BULLETIN 13 (1), 1970

ON MEASURES DETERMINED BY CONTINUOUS FUNCTIONS THAT ARE NOT OF BOUNDED VARIATION

BY
J. H. W. BURRY AND H. W. ELLIS

1. **Introduction.** In [1] it was shown that a continuous function of bounded variation on the real line determined a Method II outer measure for which the Borel sets were measurable and the measure of an open interval was equal to the total variation of f over the interval. The monotone property of measures implied that if an open interval I on which f was not of bounded variation contained subintervals on which f was of finite but arbitrarily large total variation then the measure of I was infinite. Since there are continuous functions that are not of bounded variation over any interval (e.g. the Weierstrasse nondifferentiable function) the general case was not resolved.

In this note we prove the existence, for an arbitrary finite interval (a, b), of a continuous function f that is not of bounded variation over (a, b) but is such that $\mu_f(a, b)=0$ for the corresponding measure μ_f.

We shall call a collection $\{(a_i, b_i): i=1, 2, \ldots, n\}$ of intervals covering (a, b) an f-null $C(d)$ covering if each interval has length less than d and if $f(b_i)=f(a_i)$; $i=1, 2, \ldots, n$. A constant function f has an f-null $C(d)$ covering for every $d>0$. A strictly increasing function has no such covering for any d. Any function that is continuous and not constant on (a, b) and has f-null $C(d)$ coverings for every $d>0$ will serve as the function in the preceding paragraph. We prove below that such functions exist.

2. **f-Null $C(d)$ coverings.** As in [1] (with the added assumption that f is a continuous function), we define $l(a, b)=|f(b)-f(a)|$ on open intervals, use as covering classes the collections $C(d)$ of open intervals of length less than d and define

$$\mu^*_{f,d}(A) = \inf\{\textstyle\sum l(a_i, b_i); (a_i, b_i) \in C(d), \cup(a_i, b_i) \supset A\}$$

on $\mathscr{P}(\mathbf{R})$, the subsets of $\mathbf{R}$. Then $\mu^*_{f,d}$ is a Method I outer measure in the sense of Munroe [2]. A Method II outer measure was then obtained by setting

$$\mu^*_f(A) = \lim_{d\to 0} \mu^*_{f,d}(A), \quad A \in \mathscr{P}(\mathbf{R}).$$

The Borel sets are Caratheodory measurable for μ^*_f.

Let $g: x \to \sin kx, k>0: b-a>2\pi/k$. Then there exist g-null $C(d)$ coverings of (a, b) if $d>2\pi/k$ but no g-null $C(d)$ coverings of $(0, 5\pi/2k)$ if $d<2\pi/k$.

Received by the editors June 20, 1969.

https://doi.org/10.4153/CMB-1970-025-0 Published online by Cambridge University Press

We make the following observations

(1.1) *If there are g-null $C(d)$ coverings of (a, b) for every $d>0$, then $\mu_g(a, b)=0$.*

(1.2) *If there are g-null $C(d)$ coverings of (a, b) for every $d>0$ then for every a', b', $a\le a'<b'\le b$, either g is constant or g is not of bounded variation on (a', b').*

To prove 1.2 we note that if there is a subinterval (a', b') on which g is of finite positive total variation, then $\mu_g(a, b)\ge\mu_g(a', b')=$ total variation of g over (a', b') >0, contradicting 1.1.

(1.3) LEMMA. *Let h be a continuous function with $|h'(x)|<K<\infty$ on the finite interval (a, b). Suppose that h is monotone on each of the intervals (x_i, x_{i+1}), $a=x_0<x_1<\cdots<x_n=b$. Define*

$$(1.4)\quad f(x) = h(x)+\beta\sum_{i=0}^{n-1}\sin 2\pi k_i\,(x-x_i)/(x_{i+1}-x_i)\,\chi[x_i, x_{i+1}],\quad k_i\in N,\ \beta>0.$$

Then if $k_i>(x_{i+1}-x_i)K/\beta$ (or in particular if each $k_i>(b-a)K/\beta$), there exists an f-null $C(2\beta/K)$ covering of (a, b).

Proof. We first consider the special case where h is nondecreasing on (a, b) and let

$$f(x) = h(x)+\beta\sin 2\pi k(x-a)/(b-a),\quad k\in N.$$

Set

$$M_i = a+\frac{b-a}{4k}(4i+1),\qquad m_i = a+\frac{b-a}{4k}(4i+3);\quad i = 0, 1, \ldots, k-1.$$

Since h is nondecreasing

$$\text{(i)}\qquad \begin{aligned} f(M_i) &= h(M_i)+\beta \le h(M_{i+1})+\beta = f(M_{i+1}),\\ f(m_i) &\le f(m_{i+1}); \end{aligned}$$

$$\text{(ii)}\qquad f(m_i) = h(m_i)-\beta < h(M_{i+1}) < f(M_{i+1}),\quad i = 0, 1, \ldots, k-1.$$

$$\begin{aligned} f(m_i)-f(M_i) = h(m_i)-h(M_i)-2\beta &= \int_{M_i}^{m_i} h'(t)\,dt-2\beta\\ &< K(m_i-M_i)-2\beta\\ &= \frac{K(b-a)}{2k}-2\beta \end{aligned}$$

Thus

$$\text{(iii)}\qquad f(m_i) < f(M_i)\quad\text{if } k > (b-a)K/4\beta.$$

By a similar argument

$$\text{(iv)}\qquad f(a) > f(m_0),\quad f(b) < f(M_{k-1})\quad\text{if } k > 3(b-a)K/4\beta.$$

