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EXISTENCE AND MULTIPLICITY RESULTS FOR SEMICOERCIVE UNILATERAL PROBLEMS

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In this paper, we investigate a general class of variational inequalities. Existence and multiplicity results are obtained by using minimax principles for lower semicontinuous functions due to A. Szulkin.

1. INTRODUCTION

The aim of this paper is the study of problems in mechanics characterised by a general mechanics law which may be written in the form

$$0 \in \operatorname{grad} W(u) + \partial\Phi(u), \ u \in U,$$

that is, the law is the sum of a potential law and a superpotential law. W is the potential and Φ is a proper convex function and is the superpotential. We denote by $\partial\Phi$ the convex subdifferential of Φ. U is the space of all fields of possible displacements. In this paper it will be assumed that W can be written in the following form

$$W(u) = \langle u, Tu\rangle/2 + Cu,$$

where T is linear symmetric and C is $C^1(U, \mathbb{R})$.

As an example, we shall consider the following problem: let $T > 0$ and let $H^1(\Pi, \mathbb{R})$ be the Sobolev space obtained by completing the set of C^∞ real-valued T-periodic functions on $\Pi := \mathbb{R}/T\mathbb{Z}$ with the norm

$$\|u\| = \int_0^T |u|^2 + |\dot{u}|^2 \, dt.$$

Let K be the closed convex cone defined by

$$K := \{u \in H^1(\Pi, \mathbb{R}) : u(x) \geqslant 0 \text{ on } [0, T]\}.$$

We consider the following periodic unilateral problem

$$[\mathrm{P}_0] \qquad u \in K: \int_0^T \dot{u}.(\dot{v} - \dot{u})dt + \int_0^T \nabla_u V(t, u).(v - u)dt \geqslant 0, \quad \forall\, v \in K,$$

when V is α-positively homogeneous with respect to u and such that

(a) $\forall\, u \in \mathbb{R} \setminus \{0\}$: $\int_0^T V(t, u)dt > 0$,

(b) $\exists v \in \mathbb{R}^+$: $V(., v) < 0$ on a non zero measure subset.

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In this case,

$$W(u) := \frac{1}{2}\int_0^T |\dot{u}|^2\, dt + \int_0^T V(t,\, u) dt,$$

where the former term correspond to the kinetic energy and the latter to the potential energy. $\Phi(u) := I_K$ characterises the constraints on the displacement field.

We now detail the framework of our paper.

Let $\langle X,\, X^*\rangle$ be a dual system of real Hilbert spaces, let $T\colon X \to X^*$ be a symmetric bounded linear operator and let $C \in C^1(X, \mathbb{R})$. C is assumed to be β-positively homogeneous that is, $\langle C'(u),\, u\rangle = \beta C(u)$) and strongly continuous (that is C maps weakly converging sequences into converging sequences). Let $\Phi\colon X \to (-\infty, +\infty]$ be a proper convex functional. Φ is assumed to be α-positively homogeneous and weakly lower semicontinuous.

We are looking for *non trivial solutions*, that is, $x^* \notin \operatorname{Ker} T$, of the following variational inequality:

[P] $$x^* \in X : \langle v - x^*,\, Tx^* + C'(x^*)\rangle + \Phi(v) - \Phi(x^*) \geqslant 0, \quad \forall\, v \in X.$$

In case of bilateral problems (that is, $\Phi = 0$), the first result concerning this problem is due to Lassoued [**2**]. Recently, using a version of the Ljusternik-Schnirelman theory on C^1-manifolds due to Szulkin [**5**] Ben Naoum, Troestler and Willem obtained a general abstract existence and multiplicity theory for bilateral problems [**1**], where homogeneous second order differential equations were considered. For a basic work on critical point theory and its applications to bilateral problems we shall refer to [**3**]. In this paper, we use a version of minimax primciples for lower semicontinuous functions due to Szulkin, to get new results for the variational inequality [P] and related unilateral problems.

2. Existence result

Theorem 2.1. *If the following conditions hold true: $\alpha < \beta < 2$ and*

(1) *T is semicoercive, that is, there exists $c > 0$ such that*

$$\langle x,\, Tx\rangle \geqslant c.\, \|Px\|^2 \quad \text{for each} \quad x \in X$$

with $P = I - Q$, where I denotes the identity mapping and Q denotes the orthogonal projection of X onto $\operatorname{Ker}(T)$.

