EUCLIDEAN LIE ALGEBRAS

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Introduction. Our aim in this paper is to study a certain class of Lie algebras which arose naturally in (**4**). In (**4**), we showed that beginning with an indecomposable symmetrizable generalized Cartan matrix (A_{ij}) and a field Φ of characteristic zero, we could construct a Lie algebra $E((A_{ij}))$ over Φ patterned on the finite-dimensional split simple Lie algebras. We were able to show that $E((A_{ij}))$ is simple providing that (A_{ij}) does not fall in the list given in (**4**, Table). We did not prove the converse, however.

The diagrams of the table of (**4**) appear in Table 2. Call the matrices that they represent *Euclidean* matrices and their corresponding algebras *Euclidean* Lie algebras. Our first objective is to show that Euclidean Lie algebras are not simple. This involves a close look at the root systems of Euclidean Lie algebras (§ 1) and the construction (§ 2) of a certain module endomorphism of E (E treated as an E-module in the customary way). Along the way we discover that the set of null roots Z is a group and the subgroup Z^* of (**4**, § 6) is of index 1, 2, or 3. We call $[Z:Z^*]$ the tier number, r, of our Lie algebra.

Our second objective is to describe certain simple epimorphic images of a Euclidean Lie algebra. By the results of (**4**, § 7), every proper ideal of E is of finite codimension. For each $\mu \in \Phi - \{0\}$ there is an ideal of minimal codimension and the quotient, $E(\mu)$, of E by this ideal is a finite-dimensional central simple Lie algebra over Φ. For the 1-tiered algebras we have:

(i) $E(\mu) \simeq E(\nu)$ for all $\mu, \nu \in \Phi - \{0\}$,

(ii) $E \simeq \Phi\langle x\rangle \otimes_\Phi E(1)$, where $\Phi\langle x\rangle$ is the associative algebra of finite Laurent series in an indeterminate x over Φ, and

(iii) $E(1)$ is split.

In § 4 we show that (i), and hence (ii), cannot hold in general for 2-tiered algebras. Indeed the identity of the $E(\mu)$s when E is 2-tiered or 3-tiered is rather obscure and our efforts are concentrated in working out the *type* of each $E(\mu)$. The procedure is essentially to calculate $\dim E(\mu)$ (which is independent of μ) and, although this is not very sophisticated, it does involve securing some further results on the root systems which are bound to be important in any further investigations. Except for $F_{4,2}$, which does not lend itself to this procedure, we can say that for any $\mu, \nu \in \Phi - \{0\}$, $E(\mu)$ and $E(\nu)$ are of the same type, this type being given in Table 2.

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Notation and conventions. The notation used here agrees with that of (**4**) as closely as possible. In particular, we denote $\{1, \ldots, l\}$ by $\mathbf{L}$, $\{0, 1, \ldots, l\}$ by $\mathbf{L}^*$, the integers by $\mathbf{Z}$, and the rationals by $\mathbf{Q}$. We will fix an arbitrary field Φ of characteristic zero from the outset and all algebras will be assumed to be over Φ unless explicitly stated otherwise. For each Euclidean matrix $X_{l,r}$ we will assume that $E(X_{l,r})$ is a copy of the Euclidean Lie algebra associated with $X_{l,r}$ fixed once and for all. Often we will simply write $X_{l,r}$ for $E(X_{l,r})$.

Remark. The use of the adjective "Euclidean" in the present context comes from the fact that the Weyl group of a Euclidean Lie algebra is isomorphic to the Coxeter group with corresponding diagram (see **1**) which in turn is the group generated by the reflections in the sides of a Euclidean simplex. In this terminology, the classical simple Lie algebras would be called spherical Lie algebras.

1. Root systems of Euclidean matrices. We recall a few definitions from (**4**). A generalized Cartan matrix is a square integral matrix (A_{ij}), $i, j, \in \mathbf{L}$, with the properties $A_{ii} = 2$ for all $i \in \mathbf{L}$, $A_{ij} \leqq 0$ if $i \neq j$, $A_{ij} = 0$ if and only if $A_{ji} = 0$. A Euclidean matrix is a singular generalized Cartan matrix with the property that removal of any row and the corresponding column leaves a (not necessarily indecomposable) Cartan matrix.

Let (A_{ij}), $i, j \in \mathbf{L}^*$, be a Euclidean matrix. It is obvious from the diagrams (Table 2) that (A_{ij}) is symmetrizable, i.e., there are non-zero rational numbers $\epsilon_0, \epsilon_1, \ldots, \epsilon_l$ such that $A_{ij}\epsilon_j = A_{ji}\epsilon_i$, $i, j \in \mathbf{L}^*$. Since the diagram is connected, the ϵ_i are uniquely determined up to a scalar factor. Let us scale them so that

(a) $2\epsilon_i$ is a positive integer for each i,

(b) $(2\epsilon_0, \ldots, 2\epsilon_l) = 1$.

Put $a_{ij} = A_{ij}\epsilon_j$. Note that $a_{ij} \leqq 0$ if $i \neq j$. We are going to show that (a_{ij}) is positive-semidefinite of rank l.

We can suppose that the rows and columns of (A_{ij}) are numbered so that (A_{ij}), $i, j > 0$, is indecomposable. Then (A_{ij}), $i, j > 0$, is an indecomposable Cartan matrix and (a_{ij}), $i, j > 0$, is *connected* in the sense that for any $i \in \mathbf{L}$ there is a $j \in \mathbf{L} - \{i\}$ such that $a_{ij} \neq 0$.

We have

$$(1) \qquad A_{ij} = \frac{a_{ij}}{\epsilon_j} = \frac{2a_{ij}}{A_{jj}\epsilon_j} = \frac{2a_{ij}}{a_{jj}}, \qquad i, j \in \mathbf{L}^*.$$

(a_{ij}), $i, j > 0$, is, up to a scalar factor, the unique symmetric matrix satisfying equations (1), $i, j \in \mathbf{L}$. Now, since (A_{ij}), $i, j > 0$, is an indecomposable Cartan matrix, there is a fundamental system of roots $\beta_1, \ldots, \beta_l$ for some simple Lie algebra B such that

$$A_{ij} = \frac{2k(\beta_i, \beta_j)}{k(\beta_j, \beta_j)},$$

where k is the Killing form on B. Thus, $k(\alpha_i, \alpha_j) = \mu a_{ij}$, $i, j \in \mathbf{L}$, for some real

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number μ, and since $k(\alpha_1, \alpha_1)$ and a_{11} are positive, $\mu > 0$. Thus (a_{ij}), $i, j > 0$, is positive-definite.

Let $A_0 = \mathbf{Q}\alpha_0 \oplus \ldots \oplus \mathbf{Q}\alpha_l$, where $\alpha_0, \ldots, \alpha_l$ are the fundamental roots of $E = E((A_{ij}))$ (or $L = L((A_{ij}))$), and $\mathbf{Q}$ is identified with the prime subfield of Φ. Define a bilinear form σ on A_0 by a bilinear extension of $\sigma(\alpha_i, \alpha_j) = a_{ij}$, $i, j \in \mathbf{L}^*$ (see **4**, Theorem 3). (A_{ij}), and hence (a_{ij}), is singular and thus A_0 has a radical $N_0 \neq (0)$ with respect to σ. Since σ, restricted to

$$\mathbf{Q}\alpha_1 \oplus \ldots \oplus \mathbf{Q}\alpha_l,$$

is positive-definite, σ is positive semi-definite and $\dim N_0 = 1$. Thus (a_{ij}) is positive semi-definite of rank l. We will often use $(\ ,\ )$ instead of $\sigma(\ ,\ )$.

Each element of A_0 induces a linear functional on the subalgebra H of E. In fact, if $\beta = \sum_{i \in \mathbf{L}^*} \lambda_i \alpha_i$, then

$$\beta(h_j) = \sum_{i \in \mathbf{L}^*} \lambda_i A_{ij} \quad \text{for all } j \in \mathbf{L}^*.$$

However,

$$\sum_i \lambda_i A_{ij} = \frac{1}{\epsilon_j} \sum_i \lambda_i a_{ij} = \frac{1}{\epsilon_j} \sigma(\beta, \alpha_j).$$

Thus $\beta(h_j) = 0$ if and only if $\sigma(\beta, \alpha_j) = 0$, and $\beta \mid H = 0$ (i.e., β is *null* on H) if and only if $\beta \in N_0$. If $\beta \in A_0$ and r_i is one of the fundamental reflections of the Weyl group, W, then $\beta r_i = \beta - \beta(h_i)\alpha_i$. Thus β is fixed by W if and only if β is null on H. Note also that while every element of N_0 is isotropic, it is conversely true that if $\beta \in A_0$ is isotropic, then $\beta \in N_0$.

Let ξ be a vector spanning N_0. $\xi = \sum_{i \in \mathbf{L}^*} \xi_i \alpha_i$ and clearly $\xi_0 \neq 0$. We can suppose that $\xi_0 > 0$. Following Coxeter (**1**, p. 175), we have

$$0 = \sigma(\xi, \xi) = \sum_{i,j} \xi_i \xi_j a_{ij} = \sum_i \xi_i^2 a_{ii} + \sum_{i \neq j} \xi_i \xi_j a_{ij} \geqq \sum_i \xi_i^2 a_{ii} + \sum_{i \neq j} |\xi_i| |\xi_j| a_{ij}$$

(since $a_{ij} \leqq 0$ if $i \neq j$). Thus $\sum |\xi_i| \alpha_i \in N_0$, whence $\sum_i |\xi_i| \alpha_i = \sum_i \xi_i \alpha_i$ and each $\xi_i \geqq 0$. A minor modification of (**4**, Lemma 10) yields $\xi_i > 0$ for all $i \in \mathbf{L}^*$. Scale ξ so that each ξ_i is a positive integer and $(\xi_0, \ldots, \xi_l) = 1$. Summing up, we have the following result.

LEMMA 1. *If (A_{ij}) is a Euclidean matrix and $(a_{ij}) = (A_{ij}\epsilon_j)$, where the ϵ_j are defined as above, then:*

(i) *(a_{ij}) is positive semi-definite of rank l;*

(ii) *the bilinear form σ defined on $A_0 = \mathbf{Q}\alpha_0 \oplus \ldots \oplus \mathbf{Q}\alpha_l$ by $\sigma(\alpha_i, \alpha_j) = a_{ij}$ possesses a 1-dimensional radical N_0;*

(iii) *for $\beta \in A_0$, the following are equivalent:*

  (a) *$\beta \in N_0$,*

  (b) *β is null on H,*

  (c) *β is isotropic,*

  (d) *β is fixed by the Weyl group;*

(iv) *$N_0 = \mathbf{Q}\xi$, where $\xi = \sum_{i \in \mathbf{L}^*} \xi_i \alpha_i$ can be chosen so that the ξ_i are positive integers and $(\xi_0, \ldots, \xi_l) = 1$.*

In (**4**), Δ is carelessly used both as the root system for L and for E. We correct this by denoting the root system of L by Υ and that of E by Δ. Thus $\beta \in A$ is in Υ if and only if $L_\beta \neq (0)$ and $\beta \in \Delta$ if and only if the image of L_β in E is not zero. Υ is closed under the action of W, and hence the Weyl roots $\Delta_W = \{\alpha_0, \ldots, \alpha_l\}W$ are contained in Υ. Let

$$\Upsilon_R = \{\beta \in \Upsilon \mid L_\beta \subseteq R = \text{radical of } L\}.$$

Since $R\theta(w) = R$ for all $w \in W$ and $L_\beta\theta(w) = L_{\beta w}$ for all $\beta \in \Upsilon$ (**4**, Theorem 2), we have $\Upsilon_R W = \Upsilon_R$. Thus $\Delta W = (\Upsilon - \Upsilon_R)W = \Delta$ and $\Delta_W \subseteq \Delta$.