Now

$$\begin{aligned} m_{i+1}-M_i &= 3(b-a)/2k,\\ f(m_{i+1})-f(M_i) &= h(m_{i+1})-h(M_i)-2\beta \le 3K(b-a)/2k-2\beta, \end{aligned}$$

$$\text{(v)}\qquad f(m_{i+1}) < f(M_i)\quad\text{if } k > 3K(b-a)/4\beta.$$

https://doi.org/10.4153/CMB-1970-025-0 Published online by Cambridge University Press

We now assume that $k > K(b-a)/\beta$ so that (i)–(v) hold. If $i<k-1$, $f(m_i)<f(M_i) \leq f(M_{i+1})$ by (i) and (ii). The intermediate value theorem for continuous functions then gives a point ξ_i, $m_i<\xi_i<M_{i+1}$ with $f(M_i)=f(\xi_i)$,

$$\xi_i - M_i < M_{i+1} - M_i = (b-a)/k < \beta/K.$$

Using (v), $f(m_i) \leq f(m_{i+1}) < f(M_i) = f(\xi_i)$ and there is a point η_{i+1}, $m_i<\eta_{i+1}<\xi_i$ with $f(m_{i+1})=f(\eta_{i+1})$, $\eta_{i+1}-m_{i+1}<\beta/K$, $i=0, 1, 2, \ldots, k-2$.

Using (iv), $f(m_0)<f(a)<f(M_0)$ and there is a point ξ', $M_0<\xi'<m_0$ with $f(\xi') = f(a)$, $\xi'-a<m_0-a=3(b-a)/4K<3\beta/4K$ and in a similar way, a point η', $M_{k-1} < \eta' < m_{k-1}$, with $f(\eta')=f(b)$, $b-\eta'<3\beta/K$. The intervals (a, ξ'), (η', b); (M_i, ξ_i), $i=0, 1, 2, \ldots, k-2$; (η_i, m_i), $i=1, 2 \ldots, k-1$ form an f-null $C(\beta/K)$ covering of (a, b).

The construction is similar if h is nonincreasing on (a, b). In the general case, assuming each $k_i > (b-a)K/\beta$, we obtain coverings of the open intervals (x_i, x_{i+1}) by suitable intervals in $C(\beta/K)$ as above. At each x_i, $i \neq 0, n$, there is an interval (x_i, ξ_i') in the covering of (x_i, x_{i+1}) and an interval (η_i, x_i) in the covering of (x_{i-1}, x_i) with

$$f(\xi_i') = f(x_i) = f(\eta_i'), \qquad x_i - \eta_i' < \beta/K, \quad \xi_i' - x_i < \beta/K.$$

We replace each such pair by the single interval (η_i', ξ_i'), $\xi_i'-\eta_i'<2\beta/K$ and obtain the required f-null $C(2\beta/K)$ covering of (a, b).

3. **A continuous function with f-null $C(d)$ coverings for every $d>0$ on a finite open interval.** We begin with an arbitrary function h defined and continuous on the finite interval $[a, b]$ and satisfying the hypotheses of the lemma on (a, b). We assume that $\max(|h'(x)|)>1$ so that $K>1$.

We define f_1 by (1.4) with $\beta=\frac{1}{2}$ and each $k_i>2K(b-a)$. By the lemma there exists an f_1-null $C(2\beta/K)$ covering of (a, b) and so a $C(1)$ covering.

Fixing an f_1-null $C(1)$ covering by the construction in the proof of the lemma we let $\{x_i\}$, $a=x_0<x_1<\cdots<x_n=b$ consist of the points of (a, b) at which relative maxima and minima of f_1 occur, together with any additional end points of the $C(1)$ covering. Then f_1 satisfies the hypotheses of the lemma and a simple computation shows that $\max|(f_1'(x))|>1$. Defining f_2 by (1.4) with f_1 replacing h and $\beta=1/2^2$ and taking $k_i>4K(b-a)$, $i=0, 1, \ldots, n-1$ there exists an f_2-null $C(\frac{1}{2})$ covering of (a, b).

Proceeding by induction, having defined f_i, and an f_i-null $C(2^{1-i})$ covering, $i \leq k$, f_k satisfies the hypotheses of the lemma with the points x_i consisting of the relative maxima and minima of f_k together with the end points of all the covering intervals for $i \leq k$ and with $\max|f_k'(x)|>1$. As in the preceding paragraph we use the lemma to obtain f_{k+1} and a corresponding f_{k+1}-null $C(2^{-k})$ covering of (a, b). Note in particular that $f_{k+1}(x)=f_i(x)$ at all of the end points of the f_i-null $C(2^{1-i})$ coverings.

The sequence $\{f_i\}$ converges uniformly to a continuous function f on $[a, b]$. For each $i, f(x)=f_i(x)$ at each end point of the f_i-null $C(2^{1-i})$ covering intervals of (a, b) for f_i so that this covering is also an f-null $C(2^{1-i})$ covering of (a, b). Since an f-null $C(d)$ covering is a $C(d')$ covering if $d'>d$, there exist f-null $C(d)$ coverings of (a, b) for every $d>0$. By (1.1), $\mu_f(a, b)=0$. The function f is clearly not constant on any subinterval of (a, b), and so is not of bounded variation over any subinterval of (a, b).

References

1. H. W. Ellis and R. L. Jeffery, *On measures determined by functions with finite right and left limits everywhere*. Canad. Math. Bull. (2) **10** (1967), 207–225.

2. M. E. Munroe, *Introduction to measure and integration*, Addison-Wesley, Cambridge, Mass. 1953.

Queen's University,
Kingston, Ontario