(2) $\dim \operatorname{Ker} T < +\infty$,

(3) $\exists z \in X : \langle C'(z),\, z\rangle + \Phi(z) < 0$,

(4) $C(u) > 0,\ \forall\, u \in \operatorname{Ker} T,\ u \neq 0$,

then problem [P] has a nontrivial solution.

PROOF: Let $J\colon X \to (-\infty, +\infty]$ be the functional defined by

$$J(u) := \frac{1}{2}\langle u, Tu\rangle + C(u) + \Phi(u).$$

Let $X = \overline{\bigcup_{n\in \mathbb{N}} X_n}$, where for each n, $X_n := \{x \in X\colon \|x\| \leqslant n\}$ is a weakly compact convex set in X. Since J is weakly lower semicontinuous, it reaches its minimum on each X_n, let us say at u_n.

We have $J(u_n) \leqslant J(v)$, for each $v \in X_n$.

Let $v \in X_n$, $tv + (1-t)u_n \in X_n$ for each $t \in [0, 1]$ and since Φ is convex, we get

$$\Phi(v) - \Phi(u_n) + \frac{1}{2}[\langle T(u_n + t(v-u_n)), u_n + t(v-u_n)\rangle - \langle Tu_n, u_n\rangle]/t$$
$$+ [C(u_n + t(v-u_n)) - C(u_n)]/t \geqslant 0, \text{ for all } v \in X_n.$$

Computing the limit as $t \to 0^+$ we get

$$\langle v - u_n, Tu_n + C'(u_n)\rangle + \Phi(v) - \Phi(u_n) \geqslant 0, \text{ for all } v \in X_n. \tag{2.1}$$

We show first that the sequence $\{u_n\}$ is bounded.

(a) If $0 < \beta < 2$. Suppose on the contrary that $\{u_n\}$ is unbounded. Passing possibly to a subsequence, we can suppose that $w - \lim_{n\to\infty} x_n = x^*$, where $x_n := u_n/\|u_n\|$.

Put $v = 0$ in (2.1); we obtain

$$\langle u_n, Tu_n\rangle + \beta.C(u_n) + \Phi(u_n) \leqslant 0$$

which implies:

$$\langle x_n, Tx_n\rangle + \beta.C(x_n).\|u_n\|^{\beta-2} + \Phi(x_n)\|u_n\|^{\alpha-2} \leqslant 0. \tag{2.2}$$

Taking the limit as $n \to +\infty$ in (2.2), we get

$$\langle x^*, Tx^*\rangle \leqslant \liminf\langle x_n, Tx_n\rangle \leqslant 0,$$

and since $\langle x^*, Tx^*\rangle \geqslant 0$ we obtain

$$\langle x^*, Tx^*\rangle = 0,$$

and thus $Tx^* = 0$ and $c.\liminf \|Px_n\|^2 \leqslant \liminf\langle x_n, Tx_n\rangle \leqslant 0$. Going if necessary to a subsequence we can assume that $Px_n \to 0$, $x_n \to x^* \in \operatorname{Ker} T$, since $\dim \operatorname{Ker} T < \infty$. Moreover $\|x^*\| = 1$.

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By positivity, $\langle u_n, Tu_n \rangle \geqslant 0$ for each $n \in \mathbb{N}$, and thus we have from (2.1)

$$\langle x, Tu_n \rangle + \langle x, C'(u_n) \rangle + \Phi(x) \geqslant \beta C(u_n) + \Phi(u_n), \text{ for each } x \in X_n. \tag{2.3}$$

Choosing $x = 0$, we obtain

$$\Phi(u_n) + \beta C(u_n) \leqslant 0.$$

Dividing by $\|u_n\|^\beta$,

$$\beta C(x_n) + \Phi(x_n) \|u_n\|^{\alpha-\beta} \leqslant 0$$

and taking the limit, we obtain

$$C(x^*) \leqslant 0,$$

and since $x^* \in \operatorname{Ker} T$, this is a contradiction to assumption (4).