In Table 2, the ξs for the Euclidean matrices are given. Note particularly that with our choice of subscripts, $\xi_0 = 1$ (also (A_{ij}), $i, j > 0$, is indecomposable). An immediate and important consequence of $\xi_0 = 1$ is that the set Z of null roots of E is a subset of $\mathbf{Z}\xi$.

PROPOSITION 1. (i) *If* $\beta = \sum_{i \in \mathbf{L}^*} \lambda_i \alpha_i$, $\lambda_i \in \mathbf{Q}$, *and* $\lambda_i \geqq 0$ *for all* i, *and if* $\beta \notin N_0$, *then there is a* $j \in \mathbf{L}^*$ *such that* $\beta(h_j) > 0$;

(ii) $\Delta = Z \cup \Delta_W$ *(disjoint union)*;

(iii) $\dim E_\beta = 1$ *if* $\beta \in \Delta - Z$.

Proof. (i) If $\beta(h_j) \leqq 0$ for all $j \in \mathbf{L}^*$, then $\sigma(\beta, \alpha_j) \leqq 0$ for all $j \in \mathbf{L}^*$. Thus $\sigma(\beta, \beta) \leqq 0$ and $\beta \in N_0$, contrary to hypothesis. (ii) Since W is a subgroup of the group of isometries of A_0 relative to σ, and $\Delta_W = \{\alpha_0, \ldots, \alpha_l\}W$, we have $Z \cap \Delta_W = \phi$. We see that $\Delta = Z + \Delta_W$ as soon as we can show that every positive root is in $Z + \Delta_W$. This is clear for the roots of height 1, $(\alpha_0, \ldots, \alpha_l)$. If β is of height s (> 1) and $\beta \notin Z$, then by (i) there is a j such that $\beta(h_j) > 0$. Thus $\beta r_j = \beta - \beta(h_j)\alpha_j$ is a non-null root of height less than s. If it were not positive, we would have $\beta \in \mathbf{Z}\alpha_j$, whence $\beta = \alpha_j$ and $\operatorname{ht}\beta = 1 < s$. Thus βr_j is positive. The proof is completed by using induction on the height. (iii) See (**4**, Theorem 2) for the proof.

The argument of (ii) is equally valid for Υ. A special consequence is the following result.

PROPOSITION 2. *If* $L = L((A_{ij}))$ *where* (A_{ij}) *is Euclidean, then the radical* R *of* L *coincides with the centre of* L.

Proof. By (**4**, Proposition 8 (ii)), the centre C of L is contained in R. Now let $a \in R$ be homogeneous. By (**4**, Proposition 8 (i)), $a \in L_\beta$ for some $\beta \notin \Delta_W$, i.e., for some null root β. Since for $i \in \mathbf{L}^*$, $\beta \pm \alpha_i$ is not null, and since R is an ideal, we have $[ae_i] = 0 = [af_i]$, $i \in \mathbf{L}^*$. Thus $a \in C$. Since R is a homogeneous ideal, $R \subseteq C$.

For $i \in \mathbf{L}^*$ let K_i be the 3-dimensional split simple algebra $\Phi e_i + \Phi h_i + \Phi f_i$.

LEMMA 2. *If* β *is a non-null root and* a $(\neq 0)$ *is in* E_β, *then the* K_i*-module* M *generated by* a *is irreducible and there are non-negative numbers* u, d *such that:*

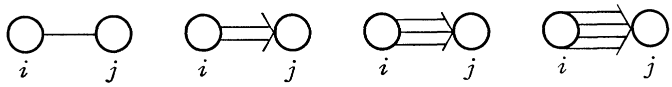
(i) $\beta(h_i) = d - u$,

(ii) $\beta + u\alpha_i, \beta + (u-1)\alpha_i, \ldots, \beta - d\alpha_i$ *are roots.*

Proof. If M is shown to be irreducible, then (i) and (ii) are well-known consequences. By Proposition 1, $\beta \in \Delta_W$ and $\dim E_\beta = 1$. Without loss of generality, $\beta > 0$. Let U_i be the universal enveloping algebra of K_i and let it act on E via the adjoint representation. $U_i = B_i U_i^- U_i^+$, where B_i, U_i^-, and U_i^+ are the subalgebras of U_i generated by h_i, f_i, and e_i, respectively. $aB_iU_i^- \subseteq \Phi f_i + \Phi h_i + \sum_{0<j\leqq \mathrm{ht}\,\beta} E(j)$, where $E(j)$ is the subspace of elements of degree j in the coarse grading of E. Thus $\dim aB_iU_i^-$ is finite. From the local nilpotency of ad e_i (**4**, Proposition 3), we have $\dim aB_iU_i^-U_i^+$ finite, i.e., M is of finite dimension. Thus $M = \sum_{j=1}^{p} \oplus M_j$, where the M_j are irreducible K_i-modules. $a = \sum_{j=1}^{p} a_j$, where each $a_j \in M_j$. Since $[ah_i] = \beta(h_i)a$, we see that $[a_jh_i] = \beta(h_i)a_j$. However, $M \subseteq \sum_{k=-\infty}^{\infty} E_{\beta+k\alpha_i}$ and hence the only non-zero elements $b \in M$ such that $[bh_i] = \beta(h_i)b$ are the elements of E_β. Since $\dim E_\beta = 1$, we must have $p = 1$ and M is irreducible.

LEMMA 3. (i) *If for some $j \in \mathbf{L}^*$ and some $k \in \mathbf{Z}$ we have $\alpha_j + k\xi \in \Delta$, then $\alpha_j + \mathbf{Z}k\xi \subseteq \Delta$.*

(ii) *If $\alpha_i + k\xi \in \Delta$ and vertices i and j of the diagram are joined in any of the ways shown, then $\alpha_j + k\xi \in \Delta$.*



Proof. (i) $\alpha_j + k\xi \in \Delta - Z$ and hence there is an $i \in \mathbf{L}^*$ and a $w \in W$ such that $\alpha_i w = \alpha_j + k\xi$. Thus $\alpha_i + k\xi = (-\alpha_i + k\xi)r_i = -\alpha_j w^{-1} r_i \in \Delta$ and $(\alpha_i + k\xi)w = \alpha_j + k\xi + k\xi = \alpha_j + 2k\xi \in \Delta$. Consequently, $\alpha_j + k\xi \in \Delta \Rightarrow \alpha_j + 2k\xi \in \Delta$. Apply this for j replaced by i and we have $\alpha_i + 2k\xi \in \Delta$, so that $(\alpha_i + 2k\xi)w = \alpha_j + 3k\xi \in \Delta$. In this fashion we see that $\alpha_j + \mathbf{N}k\xi \subseteq \Delta$. However, $\alpha_j + nk\xi \in \Delta \Rightarrow -\alpha_j - nk\xi \in \Delta \Rightarrow (-\alpha_j - nk\xi)r_j = \alpha_j - nk\xi \in \Delta$. Thus $\alpha_j + \mathbf{Z}k\xi \subseteq \Delta$.

(ii) The hypotheses imply that $|A_{ij}| \geqq |A_{ji}|$ and also that not both A_{ij} and A_{ji} are -2. Thus $A_{ji} = -1$. $(\alpha_i + k\xi)(h_j) = A_{ij} < 0$ and by Lemma 2, $\alpha_i + \alpha_j + k\xi \in \Delta$. On the other hand, $(\alpha_i + \alpha_j + k\xi)(h_i) = 2 + A_{ji} = 1 > 0$ and $\alpha_i + \alpha_j + k\xi \notin Z$, hence by Lemma 2 again, $\alpha_j + k\xi \in \Delta$.

The *weight* of a root β is defined to be $\sigma(\beta, \beta)$. From $A_{ji}\sigma(\alpha_i, \alpha_i) = A_{ij}\sigma(\alpha_j, \alpha_j)$ it is easily seen that if i and j are joined by s lines *and* an arrowhead from i to j, then the weight of α_i is s times the weight of α_j.

By inspecting the diagrams in Table 2 and using Lemma 3 (ii), we see that with the possible exceptions of $C_{l,1}$ and $A_{1,1}$, if $\alpha_i + s\xi$ is a root where α_i has maximal weight amongst the fundamental roots, then $\alpha_j + s\xi$ is a root for all $j \in \mathbf{L}^*$ (and hence $\beta + s\xi$ is a root whenever β is a root). Actually, $C_{l,1}$ and $A_{1,1}$ are not exceptions, as can be seen directly from the symmetry of their diagrams. The calculations (a) of Table 1 show that if α_i is a fundamental root of maximal weight for the Lie algebra $X_{l,r}$, then $\alpha_i + r\xi$ is also a root. Calculation (c) shows that, for the cases when $r > 1$, $\alpha_i + t\xi$ is not a root ($1 \leqq t < r$) by deriving an evidently impossible root from the assumption that it is.

Calculation (b) shows that there are some $j \in \mathbf{L}^*$ such that $\alpha_j + \xi$ is a root.

Our choice of α_0 is characterized up to symmetries of the diagram by:

(1) removal of vertex 0 does not disconnect the diagram,

(2) $\xi_0 = 1$,

(3) (if necessary) $\alpha_0 + \xi$ is a root.

(3) is used only when (1) and (2) fail to characterize a fundamental root (up to symmetries of the diagram).

