q_n\}$ of subgroups of H which satisfies the conditions (8). We observe first that, by (6), HK and

$$M_n = H_n \cap HK, \qquad P_n = M_n K, \qquad Q_n = H_n K \cap H$$

are subgroups of G and moreover P_n is open.

LEMMA 4.1. $H{:}\, Q_n = HK{:}\, P_n$.

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Proof. One checks easily that any complete system of representatives of left cosets of Q_n in H is also a complete system of representatives of left cosets of P_n in HK.

LEMMA 4.2. *For $n > n_0$, $Q_n = H$, $P_n = HK$.*

Proof. By Lemma 4.1 it is enough to show that $Q_n = H$. Since (H, H) is tractable this follows if we show that conditions (8) are satisfied. To prove that Q_n has finite index, we note that, by Lemmas 3.1 and 3.3,

$$(HK: P_n)\mu(P_n/M_n) = \mu(HK/M_n) = \mu(HK/H_n) \leqslant \mu(G/H_n) < \infty.$$

Now P_n is for sufficiently large n a non-empty open set, so $\mu(P_n/M_n) > 0$, and by Lemma 4.1, $H_n: Q_n = HK: P_n < \infty$.

To show that (8) (i) holds we have to show that if $h \in H$, then, for sufficiently large n, $h \in Q_n$. To see this we note that $K \in \mathbf{N}$, so for sufficiently large n, $hK \cap H_n \neq \phi$, that is, $h \in H_nK$. This proves our lemma.

We are now in a position to prove (7). By Theorem 2, since K is compactly generated

(9) $$\limsup \mu(K/K \cap H_n) \leqslant \mu(K/K \cap H).$$

From Lemma 3.5, we have $\mu(HK/H) = \mu(K/K \cap H)$. If, in Lemma 3.5 we replace H by M_n so that HK is replaced by P_n, we find that $\mu(P_n/M_n) = \mu(K/K \cap H_n)$. From Lemma 3.1, we have $\mu(P_n/H_n) = \mu(P_n/M_n)$ since $P_n \cap H_n = M_n$. Hence $\mu(P_n/H_n) = \mu(K/K \cap H_n)$ and substituting in (9) we derive

(10) $$\limsup \mu(P_n/H_n) \leqslant \mu(HK/H).$$

For sufficiently large n we have, by Lemma 4.2,

(11) $$\mu(HK/H_n) = \mu(P_n/H_n).$$

Using (10), (11), and Lemmas 3.3 and 3.4,

$$\begin{aligned}\mu(G/H) = (G: HK)\mu(HK/H) &\geqslant (G: HK) \limsup \mu(P_n/H_n) \\ &= (G: HK) \limsup \mu(HK/H_n) \geqslant \limsup \mu(G/H_n).\end{aligned}$$

This completes our proof.

5. A result of Chabauty.

We shall give now an alternative proof of a theorem of Chabauty **(1)** which combined with (1) yields another kind of answer to our question II.

THEOREM 4. *If $\{H_n\}$ is a sequence of lattices,* $\lim H_n = H$ *and there exists a set S of finite measure which is an H_n-covering for each n, then*

$$\limsup \mu(G/H_n) \leqslant \mu(G/H).$$

Proof. Let F, F_n denote the H and H_n-fundamental domains so that

$$\mu(G/H_n) = \mu(F_n), \qquad \mu(G/H) = \mu(F).$$

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By Lemma 2.4 we may assume $F_n \subset S$. From $S \subset HF$ follows that we can cover S, except for a set of arbitrarily small measure, by a finite union $h_1F \cup \ldots \cup h_mF$, $h_i \in H$. Since $H = \lim H_n$ it follows that these sets in turn can be approximated by unions

$$h_1^{(n)}F \cup \ldots \cup h_m^{(n)}F, \qquad \text{where} \qquad h_i^{(n)} \in H_n.$$

Therefore, for sufficiently large n, an arbitrarily small part of S remains uncovered by H_nF. Hence, by $F_n \subset S$, we have $\lim\,[\mu(F_n) - \mu(F_n \cap H_nF)] = 0$. Since

$$\mu(F_n \cap H_nF) = \mu\Big(\bigcup_{H_n} (F_n \cap h\,F)\Big) \leqslant \sum_{H_n} \mu(F_n \cap h\,F) = \sum_{H_n} \mu(h^{-1}F_n \cap F)$$
$$= \mu(H_nF_n \cap F) = \mu(F)$$

the theorem follows.

6. Examples. In this section we give three examples illustrating different possible properties of convergent sequences of discrete subgroups.