Thus the sequence $\{u_n\}$ is bounded. Without loss of generality, we can suppose that

$$u^* = w - \lim_{n\to\infty} u_n.$$

For $y \in X$, there exists $m \in \mathbb{N}$ such that $y \in X_n$ for all $n \geqslant m$. Hence $J(u_n) \leqslant J(y)$, for all $n \geqslant m$ and since J is weakly lower semicontinuous, we get

$$J(u^*) \leqslant J(y),$$

and therefore $J(u^*) = \min_X J(y)$.

We have thus

$$\langle Tu^* + C'(u^*), v - u^* \rangle + \Phi(v) - \Phi(u^*) \geqslant 0, \qquad \forall\, y \in X.$$

With $v = 0$, we obtain $\beta C(u^*) + \Phi(u^*) \leqslant 0$, and thus by assumption (4)

$$u^* \notin \operatorname{Ker} T \setminus \{0\}.$$

Now,

$$J(u^*) \leqslant J(v), \quad \text{for all} \quad v \in X,$$

and thus

$$J(u^*) \leqslant C(v), \quad \text{for all} \quad v \in \operatorname{Ker} T.$$

By assumption (3), we get

$$J(u^*) \leqslant C(z) + \Phi(z) < 0,$$

and thus $u^* \neq 0$. ∎

Corollary 2.1. *Let K be a nonempty closed convex cone of X. If the following conditions hold true: $\alpha < \beta < 2$ and*

(1) *T is semicoercive.*
(2) $\dim \operatorname{Ker} T < +\infty$,
(3) $\exists z \in K \colon \langle C'(z), z\rangle < 0$,
(4) $C(u) > 0$, $\forall\, u \in K \cap \operatorname{Ker} T$, $u \neq 0$,

then there exists $u \in K \setminus \operatorname{Ker} T$ such that

$$\langle v - x^*, Tx^* + C'(x^*)\rangle \geqslant 0, \quad \forall\, v \in K.$$

3. Multiplicity result

We shall assume that $C \in C^1(X, \mathbb{R})$ is even, β-positively homogeneous and strongly continuous, and $\Phi \colon X \to (-\infty, +\infty]$ is even, α-positively homogeneous and strongly continuous.

In order to obtain a multiplicity result to prove that [P] has many pairs of solutions $(-x^*, x^*)$, we verify the assumptions of Theorem 4.4 in [4] due to Szulkin.

Let us recall some definitions.

Let X be a real Banach space and J a function on X satisfying: $J = f + g$, where $f \in C^1(X, \mathbb{R})$ and $g \colon X \to (-\infty, +\infty]$ is convex, proper and lower semicontinuous. We say that J satisfies the Palais-Smale condition in the sense of Szulkin (PS), if $\{u_n\}$ is a sequence such that $J(u_n) \to c \in \mathbb{R}$, $z_n \in f'(u_n) + \partial g(u_n)$ where $z_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

Theorem 3.1. (Szulkin [4].) *Suppose that J is defined as above and satisfies (PS), $J(0) = 0$ and f, g are even. Assume also that*

(1) *there exists a subspace X_1 of X, of finite codimension, and numbers γ, $\rho > 0$ such that $J\,|_{\partial B_\rho \cap X_1} \geqslant \gamma$,*
(2) *there is a finite dimensional subspace X_2 of X, $\dim X_2 > \operatorname{codim} X_1$ such that $J(u) \to -\infty$ as $\|u\| \to \infty$, $u \in X_2$.*

Then J has at least $\dim X_2 - \operatorname{codim} X_1$ distinct pairs of nonzero critical points $(-x^*, x^*)$, that is $0 \in f'(x^*) + \partial g(x^*)$.

Corollary 3.1. (Szulkin [4].) *Suppose that the hypotheses of Theorem 3.1 are satisfied with (2) replaced by*

(2′) *for any positive integer k there is a k-dimensional subspace X_2 of X such that $J(u) \to -\infty$ as $\|u\| \to +\infty$.*

Then J has infinitely many distinct pairs of nonzero critical points.