TABLE 1

$A_{l,1}$	(a)	$\alpha_1 r_2 r_3 \ldots r_l r_0 = \alpha_0 + \xi$
$B_{l,1}$	(a)	$\alpha_1 r_2 r_3 \ldots r_l r_{l-1} \ldots r_3 r_2 r_0 = \alpha_0 + \xi$
$C_{l,2}$	(a)	$\alpha_l r_{l-1} \ldots r_3 r_1 r_2 r_0 r_1 r_3 \ldots r_l = \alpha_l + 2\xi$
	(b)	$\alpha_2 r_1 r_3 \ldots r_l r_{l-1} \ldots r_3 r_1 r_0 = \alpha_0 + \xi$
	(c)	$(\alpha_l + \xi) r_l r_{l-1} \ldots r_3 r_1 r_2 = \alpha_0 - \alpha_2$
$C_{l,1}$	(a)	$\alpha_l r_{l-1} \ldots r_1 r_0 = \alpha_0 + \xi$
$B_{l,2}$	(a)	$\alpha_1 r_0 r_2 r_3 \ldots r_l r_{l-1} \ldots r_2 r_1 = \alpha_1 + 2\xi$
	(b)	$\alpha_l r_{l-1} r_{l-2} \ldots r_1 r_0 = \alpha_0 + \xi$
	(c)	$(\alpha_1 + \xi) r_1 \ldots r_l = \alpha_0 - \alpha_l$
$BC_{l,2}$	(a)	$\alpha_0 r_1 \ldots r_l r_{l-1} \ldots r_0 = \alpha_0 + 2\xi$
	(b)	$\alpha_1 r_0 r_2 \ldots r_l r_{l-1} \ldots r_1 = \alpha_1 + \xi$
	(c)	$(\alpha_0 + \xi) r_0 r_1 \ldots r_{l-1} = 2\alpha_l$
$D_{l,1}$	(a)	$\alpha_1 r_2 r_3 \ldots r_l r_{l-2} \ldots r_2 r_0 = \alpha_0 + \xi$
$E_{6,1}$	(a)	$\alpha_1 r_2 r_3 r_4 r_5 r_6 r_2 r_3 r_5 r_2 r_1 r_0 = \alpha_0 + \xi$
$E_{7,1}$	(a)	$\alpha_1 r_2 r_3 \ldots r_7 r_3 r_2 r_5 r_3 r_4 r_6 r_5 r_3 r_2 r_1 r_0 = \alpha_0 + \xi$
$E_{8,1}$	(a)	$\alpha_1 r_2 r_3 \ldots r_8 r_5 r_4 r_7 r_5 r_6 r_3 r_2 r_4 r_5 r_7 r_8 r_3 r_4 r_5 r_6 r_7 r_5 r_4 r_3 r_2 r_1 r_0 = \alpha_0 + \xi$
$F_{4,1}$	(a)	$\alpha_1 r_2 r_3 r_2 r_4 r_3 r_2 r_1 r_0 = \alpha_0 + \xi$
$F_{4,2}$	(a)	$\alpha_4 r_3 r_2 r_1 r_0 r_3 r_2 r_1 r_3 r_2 r_3 r_4 = \alpha_4 + 2\xi$
	(b)	$\alpha_1 r_2 r_3 r_4 r_2 r_3 r_2 r_1 r_0 = \alpha_0 + \xi$
	(c)	$(\alpha_4 + \xi) r_4 r_3 r_2 r_1 r_0 = \alpha_3 + \alpha_2 - \alpha_0$
$G_{2,1}$	(a)	$\alpha_1 r_2 r_1 r_0 = \alpha_0 + \xi$
$G_{2,3}$	(a)	$\alpha_2 r_1 r_0 r_2 r_1 r_2 = \alpha_2 + 3\xi$
	(b)	$\alpha_1 r_2 r_1 r_0 = \alpha_0 + \xi$
	(c)	$(\alpha_2 + \xi) r_2 r_1 = \alpha_0 - \alpha_1$
		$(\alpha_2 + 2\xi) r_2 r_1 r_0 = \alpha_2 + \alpha_1 - \alpha_0$
$A_{1,1}$	(a)	$\alpha_1 r_0 = \alpha_0 + \xi$
$A_{1,2}$	(a)	$\alpha_0 r_1 r_0 = \alpha_0 + 2\xi$
	(b)	$\alpha_1 r_0 r_1 = \alpha_1 + \xi$
	(c)	$(\alpha_0 + \xi) r_0 = 2\alpha_0$

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Parts (ii) and (iii) of the following theorem have now been established.

THEOREM 1. (i) $Z = \mathbf{Z}\xi$.

(ii) *There is a positive integer* r *with the property that* $\Delta + \mathbf{Z}r\xi = \Delta$. *The minimum value for* r *is* 1, 2, *or* 3. r *is called the tier number.*

(iii) *For some* $i \in \mathbf{L}^*$, $\alpha_i + \mathbf{Z}\xi \subseteq \Delta$.

(iv) *The number of equivalence classes of* Δ *modulo* $\mathbf{Z}r\xi$ *is finite.*

Proof. (i) We know that $Z \subseteq \mathbf{Z}\xi$. By (iii), there exists a j such that $\alpha_j + k\xi \in \Delta$ for all $k \in \mathbf{Z}$. Since $(\alpha_j + k\xi)(h_j) = 2$, Lemma 2 shows that $k\xi \in \Delta$. Thus $\mathbf{Z}\xi \subseteq Z$.

(iv) It will be sufficient to show that there are only finitely many classes of Δ modulo $\mathbf{Z}\xi$. Since $\Delta = Z \cup \{\alpha_0, \ldots, \alpha_l\}W$, each element of Δ has weight equal to 0, 1, 2, 3, or 4. Any two elements in the same class have the same weight and there is precisely one representative for each class of Δ in $\mathbf{Z}\alpha_1 \oplus \ldots \oplus \mathbf{Z}\alpha_l$. Since σ is positive-definite on the lattice $\mathbf{Z}\alpha_1 \oplus \ldots \oplus \mathbf{Z}\alpha_l$, the number of elements of weight 0, 1, 2, 3, and 4 in the lattice is finite. Thus the number of classes is finite.

2. Shift mappings. In (**4**) we have seen that if $E = E((A_{ij}))$ is not simple, then the set Z of null roots of E is not trivial. Furthermore, there is a non-trivial cyclic group Z^* of roots in Z such that for each $\beta \in Z^*$ there is a homogeneous bijective module endomorphism of degree β of E (E treated as an E-module in the customary way). This mapping is unique (for each β) up to a scalar factor. Clearly a necessary condition for a root β to be in Z^* is $\beta + \Delta = \Delta$.

Consider $E = E((A_{ij}))$ when (A_{ij}) is Euclidean. We have $Z = \mathbf{Z}\xi$. However $\xi + \Delta \nsubseteq \Delta$ in general and the first candidate for an element of Z^* amongst the positive roots is $\zeta \equiv r\xi$ (r is the tier number). Suppose that we can establish a bijective endomorphism $'$: $E \rightarrow E$ of degree ζ such that $[ab]' = [a'b]$ for all $a, b \in E$. Then for $a \in E$ and $i \in \mathbf{Z}$ we define $a^{(i)}$ by $a^{(0)} = a$, $a^{(i)} = (a^{(i-1)})'$ for $i > 0$, $a^{(-1)} =$ pre-image of a under $'$, $a^{(-i)} = (a^{(-i+1)})^{(-1)}$ for $i < -1$. E is not simple (for example the smallest ideal containing $h_0 + h_0'$ is proper) and hence the results of (**4**, §§ 6, 7) apply. We recall these results briefly: $Z^* = [\zeta]$ and the ideal structure of E can be described in terms of the associative algebra $\Phi\langle x\rangle$ which is the algebra generated by the algebra of polynomials, $\Phi[x]$, in an indeterminate x, and x^{-1}. Let U be the universal enveloping algebra of E. Each ideal of E has the form $\langle a\rangle = aU$, where $a \in S = \sum_{i=-\infty}^{\infty} \Phi h_0^{(i)}$ and a is unique to within scalar multiples. We make S into an associative algebra by identifying it with $\Phi\langle x\rangle$ via $\sum \lambda_i h_0^{(i)} \leftrightarrow \sum \lambda_i x^i$. If we let (a) denote the (associative) ideal in S generated by a, then $\langle a\rangle \cap S = (a)$ and the correspondence $\langle a\rangle \leftrightarrow (a)$ is an isomorphism between the lattices of E and S. Finally, every non-zero ideal of E is of finite codimension.

Our construction of $'$ rests on a series of straightforward but tedious calculations.

Notation. For each $i \in \mathbf{L}^*$ we will choose, by a method described below, an element $e_i' \in E_{\alpha_i+\zeta}$. h_i' will denote $[e_i'f_i]$ and f_i' will denote $\frac{1}{2}[h_i'f_i]$.

Choose e_0' to be any non-zero element of $E_{\alpha_0+\zeta}$. For convenience in notation suppose that $A_{01} \neq 0$. Then $\alpha_0 + \alpha_1$, and hence $\alpha_0 + \alpha_1 + \zeta$, is a root. If $\alpha_0 + \alpha_1 + \zeta \notin Z$ we have $\dim E_{\alpha_0+\alpha_1+\zeta} = 1$. If $\alpha_0 + \alpha_1 + \zeta \in Z$, E is the Lie algebra $A_{1,1}$ and $E_{\alpha_0+\alpha_1+\zeta}$ is spanned by $[[[e_0e_1]e_1]e_0]$ and $[[[e_0e_1]e_0]e_1]$ which are equal, thus again $\dim E_{\alpha_0+\alpha_1+\zeta} = 1$. With this and Lemma 2, $[E_{\alpha_0+\zeta}, e_1] = E_{\alpha_0+\alpha_1+\zeta} = [E_{\alpha_1+\zeta}, e_0]$, whence $E_{\alpha_0+\zeta} = [[e_0E_{\alpha_1+\zeta}]f_1] = [e_0[E_{\alpha_1+\zeta}, f_1]]$. Define e_1' to be the unique element of $E_{\alpha_1+\zeta}$ such that $[e_0[e_1'f_1]] = \alpha_0(h_1)e_0'$, i.e.,

$$[e_0h_1'] = [e_0'h_1].$$

It is now easily checked that $[a'b] = [ab']$ for $a, b \in \{e_i, h_i, f_i \mid i = 1, 2\}$. For example, $[e_0'e_1] = [e_0e_1']$ follows from $[E_{\alpha_0+\alpha_1+\zeta}, f_1] = E_{\alpha_0+\zeta}$ and $[[e_0'e_1]f_1] = [e_0'h_1] = [e_0h_1'] = [e_0[e_1'f_1]] = [[e_0e_1']f_1]$.

The equation

$$(*) \qquad [a'b] = [ab'] \quad \text{for } a, b \in \{e_i, h_i, f_i \mid i = j, k\}$$

holds independently of the choice of e_j' and e_k' if j and k are unjoined vertices of the diagram. Thus, if the diagram has no loops, the procedure by which we obtained e_1' from e_0' can be repeated until $(*)$ holds for all $j, k \in \mathbf{L}^*$. The case of $A_{l,1}$ must be considered separately.

In this case, having chosen e_0', non-zero, in $E_{\alpha_0+\zeta}$, we let the automorphism τ of E defined by $e_i \to e_{i+1}$, $f_i \to f_{i+1}$, $h_i \to h_{i+1}$ (indices taken modulo $l + 1$) define e_i'. Namely $e_i' = e_0'\tau^i$, $i = 1, \ldots, l$. $(*)$ will hold for all j and k if $[e_0'e_1] = [e_0e_1']$. To be specific, put $e_0' = [e_0, e_1, \ldots, e_l, e_0]$.† $e_0' \neq 0$, for Lemma 2 shows that $[e_1, \ldots, e_l, e_0] \neq 0$ whence

$$[e_0'f_0] = -A_{10}[e_1, \ldots, e_l, e_0] \neq 0.$$

Now

$$[e_0e_1'] = [[e_0e_1], e_2, \ldots, e_l, e_0, e_1] - [e_1, e_2, \ldots, [e_l, e_0], e_0, e_1] - [e_1, e_2, \ldots, e_l, e_0, [e_1e_0]].$$

The first term on the right-hand side of this equation is $[e_0'e_1]$. Let b be the remaining pair of terms. We have $b \in E_{\alpha_0+\alpha_1+\zeta}$, and in order to show that $b = 0$, it suffices to show that $b \operatorname{ad} f_0 = 0$. This is easily checked.

e_i', h_i', f_i', $i \in \mathbf{L}^*$, are now defined in all cases so that $(*)$ holds. The next step is the following result.

LEMMA 4. *$'$ can be extended to a linear mapping of E into E in such a way that if $g_{i_1}, \ldots, g_{i_s} \in \{e_i, h_i, f_i \mid i \in \mathbf{L}^*\}$, then $[g_{i_1}, \ldots, g_{i_j}', \ldots, g_{i_s}] = [g_{i_1}, \ldots, g_{i_s}]' = [g_{i_1}, \ldots, g_{i_k}', \ldots, g_{i_s}]$ for any $j, k \in \{1, \ldots, s\}$.*

†$[a_1, \ldots, a_k]$ denotes $[[\ldots[[a_1a_2]a_3]\ldots]a_k]$.