Example 1. It follows from Theorem 2 that, if G is compactly generated, G/H_n compact and $\lim H_n = H$, then $\lim\sup \mu(G/H_n) \leqslant \mu(G/H)$. To show that this need not be true if G is not compactly generated, take $G = H = G_1 \times G_2 \times \ldots \times G_n \times \ldots$, the weak direct product of a countable family of cyclic groups of order 2, with the discrete topology. Define H_n to be the set of all $g = (g_1, g_2, \ldots, g_n, \ldots,) \in G$ with $g_n = e$. Then $\mu(G/H_n) = 2$, $\lim H_n = H$, $\mu(G/H) = 1$.

Example 2. In this example G is a connected Lie group, and G/H_n is compact for each n, but G/H is not compact. Let G be the group of all linear transformations

$$w = \frac{az+b}{cz+d},$$

where w, z are complex variables, a, b, c, d are real and $ad - bc > 0$. In addition to G we consider the set G_1 of inversions, that is, transformations of the form

$$w = \frac{(a\bar{z}+b)}{(c\bar{z}+d)},$$

where $\bar{z}$ is the complex conjugate, and a, b, c, d are real with $ad - bc < 0$. The set $G \cup G_1$ is a group of transformations of the upper half-plane $\Re z > 0$ on itself, and G is a normal subgroup of index 2. The topology is the natural one obtained from the variables a, b, c, d.

Let P be the point $i = \sqrt{-1}$, and let $Q = ki(1 < k < \sqrt{3})$ be a variable point on the imaginary axis. Let $C(Q)$ be the circle through Q with centre on the positive real axis and cutting the imaginary axis at an angle $\frac{1}{3}\pi$. Let $C(Q)$ cut $|z| = 1$ in R and consider the curved triangle PQR, made up of

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part of the imaginary axis and parts of the circles. As k varies between 1 and $\sqrt{3}$, the angle at R will decrease continuously from $\frac{1}{6}\pi$ to 0. Thus there will be a sequence of points $Q_7, Q_8, Q_9, \ldots$, and corresponding points $R_7, R_8, R_9, \ldots$, such that the angles at R take the values $\frac{1}{7}\pi, \frac{1}{8}\pi, \frac{1}{9}\pi, \ldots$

It is easy to see that the subgroup K_n of $G_1 \cup G$, generated by the operations of inversion in the circles PR_n, Q_nR_n and reflection in the line PQ_n is a discrete subgroup of $G \cup G_1$. Let $K_n \cap G = H_n$. Regarded as a group of transformations of the complex plane, it has as a fundamental domain the interior of the curved triangle PQ_nR_n, the reflection of this triangle in the line PQ_n, together with some of the boundary points of this region. It is one of the triangle groups well known in the theory of automorphic functions **(2; 3)**.

A H_n-fundamental domain in G is the set of all mappings t of G such that tP lies in the fundamental domain in the z-plane just described. For each n, the closure of the triangle PQ_nR_n lies in the interior of the upper half-plane, so G/H_n is compact.

The limit H of the sequence H_n has a fundamental domain which is obtained in the same way from the triangle $PQ_\infty R_\infty$, where $Q_\infty = i\sqrt{3}$, $R_\infty = -1$, and the R-angle of the curved triangle is zero. However, G/H is not compact because the closure of its fundamental domain contains the point R_∞, which is a boundary point of the upper half plane, and is not equal to tP for any $t \in G$.

Example 3. This example indicates that the conclusions of Theorem 2 cease to be true if G/H is not compact, even when G is connected and H finitely generated. The example was suggested to us in conversation by Professor Martin Kneser, and we are grateful to him for permission to include it here.

Let P, Q, R, S be four points on the real axis in the order indicated. Consider the operations t_1, t_2, t_3, t_4 of inversion in the circles on diameters SP, PQ, QR, RS. These generate a discrete subgroup H of $G \cup G_1$ which is a free product of four cyclic groups of order 2. Its fundamental domain in the upper half plane is the interior of the curved quadrilateral $PQRS$. Keep P, Q, S fixed and let R pass through a sequence of points tending to S. The group H will tend to a limit H_∞ which is generated by inversions in the circles SP, PQ, QS. The fundamental domain in the half-plane is the triangle PQS.

Now in the hyperbolic plane, the area of triangles with zero angles is a constant. Since the quadrilateral $PQRS$ is a union of two such triangles, its area is twice the area of the triangle PQS. Returning to the original group-space, we deduce without difficulty that

$$\mu(G/H \cap G) = 2\mu(G/H_\infty \cap G).$$

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