From this theorem, we obtain the following

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THEOREM 3.2. *If $\alpha > 1$, $\beta > \max\{2, 2^\alpha - 1\}$ and*

(1) *T is semicoercive*
(2) $\dim \operatorname{Ker} T < +\infty$,
(3) *there exists a subspace X_n of X, such that $n := \dim X_n > \dim \operatorname{Ker} T$ and $C(y) < 0$, for all $y \in X_n$, $y \neq 0$,*
(4) $\Phi(u) > 0$, $\forall\, u \in (\operatorname{Ker} T) \setminus \{0\}$; $\Phi(u) \geqslant 0$, $\forall\, u \in X$.

Then there exist at least $n - \dim \operatorname{Ker} T$ distinct pairs of nontrivial solutions for problem [P].

PROOF: Let $f(x) := \langle x, Tx\rangle/2 + C(x)$, $g(x) := \Phi(x)$. Let $X_1 := (\operatorname{Ker} T)^\perp$, $X_2 := X_n$

(1) For every $x \in X_1$, we have

$$\langle x, Tx\rangle/2 + \Phi(x) + C(x) \geqslant c/2.\|x\|^2 - |C(x)|$$

and since Φ and C are continuous and positively homogeneous, there exist k, $k' > 0$ such that

$$\langle x, Tx\rangle/2 + \Phi(x) + \beta C(x) \geqslant c/2.\|x\|^2 - k'\|x\|^\beta.$$

It is always possible to choose ρ such that $\tau := c\rho^2/2 - k'\rho^\beta > 0$ and thus

$$J(x) \geqslant \tau, \quad \forall\, x \in \partial B_\rho \cap X_1.$$

(2) By assumption (3), there exists $\delta > 0$ such that

$$C(x) \leqslant -\delta\|x\|^\beta, \quad \text{for all} \quad x \in X_2.$$

We have

$$J(x) \leqslant \|T\|_* \|x\|^2 - \delta\|x\|^\beta + k\|x\|^\alpha$$

and thus

$$\lim_{\substack{\|x\| \to +\infty \\ x \in X_2}} J(x) = -\infty.$$

It remains to prove that J satisfies the (PS) condition. Let $u_n \in X$ be a sequence such that $J(u_n) \to c \in \mathbb{R}$, $z_n \in f'(u_n) + \partial g(u_n)$ where $z_n \to 0$; that is also (see [4] for more details)

$$\Phi(v) - \Phi(u_n) + \langle Tu_n + C'u_n, v - u_n\rangle \geqslant -\delta_n.\|v - u_n\|, \tag{3.1}$$

where $\delta_n \to 0$.

We claim that $\{u_n\}$ is bounded. Suppose that $\{u_n\}$ is unbounded. With $v = 2u_n$ in (3.1) we get

$$\langle u_n, Tu_n\rangle + \beta C(u_n) + (2^\alpha - 1)\Phi(u_n) \geqslant -\delta_n . \|u_n\| ,$$

so that, for n large enough,

$$\beta J(u_n) - \{\langle u_n, Tu_n\rangle + \beta C(u_n) + (2^\alpha - 1)\Phi(u_n)\} \leqslant \beta(c+1) + \|u_n\| .$$

Thus

$$(\beta + 1 - 2^\alpha)\Phi(u_n) + (\beta/2 - 1)\langle u_n, Tu_n\rangle \leqslant \beta(c+1) + \|u_n\| . \tag{3.2}$$

By assumption (6) we have

$$(\beta/2 - 1)\langle u_n, Tu_n\rangle \leqslant \beta(c+1) + \|u_n\| .$$

Put $v_n := u_n / \|u_n\|$. We can suppose, by considering if necessary a subsequence, that $w - \lim\limits_{n\to\infty} v_n = v^*$.

We have

$$(\beta/2 - 1)\langle v_n, Tv_n\rangle \leqslant \beta(c+1)/\|u_n\|^2 + 1/\|u_n\| .$$

Taking the limit, we get

$$0 \leqslant \langle v^*, Tv^*\rangle \leqslant \liminf \langle v_n, Tv_n\rangle \leqslant 0,$$

and as in Theorem 2.1, going if necessary to a subsequence, we can assume that $\|v^*\| = 1$.

Since T is positive, from (3.2) we get also

$$(\beta + 1 - 2^\alpha)\Phi(u_n) \leqslant \beta(c+1) + \|u_n\| ,$$

and thus

$$(\beta + 1 - 2^\alpha)\Phi(v_n) \leqslant \beta(c+1)/\|u_n\|^\alpha + 1/\|u_n\|^{\alpha-1} .$$

By taking the limit, we get $\Phi(v^*) \leqslant 0$, which is a contradiction to assumption (4).