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Proof. (i) Suppose that $g_{i_1}, \ldots, g_{i_s} \in \{e_i \mid i \in \mathbf{L}^*\}$. We establish

$$[e_{i_1}, \ldots, e_{i_j}', \ldots, e_{i_s}] = [e_{i_1}, \ldots, e_{i_k}', \ldots, e_{i_s}]$$

by induction on s. If the result holds at $s - 1$ $(\geqq 2)$, then for each $t \in \mathbf{L}^*$ we have:

$$[e_{i_1}, \ldots, e_{i_j}', \ldots, e_{i_s}] \operatorname{ad} f_t = \sum_{u \neq j,k} [e_{i_1}, \ldots, [e_{i_u} f_t], \ldots, e_{i_j}', \ldots, e_{i_s}]$$
$$+ [e_{i_1}, \ldots, [e_{i_j}' f_t], \ldots, e_{i_s}] + [e_{i_1}, \ldots, e_{i_j}', \ldots, [e_{i_k} f_t], \ldots, e_{i_s}],$$

and a similar expression results for $[e_{i_1}, \ldots, e_{i_k}', \ldots, e_{i_s}] \operatorname{ad} f_t$. The summation terms are clearly equal by the induction hypothesis and a straightforward computation, together with the induction hypothesis, shows that

$$[e_{i_1}, \ldots, [e_{i_j}' f_t], \ldots, e_{i_s}] = [e_{i_1}, \ldots, [e_{i_j} f_t], \ldots, e_{i_k}', \ldots, e_{i_s}]$$

and $[e_{i_1}, \ldots, e_{i_j}', \ldots, [e_{i_k} f_t], \ldots, e_{i_s}] = [e_{i_1}, \ldots, e_{i_j}, \ldots, [e_{i_k}' f_t], \ldots, e_{i_s}]$. The result follows by (**4**, Proposition 9).

If each $g_{i_p} \in \{f_i \mid i \in \mathbf{L}^*\}$, we obtain the corresponding result by a similar computation. If each $g_{i_p} \in \{h_i \mid i \in \mathbf{L}^*\}$, the result is trivial.

(ii) Define $'$ from E into itself by $[g_{i_1}, \ldots, g_{i_s}]' = [g_{i_1}, \ldots, g_{i_j}', \ldots, g_{i_s}]$ whenever the gs are all es, all hs, or all fs, and by linear extension. This mapping is well-defined: Suppose that $a = \sum \lambda_{i_1, \ldots, i_s}[g_{i_1}, \ldots, g_{i_s}] = 0$, where the gs in each product are all es, all hs, or all fs. We can suppose that each product in the sum is of the same type, for example, all es. Then for any $k \in \mathbf{L}^*$, we have $0 = [ae_k'] = \sum \lambda_{i_1, \ldots, i_s}[[e_{i_1}, \ldots, e_{i_s}]e_k'] = \sum \lambda_{i_1, \ldots, i_s}[[e_{i_1}', \ldots, e_{i_s}]e_k] = [a'e_k]$. By (**4**, Proposition 9), $a' = 0$, as required. The other cases are similar.

(iii) We have:

$$[[e_{i_1}, \ldots, e_{i_s}] h_j'] = [[e_{i_1}, \ldots, e_{i_s}]' h_j] = [[e_{i_1}, \ldots, e_{i_s}] h_j]',$$
$$[[e_{i_1}, \ldots, e_{i_s}] f_j'] = [[e_{i_1}, \ldots, e_{i_s}]' f_j] = [[e_{i_1}, \ldots, e_{i_s}] f_j]',$$

and the corresponding equations hold when the es and fs are interchanged.

(iv) *The general case.* We wish to show that $[g_{i_1}, \ldots, g_{i_j}', \ldots, g_{i_s}] = [g_{i_1}, \ldots, g_{i_s}]' = [g_{i_1}, \ldots, g_{i_k}', \ldots, g_{i_s}]$, where each $g_{i_p} \in \{e_i, h_i, f_i \mid i \in \mathbf{L}^*\}$. This is true when $s = 1, 2$. Assume that it is true at $s - 1 \geqq 2$.

$$[g_{i_1}, \ldots, g_{i_{s-1}}] \in E_\beta$$

for some root β. Suppose that $\beta > 0$. Then

$$[g_{i_1}, \ldots, g_{i_{s-1}}] = \sum \lambda_{j_1, \ldots, j_u}[e_{j_1}, \ldots, e_{j_u}],$$

and

$$[(\sum \lambda_{j_1, \ldots, j_u}[e_{j_1}, \ldots, e_{j_u}])', g_{i_s}] = [(\sum \lambda_{j_1, \ldots, j_u}[e_{j_1}, \ldots, e_{j_u}]), g_{i_s}]'$$
$$= [(\sum \lambda_{j_1, \ldots, j_u}[e_{j_1}, \ldots, e_{j_u}]), g_{i_s}'].$$

The other cases are similar.

It is now easily checked that $[ab]' = [a'b] = [ab']$ for all $a, b \in E$. That $'$ is bijective follows from the fact that $'$ is homogeneous and that if M is the multiplication algebra of E and $a \in E$ is homogeneous and not zero, then $aM = E$. Thus $E' = (e_0M)' = e_0'M = E$, and if $a \in E$ is homogeneous and $a' = 0$, then $(0) = a'M = (aM)'$ and hence a must be zero.

3. 1-tiered Euclidean Lie algebras. From the results of § 2 and (**4**, § 7), we know that the ideal lattice of a Euclidean Lie algebra E is isomorphic to that of the associative algebra

$$\Phi\langle x\rangle = \left\{ \sum_{i=-\infty}^{\infty} \lambda_i x^i \,\middle|\, \text{almost all } \lambda_i = 0 \right\}$$

with its usual multiplication. An obvious method for constructing a Lie algebra with this ideal lattice is to take a central simple Lie algebra, B, over Φ and form $\Phi\langle x\rangle \otimes_\Phi B$ with the standard multiplication.†† In this section we establish that the 1-tiered algebras have this form. Later (Theorem 3) we will show that this is not in general true for the 2-tiered algebras.

Let $E = E((A_{ij}))$ be a Euclidean Lie algebra and let $\mu \in \Phi - \{0\}$. The ideal of E corresponding to the ideal of $\Phi\langle x\rangle$ generated by $x - \mu$ is maximal and thus the quotient, $E(\mu)$, of E by this ideal is simple. The elements of $E(\mu)$ are the classes formed by identification of a' with μa for all $a \in E$. By (**4**, Theorem 6), $E(\mu)$ is of finite dimension.

PROPOSITION 3. *If (A_{ij}) is a Euclidean matrix, and if F is a Lie algebra over Φ with generators $E_0, \ldots, E_l, H_0, \ldots, H_l, F_0, \ldots, F_l$ satisfying the relations*

(1) $[E_iH_j] = A_{ij}E_i$, $[F_iH_j] = -A_{ij}F_i$, $[E_iF_j] = \delta_{ij}H_i$, $[H_iH_j] = 0$ *for all i and j and* $E_i(\operatorname{ad} E_j)^{-A_{ij}+1} = 0$, $F_i(\operatorname{ad} F_j)^{-A_{ij}+1} = 0$ *for all i and j with $i \neq j$; and if in addition,*

(2) *F has a trivial centre and*

(3) *F is of infinite dimension,*

then $F \simeq E((A_{ij}))$.

Proof. There is clearly an epimorphism of $L((A_{ij}))$ onto F. Since the radical of L is its centre, there is an induced epimorphism from $E((A_{ij}))$ onto F. Since every proper ideal of E is of finite codimension, the mapping must be a monomorphism.

THEOREM 2. *If $E = E((A_{ij}))$ is a 1-tiered Euclidean Lie algebra, then $E \simeq \Phi\langle x\rangle \otimes_\Phi E(1)$.*

Note. The 1 is a matter of convenience. It follows from this theorem that $E(\mu) \simeq E(1)$ for all $\mu \in \Phi - \{0\}$.

Proof. Inspection of the calculations (a) for the 1-tiered cases (Table 1) shows that they have the form $\alpha_j r_{i_k} \ldots r_{i_1} r_0 = \alpha_0 + \xi$, where $j, i_k, \ldots, i_1$

††$[p(x) \otimes a, q(x) \otimes b] = p(x)q(x) \otimes [ab]$.

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are all different from 0. As a matter of convenience, we will suppose that $j = l$ in what follows. Thus $\alpha_l r_{i_k} \ldots r_{i_1} = -\alpha_0 + \xi$, whence $\alpha_0 r_{i_1} \ldots r_{i_k} = -\alpha_l + \xi$.

Consider the Lie algebra $K = \Phi\langle x\rangle \otimes_\Phi E(1)$. We will identify $E(1)$ with the subalgebra of elements $1 \otimes b$, $b \in E(1)$. For $a \in E$ let $\tilde{a}$ be its canonical image in $E(1)$. The elements $E_0 = x \otimes \tilde{e}_0$, $E_1 = \tilde{e}_1, \ldots, E_l = \tilde{e}_l$, $H_0 = \tilde{h}_0$, $H_1 = \tilde{h}_1, \ldots, H_l = \tilde{h}_l$, and $F_0 = x^{-1} \otimes f_0$, $F_1 = \tilde{f}_1, \ldots, F_l = \tilde{f}_l$ satisfy the relations (1) of Proposition 3 and generate a certain subalgebra F of K. If we can show that $F = K$, then it will be clear that F also satisfies conditions (2) and (3) of the proposition, and in consequence, $K = F \simeq E$.

Let β be a non-null root and suppose that $a \in E_\beta - \{0\}$. Let $i \in \mathbf{L}^*$. By Lemma 2, $\beta r_i = \beta - \beta(h_i)\alpha_i$ is a root and by the theory of irreducible modules over a split 3-dimensional simple algebra, we have:

$$a(\operatorname{ad} f_i)^{\beta(h_i)} \in E_{\beta r_i} - \{0\} \quad \text{if } \beta(h_i) \geqq 0,$$

$$a(\operatorname{ad} e_i)^{|\beta(h_i)|} \in E_{\beta r_i} - \{0\} \quad \text{if } \beta(h_i) < 0.$$

It follows that

$$e_0(\operatorname{ad} g_{i_1})^{|\alpha_0(h_{i_1})|}(\operatorname{ad} g_{i_2})^{|(\alpha_0 r_{i_1})(h_{i_2})|} \ldots (\operatorname{ad} g_{i_k})^{|(\alpha_0 r_{i_1} \cdots r_{i_{k-1}})(h_{i_k})|}$$

is a non-zero element of $E_{-\alpha_l+\xi}$, where $g_{i_j} = e_{i_j}$ $(g_{i_j} = f_{i_j})$ for

$$(\alpha_0 r_{i_1} \ldots r_{i_{j-1}})(h_{i_j}) < 0 \quad (\geqq 0).$$

Thus

$$(x \otimes \tilde{e}_0)(\operatorname{ad} \tilde{g}_{i_1})^{|\alpha_0(h_{i_1})|} \ldots (\operatorname{ad} \tilde{g}_{i_k})^{|\alpha_0 r_{i_1} \cdots r_{i_{k-1}}(h_{i_k})|} = x \otimes \alpha \tilde{f}_l{}' = x \otimes \alpha \tilde{f}_l$$

for some $\alpha \in \Phi - \{0\}$. This shows that $x \otimes \tilde{f}_l \in F$. Successive applications of $\operatorname{ad} \tilde{h}_l$ show that $x \otimes \tilde{h}_l$ and $x \otimes \tilde{e}_l \in F$. One then obtains readily that $x \otimes \tilde{h}_i$, $x \otimes \tilde{f}_i$, and $x \otimes \tilde{e}_i$, $i \in \mathbf{L}$, are in F. (The fact that the vertex 0 is a terminal vertex is used here.) Then $[[x \otimes \tilde{e}_0, x \otimes \tilde{e}_j], \tilde{f}_j] = \beta x^2 \otimes \tilde{e}_0$, $\beta \in \Phi - \{0\}$ if vertex j is joined to vertex 0, so that $x^2 \otimes \tilde{e}_0$, $x \otimes \tilde{h}_0$, and $\tilde{f}_0$ are in F. Repetition of this performance demonstrates that for all $i \in \mathbf{L}^*$, $x^n \otimes \tilde{f}_i$, $x^n \otimes \tilde{h}_i \in F$, $n \geqq 0$, and $x^n \otimes \tilde{e}_i \in F$, $n > 0$. (At this point we have not shown that $e_0 \in F$.)