Thus $\{u_n\}$ is bounded and by considering possibly a subsequence, we may suppose that u_n is weakly convergent. Let $u^* = w - \lim u_n$. Put $v = u^*$ in (3.1). We get

$$\langle Tu_n, u^* - u_n\rangle + \langle C'u_n, u^* - u_n\rangle + \Phi(u^*) - \Phi(u_n) \geqslant -\delta_n . \|u^* - u_n\| .$$

Taking the limit, we get

$$\varlimsup_{n\to\infty} \langle Tu_n, u_n - u^*\rangle \leqslant 0.$$

The orthogonal decomposition $\overline{X} \oplus \operatorname{Ker}(T)$ allows us to write $u_n =: \overline{u}_n + \widehat{u}_n$. Thus we have

$$\varlimsup_{n\to\infty} c . \|\overline{u}_n - \overline{u}^*\|^2 \leqslant \varlimsup_{n\to\infty} \langle T(\overline{u}_n - \overline{u}^*), \overline{u}_n - \overline{u}^*\rangle \leqslant 0,$$

and $\overline{u}_n$ is strongly convergent to $\overline{u}^*$. Since $\dim \operatorname{Ker} T < +\infty$, going if necessary to a subsequence, $\widehat{u}_n$ is strongly convergent to Q^* and the conclusion follows. ∎

4. Examples

Example 4.1. Let $T > 0$ and let $X := H^1(\Pi, \mathbb{R})$. Let K be the closed convex cone defined by $K := \{u \in H^1(\Pi, \mathbb{R}) : u(x) \geqslant 0 \text{ in } [0, T]\}$. We consider the periodic unilateral problem

$$(1) \qquad u \in K\colon \int_0^T \dot{u}.(\dot{v} - \dot{u})dt + \int_0^T \nabla_u V(t, u).(v - u)dt \geqslant 0, \ \forall\, v \in K.$$

We assume that:

(a) $\forall\, u \in \mathbb{R}$, $V(\cdot, u)$ is measurable and there exist $a, b \in L^1([0, T], \mathbb{R}_+)$ such that $\forall\, t \in [0, T]$, $\forall\, u \in \mathbb{R}$, $|u| = 1$, $|V(t, u)| \leqslant a(t)$ and $|\nabla_u V(t, u)| \leqslant b(t)$.

(b) for almost all $t \in \mathbb{R}$, $V(t, .) \in C^1$,

(c) $\forall\, u \in \mathbb{R} \setminus \{0\} : \int_0^T V(t, u)dt > 0$,

(d) $\exists \nu \in \mathbb{R}^+ : V(., v) < 0$, on a non zero measure subset,

(e) V is β-positively homogeneous ($\beta < 2$) with respect to u.

Let $T\colon X \to X^*$ and $C\colon X \to \mathbb{R}$ be defined by

$$\langle Tu, v\rangle := \int_0^T \dot{u}.(\dot{v} - \dot{u})dt, \ C(u) := \int_0^T V(t, u)dt.$$

We can prove that if V satisfies (a)–(e), then all assumptions of Corollary 2.1 are satisfied [1], so that (1) has at least one non-constant solution.

Example 4.2. We consider the problem

$$(2) \qquad u \in X\colon \int_0^T \dot{u}.(\dot{v} - \dot{u})dt + \int_0^T \nabla_u V(t, u).(v - u)dt + \int_0^T g(t)\Big(|v|^3 - |u|^3\Big)dt \geqslant 0, \ \forall\, u \in X.$$

Let $T\colon X \to X^*$ and $C\colon X \to \mathbb{R}$ be defined as in Example 4.1 and put $\Phi(u) := \int_0^T g(t)\,|u|^3\, dt$. We assume that g is a positive ($g \neq 0$) bounded function.

We can prove that if V satisfies (a)–(d) and (e) with $\beta > 7$ and even, then all assumptions of Theorem 3.2 [1] are satisfied. Therefore (1) has infinitly many distinct pairs of non-constant solutions.

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