If we begin with f_0 instead of e_0, we find that $x^n \otimes \tilde{e}_i$, $x^n \otimes \tilde{h}_i \in F$, $n \leqq 0$, and $x^n \otimes \tilde{f}_i \in F$, $n < 0$. Thus $F = K$, as required.

It is interesting to note that when E is 1-tiered, $E(1) \simeq E((A_{ij})_{i,j>0})$ so that $E(1)$ is the split simple Lie algebra over Φ whose diagram is that of E with vertex 0 removed. To show this isomorphism it is enough to show that $\tilde{e}_0$, $\tilde{h}_0$, and $\tilde{f}_0$ are in the algebra F^* generated by $\tilde{e}_i$, $\tilde{h}_i$, $\tilde{f}_i$, $i \in \mathbf{L}$. However, from $\alpha_l r_{i_k} \ldots r_{i_1} = -\alpha_0 + \xi$, we have $(-\alpha_l + \xi) r_{i_k} \ldots r_{i_1} = \alpha_0$. Thus

$$f_l{}'(\operatorname{ad} g_{i_k})^{|(-\alpha_l+\xi)(h_{i_k})|} \ldots (\operatorname{ad} g_{i_1})^{|((-\alpha_l+\xi) r_{i_k} \cdots r_{i_2})(h_{i_1})|}$$

is a non-zero element of E_{α_0}. Here again

$$g_{i_j} = e_{i_j} \ (g_{i_j} = f_{i_j}) \quad \text{for } (-\alpha_l + \xi) r_{i_k} \ldots r_{i_{j+1}}(h_{i_j}) < 0 \ (\geqq 0).$$

Since none of the numbers $i_k, \ldots, i_1$ is 0, we obtain $\tilde{e}_0 \in F^*$. Likewise, $\tilde{f}_0 \in F^*$ whence $[\tilde{e}_0\tilde{f}_0] = \tilde{h}_0 \in F^*$ and $F^* = E(1)$.

With the aid of the following proposition, we can obtain interpretations of $A_{l,1}$, $B_{l,1}$, $C_{l,1}$, and $D_{l,1}$ as Lie algebras of linear transformations.

Let V be a vector space over Φ of countably infinite dimension and let v_i, $i \in \mathbf{Z}$, be a basis for V. Let S be the shift mapping defined by $v_iS = v_{i+1}$ for all $i \in \mathbf{Z}$ and let $I_l = \{T \in \mathrm{Hom}_\Phi(V, V) \mid S^{-l}TS^l = T\}$.

PROPOSITION 4. *$I_l \simeq \Phi\langle x\rangle \otimes \Phi_l$, where Φ_l is the ring of $l \times l$ matrices over Φ.*

Proof. Suppose that $v_iT = \sum_{j\in\mathbf{Z}} \mu_{ij}v_j$ ($\mu_{ij} = 0$ for all but a finite number of the $j \in \mathbf{Z}$). Then $T \in I_l$ if and only if $\mu_{ij} = \mu_{i+kl,j+kl}$ for all $k \in \mathbf{Z}$. In view of this, the matrix of T relative to the basis $\{v_i\}$ has the form:

$$\begin{matrix} & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdots & A_0 & A_1 & A_2 & A_3 & A_4 & \cdots \\ \cdots & A_{-1} & A_0 & A_1 & A_2 & A_3 & \cdots \\ \cdots & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & \cdots \\ \cdots & A_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ \cdots & A_{-4} & A_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \end{matrix}$$

where $A_i \in \Phi_l$ and all but a finite number of them are zero. Define $\omega: I_l \to \Phi\langle x\rangle \otimes \Phi_l$ by

$$T \to \sum_{i\in\mathbf{Z}} x^i \otimes A_i.$$

Clearly ω is linear, one-to-one, and onto. One checks also that it preserves multiplication.

Consider $\Phi\langle x\rangle \otimes \Phi_l$ with its usual Lie multiplication. For a Lie algebra K, let $D(K)$ represent its derived algebra. Clearly $D(\Phi\langle x\rangle \otimes \Phi_l) \simeq \Phi\langle x\rangle \otimes D(\Phi_l)$, whence we have the following result.

PROPOSITION 5. $D(I_l) \simeq A_{l-1,1}$.

Now let (,) be a bilinear form on $V_l = \Phi v_0 + \ldots + \Phi v_{l-1}$ and extend this to a bilinear mapping of V into $\Phi\langle x\rangle$ by

$$(v_iS^{tl}, v_jS^{vl})_l = (v_i, v_j)x^{t+v}.$$

Let $S_l = \{T \in \mathrm{Hom}(V_l, V_l) \mid T$ is skew relative to (,) on $V_l\}$ and $S_l^* = \{T \in \mathrm{Hom}(V, V) \mid T$ is skew relative to $(,)_l$ on $V\}$. Then $S_l^* \simeq \Phi\langle x\rangle \otimes S_l$, as is easily verified.

PROPOSITION 6. *Let (,) be a non-degenerate skew-symmetric bilinear form or a non-degenerate symmetric bilinear form of maximal Witt index on V_l. Extend it to V as above. Then in the skew case, $S_l^* \simeq C_{l/2,1}$ and in the symmetric case, $S_l^* \simeq D_{l/2,1}$ or $B_{(l-1)/2,1}$ depending on whether l is even or odd.*

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TABLE 2

All the αs have been omitted, leaving only the indices. When we write $\xi = (1, 2, \ldots, 2, 1)$ we mean $\xi = \alpha_0 + 2\alpha_1 + \ldots + 2\alpha_{l-1} + \alpha_l$.

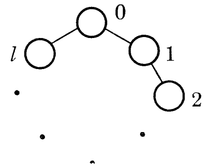
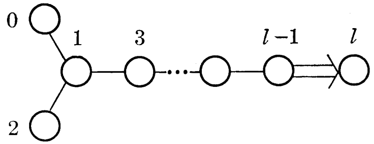
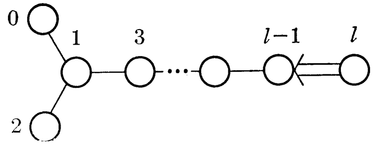
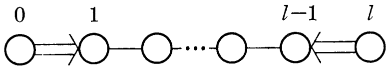
Algebra	Diagram	ξ	Fundamental roots repeating mod ξ	dim E_ξ	Type of $E(\mu)$	l
A_{l-1}		$(1, 1, \ldots, 1)$	$0, 1, \ldots, l$		A_l	$\geqq 2$
$B_{l,1}$		$(1, 2, 1, 2, 2, \ldots, 2)$	$0, 1, \ldots, l$		B_l	$\geqq 3$
$C_{l,2}$		$(1, 2, 1, 2, 2, \ldots, 2, 1)$	$0, 1, \ldots, l-1$	$l-1$	A_{2l-1}	$\geqq 3$
$C_{l,1}$		$(1, 2, \ldots, 2, 1)$	$0, 1, \ldots, l$		C_l	$\geqq 2$

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TABLE 2 (*continued*)

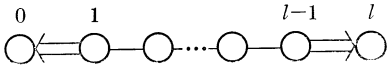
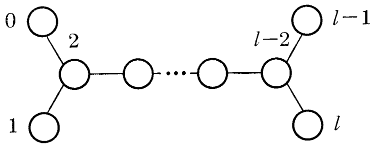
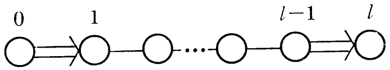
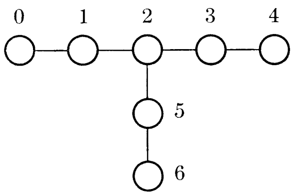
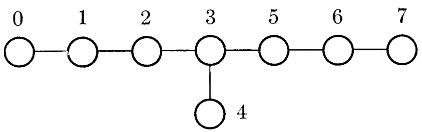
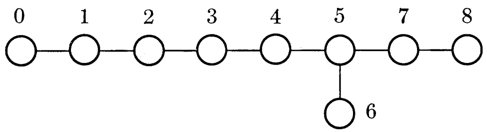
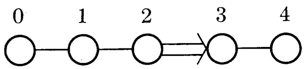
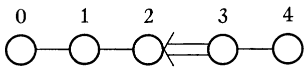
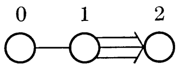
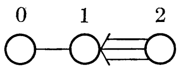
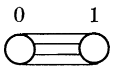
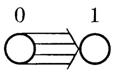
Algebra	Diagram	ξ	Fundamental roots repeating mod ξ	$\dim E_\xi$	Type of $E(\mu)$	l
$B_{l,2}$		$(1, 1, \ldots, 1)$	$0, l$	1	D_{l+1}	$\geqq 2$
$D_{l,1}$		$(1, 1, 2, \ldots, 2, 1, 1)$	$0, 1, \ldots, l$		D_l	$\geqq 4$
$BC_{l,2}$		$(1, 2, \ldots, 2)$	$1, \ldots, l$	l	A_{2l}	$\geqq 2$
$E_{6,1}$		$(1, 2, 3, 2, 1, 2, 1)$	$0, 1, \ldots, 6$		E_6	
$E_{7,1}$		$(1, 2, 3, 4, 2, 3, 2, 1)$	$0, 1, \ldots, 7$		E_7	

TABLE 2 (*Concluded*)

Algebra	Diagram	ξ	Fundamental roots repeating mod ξ	$\dim E_\xi$	Type of $E(\mu)$	l
$E_{8,1}$		(1, 2, 3, 4, 5, 6, 3, 4, 2)	0, 1, . . . , 8		E_8	
$F_{4,1}$		(1, 2, 3, 4, 2)	0, 1, . . . , 4		F_4	
$F_{4,2}$		(1, 2, 3, 2, 1)	0, 1, 2	2	E_6	
$G_{2,1}$		(1, 2, 3)	0, 1, 2		G_2	
$G_{2,3}$		(1, 2, 1)	0, 1	1 $\dim E_{2\xi} = 1$	D_4	
$A_{1,1}$		(1, 1)	0, 1		A_1	
$A_{1,2}$		(1, 2)	1	1	A_2	

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4. 2-tiered and 3-tiered algebras.

(1) *A non-isomorphism theorem.* Let E be a Euclidean Lie algebra. The isomorphism of (**4**, § 7) between the ideal lattices of E and $\Phi\langle x\rangle$ maps the ideal $\langle\sum_{i=0}^{k}\lambda_i h_0^{(i)}\rangle$ into the ideal $(\sum_{i=0}^{k}\lambda_i x^i)$ of $\Phi\langle x\rangle$. It is straightforward to show that the centroid C of E is the subalgebra of $\mathrm{Hom}_\Phi(E, E)$ generated by the shift mapping $'$ and its inverse. Thus C can be identified with $\Phi\langle x\rangle$, where the action of x on E is given by $ax = a'$ for all $a \in E$. The centroid of

$$E\Big/\left\langle\sum_{i=0}^{k}\lambda_i h_0^{(i)}\right\rangle$$

is then isomorphic to $\Phi\langle x\rangle/(\sum_{i=0}^{k}\lambda_i x^i)$. In particular, $E(\mu)$ is central simple for all $\mu \in \Phi - \{0\}$.

Fix a particular $\mu \in \Phi - \{0\}$. Let $\sim$ be the natural mapping of E onto $E(\mu)$.

LEMMA 5. $\sum_{i=0}^{r-1}\tilde{E}_{i\xi}$ *is a Cartan subalgebra of* $E(\mu)$. (r *is the tier number.*)

Proof. Let $B = \sum_{i=0}^{r-1}\tilde{E}_{i\xi}$. We will show that B is its own normalizer and is nilpotent. Suppose that $[b, B] \subseteq B$ for some $b \in \tilde{E}$. There is an element $a \in E$ of the form $\sum_{0\leqq \mathrm{ht}\,\beta<\mathrm{ht}\,\zeta} a_\beta$, where $a_\beta \in E_\beta$, such that $\tilde{a} = b$. We must have $[a, H] \subseteq \sum_{i=0}^{r-1}E_{i\xi} + \langle h_0' - \mu h_0\rangle$. This can only happen if $a \in \sum_{i=0}^{r-1}E_{i\xi}$. Thus $b \in B$.

In showing that B is nilpotent we will rely on the fact that $r = 1, 2$, or 3. If $r = 1$, then $B = \tilde{H}$, which is abelian. If $r = 2$ or 3, then $\tilde{H}$ is in the centre of B. If $r = 2$, $B/\tilde{H}$ is abelian since $[\tilde{E}_\xi, \tilde{E}_\xi] \subseteq \tilde{E}_\zeta = \tilde{H}$. Thus $B^3 = (0)$. If $r = 3$, let $B_1 = (\tilde{E}_\xi + \tilde{H})/\tilde{H}$ and $B_2 = (\tilde{E}_{2\xi} + \tilde{H})/\tilde{H}$. Then $B/\tilde{H} = B_1 + B_2$, $B_1^2 \subseteq B_2$, $B_2^2 \subseteq B_1$, and $[B_1B_2] = (0)$. It follows that $(B_1 + B_2)^3 = (0)$, so that $B^4 = (0)$.

Let Ψ be the algebraic closure of Φ. Then $\Psi \otimes_\Phi E(\mu)$ is a split finite-dimensional simple Lie algebra and $\Psi \otimes \sum_{i=0}^{r-1}E_{i\xi}$ is a Cartan subalgebra. Thus $E(\mu)_\Psi$ has dimension $E(\mu)$ and rank $\sum_{i=0}^{r-1}\dim E_{i\xi}$. Note that

$$\dim E(\mu) = \sum_{0\leqq \mathrm{ht}\,\beta<\mathrm{ht}\,\zeta}\dim E_\beta$$

which is independent of μ.

THEOREM 3. *Suppose that* E *is a Euclidean Lie algebra and* $\dim E(\mu) = m$, $\dim H = l$. *Then* $E(\mu_1)$ is not isomorphic to $E(\mu_2)$ *if* $(\mu_1^{-1}\mu_2)^{m-l}$ *is not a square.*

Proof. Let $B = \{c_j \mid j \in \Omega\}$ be a basis for $\sum_{0\leqq \mathrm{ht}\,\beta<\mathrm{ht}\,\zeta}E_\beta$, chosen so that (i) each c_j is in some root space E_{β_j} and (ii) $h_1, \ldots, h_l \in B$. Let $\mu_1, \mu_2 \in \Phi - \{0\}$ and suppose that $E(\mu_1) \simeq E(\mu_2)$. Let D be a Lie algebra isomorphic to both and for $a \in E$ let $\tilde{a}$ ($\bar{a}$) be the image of a in D under the composition of the natural mapping of E onto $E(\mu_1)$ ($E(\mu_2)$) and the given isomorphism of $E(\mu_1)$ ($E(\mu_2)$) onto D. $\{\tilde{c}_j\}$ ($j \in \Omega$) and $\{\bar{c}_j\}$ ($j \in \Omega$) are bases of D. We will derive the theorem by comparing the determinants of the Killing form on D computed relative to these two bases and using the fact that they must differ by a factor of a square.

For $\mathrm{tr}_D \operatorname{ad} \tilde{c}_i \operatorname{ad} \tilde{c}_j \neq 0$, it is necessary that $\beta_i + \beta_j = 0$ or $\beta_i + \beta_j = r\xi$, and similarly for $\mathrm{tr}_D \operatorname{ad} \bar{c}_i \operatorname{ad} \bar{c}_j$. Thus the matrix

$$\tilde{M} = (\mathrm{tr}_D \operatorname{ad} \tilde{c}_i \operatorname{ad} \tilde{c}_j)_{i,j\in\Omega} \qquad (\bar{M} = \mathrm{tr}_D (\operatorname{ad} \bar{c}_i \operatorname{ad} \bar{c}_j)_{i,j\in\Omega})$$

decomposes into two diagonal blocks, one for the h_is, $\tilde{M}_1$ ($\bar{M}_1$), and one for the remainder of the basis, $\tilde{M}_2$ ($\bar{M}_2$).

$$\tilde{M} = \left(\begin{array}{c|c} \tilde{M}_1 & \\ \hline & \tilde{M}_2 \end{array}\right) \quad \text{and} \quad \bar{M} = \left(\begin{array}{c|c} \bar{M}_1 & \\ \hline & \bar{M}_2 \end{array}\right).$$

For $c_q \in B$, $\tilde{c}_q \operatorname{ad} \tilde{h}_i \operatorname{ad} \tilde{h}_j = \beta_q(h_i)\beta_q(h_j)\tilde{c}_q$ and $\bar{c}_q \operatorname{ad} \bar{h}_i \operatorname{ad} \bar{h}_j = \beta_q(h_i)\beta_q(h_j)\bar{c}_q$. Thus $\tilde{M}_1 = \bar{M}_1$. For $c_i, c_j \in B - \{h_1, \ldots, h_l\}$, $\mathrm{tr}_D \operatorname{ad} \tilde{c}_i \operatorname{ad} \tilde{c}_j$ and $\mathrm{tr}_D \operatorname{ad} \bar{c}_i \operatorname{ad} \bar{c}_j$ are zero unless $\beta_i + \beta_j = r\xi$. Suppose that $\beta_i + \beta_j = r\xi$. For $q \in \Omega$,

$$c_q \operatorname{ad} c_i \operatorname{ad} c_j = \Bigl(\sum_{\{k\in\Omega \mid \beta_k = \beta_q\}} \lambda_k c_k\Bigr)',$$

and hence $\tilde{c}_q \operatorname{ad} \tilde{c}_i \operatorname{ad} \tilde{c}_j = \mu_1(\sum \lambda_k \tilde{c}_k)$ and $\bar{c}_q \operatorname{ad} \bar{c}_i \operatorname{ad} \bar{c}_j = \mu_2(\sum \lambda_k \bar{c}_k)$. It follows that $\mathrm{tr} \operatorname{ad} \bar{c}_i \operatorname{ad} \bar{c}_j = \mu_1^{-1}\mu_2 \,\mathrm{tr} \operatorname{ad} \tilde{c}_i \operatorname{ad} \tilde{c}_j$ and $\bar{M}_2 = \mu_1^{-1}\mu_2 \tilde{M}_2$. Therefore $\det \bar{M} = (\mu_1^{-1}\mu_2)^{m-l} \det \tilde{M}$ and $(\mu_1^{-1}\mu_2)^{m-l}$ must be a square.

To apply this theorem we require that Φ contain non-squares and $m - l$ is odd. The remainder of this section is devoted essentially to calculating m for the various Euclidean Lie algebras. These values may be computed from Table 2. Here one observes that $m - l$ can be odd (for example, in the case of $C_{l,2}$ when l is even) and this at least suffices to show that Theorem 2 is not valid (in general) for the 2-tiered algebras. We conjecture that Theorem 2 is false for all 2-tiered and 3-tiered algebras.

(2) *The type of* $E(\mu)$. Assume throughout that a fixed $\mu \in \Phi - \{0\}$ has been chosen. We would like to know the type of $E(\mu)$. This amounts to identifying $E(\mu)_\Psi$, where Ψ is the algebraic closure of Φ. The knowledge of the dimension and rank of a split simple Lie algebra is often enough to determine its identity, and we will find that this approach works on all the 2-tiered and 3-tiered algebras except $F_{4,2}$. What we must do is calculate $\rho = \sum_{i=0}^{r-1} \dim E_{i\xi}$ and $\dim E(\mu)$ for each 2-tiered and 3-tiered Euclidean Lie algebra.

Let $\bar{\Delta}_W$ denote the set of equivalence classes of Δ_W modulo $\zeta = r\xi$. $\dim E(\mu) = |\bar{\Delta}_W| + \rho$, thus we must calculate $|\bar{\Delta}_W|$ and ρ.

LEMMA 6. *Let* $k \in \{1, \ldots, r-1\}$ *and let* $\mathbf{J}_k = \{j \in \mathbf{L}^* \mid \alpha_j + k\xi \in \Delta\}$. *For each* $j \in \mathbf{J}_k$ *let* e_j^* *be a non-zero element of* $E_{\alpha_j + k\xi}$. *Then* $\{[e_j^* f_j] \mid j \in \mathbf{J}_k\}$ *spans* $E_{k\xi}$.

Proof. Let $a \in E_{k\xi}$. a can be expressed in the form $\sum_{\gamma\in\Gamma} \lambda_\gamma [a_\gamma e_{i(\gamma)}]$, where $\lambda_\gamma \in \Phi$, $i(\gamma) \in \mathbf{L}^*$, and $a_\gamma \in E_{k\xi - \alpha_{i(\gamma)}}$. Since $(k\xi - \alpha_{i(\gamma)})(h_{i(\gamma)}) = -2$,

$$a_\gamma (\operatorname{ad} e_{i(\gamma)})^2 \neq 0.$$

Thus $a_\gamma(\operatorname{ad} e_{i(\gamma)})^2 = \omega_\gamma e_{i(\gamma)}{}^*$. The $K_{i(\gamma)}$-module generated by a_γ contains $[a_\gamma e_{i(\gamma)}]$ and $\omega_\gamma e_{i(\gamma)}{}^*$, and hence $[\omega_\gamma e_{i(\gamma)}{}^*, f_{i(\gamma)}]$ is a non-zero multiple of $[a_\gamma e_{i(\gamma)}]$.

Because of its exceptional nature, it is as well to dispose of $G_{2,3}$ immediately. It is easy to list all the roots and find that $|\bar{\Delta}_W| = 24$. $\dim H = 2$, $\dim E_\xi = 1$ or 2, and $\dim E_{2\xi} = 1$ or 2 by Lemma 6. Thus $G_{2,3}(\mu)$ is of dimension 28, 29, or 30 with a rank of 4, 5, or 6, respectively. The only possibility is dimension 28 and rank 4, and hence $G_{2,3}(\mu)$ is of type D_4.

From now on all our Euclidean Lie algebras will be assumed to be 2-tiered.

LEMMA 7. *Let E be a 2-tiered Euclidean Lie algebra and let $\Pi' = \{\alpha_1, \ldots, \alpha_s\}$ be a connected subset of the fundamental roots with the property that $\alpha_i + \xi \in \Delta$, $i = 1, \ldots, s$. (The numbering on these roots is one of convenience and does not necessarily coincide with that of the diagram for E in Table 2.) Let $L(\Pi')$ be the finite-dimensional Lie algebra generated by $\{e_i, h_i, f_i \mid i = 1, \ldots, s\}$ and let γ be the highest root of $L(\Pi')$. $\gamma + \xi$ is a root. If $E_{\gamma+\xi}$ is annihilated by* $\operatorname{ad} e_1, \ldots, \operatorname{ad} e_s$, *then the $L(\Pi')$-module M generated by $E_{\gamma+\xi}$ is isomorphic to* $L(\Pi')$, considered as an *$L(\Pi')$-module and* $\dim(M \cap E_\xi) = s$.

Proof. Let a and a^* be non-zero elements of $E_{\gamma+\xi}$ and E_γ, respectively. $L(\Pi')$ is, up to isomorphism, the only e-extreme irreducible $L(\Pi')$-module with γ as the highest weight. Since M is e-extreme and has the same highest weight, there is an $L(\Pi')$-module homomorphism ϕ of M onto $L(\Pi')$ which takes a onto a^* and each weight space $M_{-\Sigma\mu_i\alpha_i}$ (which is a subspace of $E_{\gamma-\Sigma\mu_i\alpha_i+\xi}$) onto $E_{\gamma-\Sigma\mu_i\alpha_i}$ (**3**, pp. 214–215). Now $M \cap E_\xi$ is clearly contained in the space spanned by $[e_1{}^*f_1], \ldots, [e_s{}^*f_s]$, where the $e_i{}^*$ are non-zero elements of $E_{\alpha_i+\xi}$, and on the other hand $M \cap E_\xi$ is mapped onto

$$\Phi h_1 + \ldots + \Phi h_s.$$

Thus $\phi \mid M \cap E_\xi$ is injective.

Suppose that ϕ is not injective. Then there is a weight space $M \cap E_{\beta+\xi}$ on which ϕ is not injective and it is clear that $\beta \notin \mathbf{Z}\xi$ so that $\beta + \xi$ is not isotropic. Thus $M \cap E_{\beta+\xi} = E_{\beta+\xi}$ and $(E_{\beta+\xi})\phi = (0)$. Suppose that β is chosen of maximal height with this property. Let $a \operatorname{ad} f_{i_1} \ldots \operatorname{ad} f_{i_t}$ be a non-zero element of $E_{\beta+\xi}$. By assumption, $a^* \operatorname{ad} f_{i_1} \ldots \operatorname{ad} f_{i_t} = 0$. However,

$$b^* \equiv a^* \operatorname{ad} f_{i_1} \ldots \operatorname{ad} f_{i_{t-1}} \neq 0.$$

Let $b = a \operatorname{ad} f_{i_1} \ldots \operatorname{ad} f_{i_{t-1}}$. In the notation of Lemma 2, with the obvious use of $*$, $d - u = d^* - u^*$. Now $b(\operatorname{ad} e_i)^j \neq 0$ implies that $b^*(\operatorname{ad} e_i)^j \neq 0$ since ϕ is injective on any space $E_{\delta+\xi} \cap M$ when $\operatorname{ht} \delta > \operatorname{ht} \beta$. This means that $u^* \geqq u$. Combining this with $d^* = 0$ yields $d = 0$, which is a contradiction. Thus $M \simeq L(\Pi')$ as $L(\Pi')$-modules.

COROLLARY. *Under the hypotheses of Lemma 7, and in the notation used in its proof,* $[e_1{}^*f_1], \ldots, [e_s{}^*f_s]$ *are linearly independent.*

In what follows, α_0 will always represent a fundamental root chosen in the manner described in § 1. α_1 will be the unique fundamental root such that $(\alpha_0, \alpha_1) \neq 0$. Unless otherwise indicated, the labels $2, \ldots, l$ will be assigned to the remaining vertices of the diagram in some way convenient to the discussion at hand. μ will denote a fixed element of $\Phi - \{0\}$. The subgroup of W generated by $w_1, \ldots, w_s \in W$ will be denoted by $\langle w_1, \ldots, w_s \rangle$. If X is any finite-dimensional split semi-simple Lie algebra, its Weyl group will be denoted by $W(X)$. The Lie algebra L generated by $\{e_i, h_i, f_i \mid i \in \mathbf{L}\}$ will be considered as both a subalgebra of E and $E(\mu)$. Note that $W(L) = \langle r_1, \ldots, r_l \rangle$. The set of non-zero roots, $\Delta_{W(L)}$, of L is $\langle \alpha_1, \ldots, \alpha_l \rangle W(L)$. Let Γ be the set of roots $\sum_{i \in \mathbf{L}^*} \lambda_i \alpha_i \in \Delta_W$ with $\lambda_0 = 1$. $\bar{\Delta}_W$, $\bar{\Delta}_{W(L)}$, and $\bar{\Gamma}$ represent the set of equivalence classes of Δ_W, $\Delta_{W(L)}$, and Γ modulo $2\xi = \zeta$. $|\bar{\Delta}_{W(L)}| = |\Delta_{W(L)}|$ and $|\bar{\Gamma}| = |\Gamma|$.

LEMMA 8. $\bar{\Delta}_W = \bar{\Delta}_{W(L)} \cup \bar{\Gamma}$ *(disjoint union).*

Proof. Let $\bar{\beta} \in \bar{\Delta}_W$ and choose $\beta = \sum_{i \in \mathbf{L}^*} \lambda_i \alpha_i$ so that the equivalence class of β mod ζ is $\bar{\beta}$ and $\lambda_0 = 0$ or 1. If $\lambda_0 = 0$, then a trivial induction on the height of β shows that $\beta \in \Delta_{W(L)}$. If $\lambda_0 = 1$, then $\beta \in \Gamma$. The union is clearly disjoint.

As a result, $|\bar{\Delta}_W| = |\Delta_{W(L)}| + |\Gamma|$, and since $\Delta_{W(L)}$ is the root system of a finite-dimensional split simple Lie algebra and $|\Delta_{W(L)}|$ is well known, we need only determine $|\Gamma|$.

We first derive a method of calculating $|\Gamma|$ when α_0 is a root of minimal weight amongst the weights of the α_i. This is applicable to $B_{l,2}$, $C_{l,2}$, and $F_{4,2}$. Slight modifications then allow us to deal with $BC_{l,2}$ and $A_{1,2}$.

LEMMA 9. *If α_0 is of minimal weight, then $\Gamma = \alpha_0 \langle r_1, \ldots, r_l \rangle$.*

Proof. Clearly $\alpha_0 \langle r_1, \ldots, r_l \rangle \subseteq \Gamma$. To prove the reverse inclusion we use induction on the height. If $\beta \in \Gamma$ has height 1, then $\beta = \alpha_0$, hence we can suppose now that $\mathrm{ht}(\beta) < m$ implies that $\beta \in \alpha_0 \langle r_1, \ldots, r_l \rangle$, $m > 1$. Suppose that $\beta \in \Gamma$ and $\mathrm{ht}(\beta) = m$. $\beta = \alpha_0 + \gamma_1 = \alpha_0 + \sum_{i \in \mathbf{K}} \lambda_i \alpha_i$, where $\mathbf{K}$ is a subset of $\mathbf{L}$ and each $\lambda_i > 0$.

We know that there is an $i \in \mathbf{L}^*$ for which $\mathrm{ht}(\beta r_i) < \mathrm{ht}(\beta)$, and if we can choose $i > 0$, our proof will be complete. We assume therefore that $\mathrm{ht}(\beta r_i) \geqq \mathrm{ht}(\beta)$ if $i > 0$ and that $\mathrm{ht}(\beta r_0) < \mathrm{ht}(\beta)$. From

$$\beta r_0 = -\alpha_0 + \gamma_1 - 2((\gamma_1, \alpha_0)/(\alpha_0, \alpha_0))\alpha_0,$$

we have $-1 = 2(\gamma_1, \alpha_0)/(\alpha_0, \alpha_0) = \lambda_1 A_{10}$, and consequently $\lambda_1 = 1$ and $A_{10} = -1$. This rules out $B_{l,2}$.

There is no loss of generality in supposing that $\mathbf{K} = \{1, 2, \ldots, s\}$, $s \leqq l$. In what follows, we may rearrange the subscripts $2, \ldots, s$. However, we define γ_k, $k = 1, 2, \ldots, s$, by $\gamma_k = \sum_{k \leqq i \leqq s} \lambda_i \alpha_i$ bearing in mind that this is not properly defined yet if $k \geqq 2$. Put $\gamma_0 = \beta$.

Consider the following statements:

(a) $\lambda_k = 1$;

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(b) each of $\alpha_0, \ldots, \alpha_{k-1}$ is orthogonal to each of $\alpha_{k+1}, \ldots, \alpha_s$;
(c) $A_{k,k-1} = -1 = A_{k-1,k}$;
(d) $(\gamma_{k-1}, \alpha_k) \leqq 0$;
These are all true if $k = 1$. We will see that their truth up to and including k implies their truth at $k + 1$, after a possible rearrangement of the indices $k + 1, \ldots, s$.

Since γ_k has all positive coefficients in its expression in terms of the fundamental roots, there is an $i \in \mathbf{L}^*$ such that $(\gamma_k, \alpha_i) > 0$. This certainly does not happen if $i < k$ or $i \notin \mathbf{K}$, thus we know that $k \leqq i \leqq s$. If $i > k$, then $(\beta, \alpha_i) = (\alpha_0 + \ldots + \alpha_{k-1} + \gamma_k, \alpha_i) = (\gamma_k, \alpha_i) > 0$ by (a) and (b). This implies that $\mathrm{ht}(\beta r_i) < \mathrm{ht}(\beta)$, contrary to our assumption. Thus $(\gamma_k, \alpha_k) > 0$ and $(\gamma_k, \alpha_i) \leqq 0$ for $k < i \leqq s$. Now

$$0 \geqq 2(\gamma_{k-1}, \alpha_k)/(\alpha_k, \alpha_k) = 2(\alpha_{k-1} + \gamma_k, \alpha_k)/(\alpha_k, \alpha_k) = A_{k-1,k} + 2(\gamma_k, \alpha_k)/(\alpha_k, \alpha_k),$$

whence $0 < 2(\gamma_k, \alpha_k)/(\alpha_k, \alpha_k) \leqq -A_{k-1,k} = 1$.

$$1 = 2(\gamma_k, \alpha_k)/(\alpha_k, \alpha_k) = 2 + \sum_{k<i\leqq s} \lambda_i A_{ik}.$$

This means that there is only one i, $k < i \leqq s$, such that $A_{ik} \neq 0$. Call it $k + 1$. Then $A_{k+1,k} = -1$. We have $\lambda_{k+1} = 1$. (b), (c), (d) are now seen to be true at $k + 1$ (note that $A_{k,k+1} = -1$ since α_0 has minimal weight and $B_{l,2}$ has been ruled out, so that as we move away from α_0 we always meet arrows head on).

We conclude that $\beta = \alpha_0 + \alpha_1 + \ldots + \alpha_s$ and for $i, j \in \{1, \ldots, s\}$ we have:

$$A_{ij} = A_{ji} = \begin{cases} 0 & \text{if } |i - j| > 1, \\ -1 & \text{if } |i - j| = 1. \end{cases}$$

Thus $\beta r_s = \alpha_0 + \alpha_1 + \ldots + \alpha_{s-1}$ and $\mathrm{ht}(\beta r_s) < \mathrm{ht}(\beta)$, a contradiction.

For $w \in \langle r_1, \ldots, r_l\rangle$ we will let $l(w)$ denote the reduced length of w, i.e., the minimum integer k such that w may be written in the form $r_{i_1} \ldots r_{i_k}$.

LEMMA 10. *If α_0 is of minimal weight, then*

$$|\Gamma| = |\langle r_1, \ldots, r_l\rangle|/|\langle r_2, \ldots, r_l\rangle|.$$

Proof. We know that $\Gamma = \alpha_0\langle r_1, \ldots, r_l\rangle$. Let V be the subgroup of $\langle r_1, \ldots, r_l\rangle$ leaving α_0 fixed. Clearly $V \supseteq \langle r_2, \ldots, r_l\rangle$, and we complete the lemma by showing that we have equality here.

Use induction on the length $l(w)$ of $w \in V$. Thus suppose that for some $m > 0$ we have shown that $w \in V$ and $l(w) < m$ imply that $w \in \langle r_2, \ldots, r_l\rangle$, and suppose now that $w \in V$ and $l(w) = m$. By a well-known result (see for example **3**, Theorem 8.2), there is an $i \in \mathbf{L}$ such that $\alpha_i w \in \Delta^-$. This i cannot be 1, for then $\alpha_1 w = \sum_{i\in\mathbf{L}} \lambda_i \alpha_i$ with each $\lambda_i \leqq 0$ and

$$0 > (\alpha_0, \alpha_1) = (\alpha_0 w, \alpha_1 w) = (\alpha_0, \textstyle\sum_{i\in\mathbf{L}} \lambda_i \alpha_i) \geqq 0.$$

Thus $\alpha_i w \in \Delta^-$ for some $i > 1$. It follows (**2**, Lemma 2.2) that $l(r_i w) < l(w)$. However, $r_i w \in V$. By the induction assumption, $r_i w \in \langle r_2, \ldots, r_l \rangle$, whence $w \in \langle r_2, \ldots, r_l \rangle$.

PROPOSITION 7. *If E is a 2-tiered Euclidean Lie algebra, and α_0 is of minimal weight amongst the weights of the α_i, $i \in \mathbf{L}^*$, then*

$$|\bar{\Delta}_W| = |\Delta_{W(L)}| + |\langle r_1, \ldots, r_l \rangle| / |\langle r_2, \ldots, r_l \rangle|.$$

We now turn our attention to the remaining 2-tiered Lie algebras, $BC_{l,2}$ and $A_{1,2}$. In both these cases, α_0 is the unique fundamental root of maximum weight and this weight is 4.

LEMMA 11. *In the cases $BC_{l,2}$ and $A_{1,2}$, we have:*

$$\bar{\Gamma} = \overline{\Delta_{W(L)} + \xi} \cup \overline{\alpha_0 \langle r_1, \ldots, r_l \rangle} \quad (\textit{disjoint union}).$$

Proof. It is clear that $\overline{\Delta_{W(L)} + \xi} \cup \overline{\alpha_0 \langle r_1, \ldots, r_l \rangle} \subseteq \bar{\Gamma}$. The union is disjoint since the elements of $\Delta_{W(L)} + \xi$ are of weight less than 4 whereas those of $\alpha_0 \langle r_1, \ldots, r_l \rangle$ have weight equal to 4.

Let $\bar{\beta} \in \bar{\Gamma}$ and choose β so that β defines the equivalence class $\bar{\beta}$ and $\beta \in \Gamma$. If β has weight less than 4, then $\beta \in \{\alpha_1, \ldots, \alpha_l\} W$, and since $\alpha_i + k\xi$ is a root for all $k \in \mathbf{Z}$ when $i \in \mathbf{L}$, $\beta - \xi$ is a root. $\beta - \xi = \sum_{i \in \mathbf{L}} \lambda_i \alpha_i$ so that, as in Lemma 8, $\beta - \xi \in \Delta_{W(L)}$. Thus $\beta \in \overline{\Delta_{W(L)} + \xi}$. If β has weight 4, which is the only other possibility, $\beta \in \alpha_0 W$. We must still show that $\beta \in \alpha_0 \langle r_1, \ldots, r_l \rangle$. If β has height 1, this is obvious. If β is of height greater than 1, there is an $i \in \mathbf{L}^*$ such that $0 < \mathrm{ht}(\beta r_i) < \mathrm{ht}(\beta)$. If $i = 0$, we obtain a contradiction since $\beta r_0 \in Z\alpha_1 + \ldots + Z\alpha_l$ and this means that $\beta r_0 \in \langle \alpha_1, \ldots, \alpha_l \rangle W$, which in turn means that βr_0 has weight less than 4. Thus $0 < \mathrm{ht}(\beta r_i) < \mathrm{ht}(\beta)$ for some $i \in \mathbf{L}$ and we complete our proof by induction.

The proof given in Lemma 10 can be used without change to show that $|\alpha_0 \langle r_1, \ldots, r_l \rangle| = |\langle r_1, \ldots, r_l \rangle| / |\langle r_2, \ldots, r_l \rangle|$.

PROPOSITION 8. *If E is of type $BC_{l,2}$ or $A_{1,2}$, then*

$$|\bar{\Delta}_W| = 2|\Delta_{W(L)}| + |\langle r_1, \ldots, r_l \rangle| / |\langle r_2, \ldots, r_l \rangle|.$$

We are now ready for a case by case discussion of the 2-tiered Lie algebras.

(1) $C_{l,2}(\mu)$. Consider $\Pi' = \{\alpha_1, \ldots, \alpha_{l-1}\}$. It is an easy matter to check that $\alpha_1 + \ldots + \alpha_{l-1} + \xi + \alpha_i$ is not a root for any $i = 1, \ldots, l-1$. For example, if $\alpha_1 + 2\alpha_2 + \ldots + \alpha_{l-1} + \xi$ is a root, we find, by applying $r_l r_{l-1} \ldots r_3 r_0$, that

$$\alpha_0 + \alpha_1 + 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{l-1} + \alpha_l + \xi = \alpha_2 - \alpha_1 + 2\xi$$

is a root. This means that $\alpha_2 - \alpha_1$ is a root, which is false.

By the corollary of Lemma 7, $[e_1{}^* f_1], \ldots, [e_{l-1}{}^* f_{l-1}]$ are linearly independent, where $e_i{}^*$ is a non-zero element of $E_{\alpha_i + \xi}$, $i = 1, \ldots, l-1$. Hence $\dim E_\xi \geqq l - 1$. On the other hand, since $\alpha_i + \xi$ is a root if and only if

$i \in \{0, 1, \ldots, l-1\}$, we see that $[e_0{}^*f_0]$, $[e_1{}^*f_1]$, $\ldots$, $[e_{l-1}{}^*f_{l-1}]$ span E_ξ, so that $\dim E_\xi \leqq l$. We now show that $[e_0{}^*f_0]$ and $[e_2{}^*f_2]$ are linearly dependent. Since $\alpha_0 + \alpha_2 + \xi = \alpha_l r_{l-1} \ldots r_3 r_1 r_2 r_0$, it is a root. Let b be a non-zero element of $E_{\alpha_0+\alpha_2+\xi}$. $2\alpha_0 + \alpha_2 + \xi$ and $\alpha_0 + 2\alpha_2 + \xi$ are of weight 5 and thus are not roots. This means that $[be_0] = 0 = [be_2]$. By Lemma 2, $[bf_0] \neq 0$ and $[bf_2] \neq 0$. Put $e_0{}^* = [bf_2]$ and $e_2{}^* = [bf_0]$. We see immediately that $[e_2{}^*f_2] = [e_0{}^*f_0]$ which is what we wanted.

Using Proposition 7, we have:

$$\begin{aligned} \dim C_{l,2}(\mu) &= \dim C_l + |W(C_l)|/|W(A_1) \times W(C_{l-2})| + (l-1) \\ &= (2l-1)(2l+1). \end{aligned}$$

The only split simple Lie algebra of rank $2l-1$ and dimension $(2l-1)(2l+1)$ is A_{2l-1}.

(2) $B_{l,2}(\mu)$. In this case, $\alpha_i + \xi$ is a root if and only if $i = 0$ or l. Consequently, $\dim E_\xi \leqq 2$. Amazing as it may seem, $\dim E_\xi = 1$, the argument being essentially the same as in (1). The dimension of $B_{l,2}(\mu)$ may then be computed and it is $2l^2 + 3l + 1$. Thus $B_{l,2}(\mu)$ is of type D_{l+1}.

(3) $BC_{l,2}(\mu)$. Put $\Pi' = \{\alpha_1, \ldots, \alpha_{l-1}\}$. Apply the corollary of Lemma 7 to conclude that $\dim E_\xi \geqq l-1$. On the other hand $\dim E_\xi \leqq l$. From $\dim BC_{l,2}(\mu) = \dim H + \dim E_\xi + 2(\dim B_l - l) + |W(B_l)|/|W(B_{l-1})|$ we obtain:

$$\dim BC_{l,2}(\mu) = \begin{cases} 4l^2 + 4l - 1 & \text{with rank } 2l-1 \text{ or} \\ 4l^2 + 4l & \text{with rank } 2l. \end{cases}$$

It is not hard to see that $2l-1$ can only divide $4l^2 + 4l - 1$ if $l = 1$ and hence the only possibility is the second. Thus $BC_{l,2}(\mu)$ is of type A_{2l}.

(4) $A_{1,2}(\mu)$. $A_{1,2}(\mu)$ is of type A_2.

(5) $F_{4,2}(\mu)$. One checks that $F_{4,2}(\mu)$ is of rank 6 and dimension 78. Unfortunately, B_6, C_6, and E_6 all have dimension 78. To resolve this difficulty one has to take a little less superficial approach to the structure of the 2-tiered algebras. We hope soon to publish some results in this direction which show among other things that $F_{4,2}$ is of type E_6.†††

THEOREM 4. *Let E be the Euclidean Lie algebra whose diagram is $X_{l,2}$. For $\mu \in \Phi - \{0\}$, $X_{l,2}(\mu)$ is finite-dimensional central simple of type given by:*

$X_{l,2}$	$C_{l,2}$	$B_{l,2}$	$BC_{l,2}$	$A_{1,2}$
Type	A_{2l-1}	D_{l+1}	A_{2l}	A_2

†††*Added in proof.* We have recently shown (Robert V. Moody, *Simple quotients of Euclidean Lie algebras*, Can. J. Math., to appear) that if $X_{l,2}$ is a 2-tiered algebra, then the shift map can be chosen so that $X_{l,2}(\mu)$ splits over $\Phi(\sqrt{\mu})$ for all $\mu \in \Phi - \{0\}$, and $X_{l,2}(\mu) \simeq X_{l,2}(\nu)$ if and only if $\mu\nu^{-1}$ is a square. The type of the split algebra $\Phi(\sqrt{\mu}) \otimes X_{l,2}(\mu)$ is given by the table of Theorem 4 for $X_{l,2} \neq F_{4,2}$, while the type of $\Phi(\sqrt{\mu}) \otimes F_{4,2}(\mu)$ is E_6.

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