43

Models of the QCD effective action

43.1 Introduction

Our purpose is to briefly present the general features of different models of the low-energy hadronic interactions based on the effective action of QCD using a well-defined set of approximations. In this chapter, we shall follow closely the discussions in [500]. The chiral symmetry of the underlying QCD theory implies that the generating functional $\Gamma(v, a, s, p)$ of the Green's functions of quark currents:

$$e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int \mathcal{D}G_\mu \, \det \not{D} \, \exp\left(-i \int d^4x \, \frac{1}{4} \vec{G}_{\mu\nu} \vec{G}^{\mu\nu}\right) , \tag{43.1}$$

with: $\not{D}$ the Dirac operator

$$\not{D} = \gamma^\mu(\partial_\mu + i g_s G_\mu) - i\gamma^\mu(v_\mu + \gamma_5 a_\mu) + i(s - i\gamma_5 p) ; \tag{43.2}$$

G_μ is the gluon field, $\vec{G}^{\mu\nu}$ the gluon field strength tensor; and v_μ, a_μ, s, p external field sources; the normalization factor Z is such that $\Gamma(0, 0) = 1$, admits a low-energy representation:

$$e^{i\Gamma(v,a,s,p)} = \frac{1}{Z} \int \mathcal{D}U \, \exp\left[i \int d^4x \, \mathcal{L}_{\rm eff}(U; v, a, s, p)\right] , \tag{43.3}$$

in terms of an effective Lagrangian $\mathcal{L}_{\rm eff}(U; v, a, s, p)$ with $U(x)$ a 3×3 unitary matrix containing the octet of pseudoscalar fields (π, K, η). However, the single term in $\mathcal{L}_{\rm eff}$ which is known from first principles, is the one associated with the existence of anomalies in the fermionic determinant [510]. The corresponding effective action is the Wess and Zumino [508,509] functional that we have discussed in the previous section. All possible other terms in $\mathcal{L}_{\rm eff}$, are not fixed by symmetry requirements alone. The desire is to build some effective dynamical QCD models with a mimimum set of parameters that can fix the different coupling constants of the effective chiral Lagrangian, and that are needed for making progress in the phenomenology of non-leptonic flavour dynamics. In the following, we shall list the following models:

- QCD in the large–N_c limit.
- Low-lying resonances dominance models.
- The constituent chiral quark model.

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- Effective action approach models.
- The extended Nambu and Jona-Lasinio Model (ENJL model.)

43.2 QCD in the large–N_c limit

43.2.1 Large N_c counting rules for mesons

The study of QCD in the limit of large N_c was suggested by t'Hooft [520], soon after the discovery of asymptotic freedom, as an attempt to get an insight into the non-perturbative aspects of QCD. The large N_c limit of QCD corresponds to the case where the number of colours is sent to infinity and the QCD coupling α_s sent to zero in such a way that:

$$N_c\alpha_s = \text{constant}\ . \tag{43.4}$$

Therefore the Green's function of the theory is proportional to a power of N_c [520–522]. Denoting by $\mathcal{G}_{qw}$ the general connected Green's function containing q quark currents and w winding number densities:

$$\mathcal{G}_{qw} = \langle 0|\mathcal{T} J_1(x-1)\cdots J_q(x_q)Q(y_1)\cdots Q(y_w)|0\rangle_{\text{connect}} \tag{43.5}$$

with:

$$J_i = \bar\psi\Gamma_i\psi\ , \qquad Q(x) = \frac{g^2}{8\pi^2}\mathbf{Tr}\,(G_{\mu\nu}\tilde G^{\mu\nu})\ , \tag{43.6}$$

where Γ_i is neutral colour matrices acting on the spin and quark flavours. For large N_c, the Green's function behaves as:

$$\begin{aligned}\mathcal{G}_{qw} &= \mathcal{O}\big(N_c^{2-w}\big)\ , \qquad q=0\\ &= \mathcal{O}\big(N_c^{1-w}\big)\ , \qquad q\neq 0\ .\end{aligned} \tag{43.7}$$

This counting rule holds only for generic momenta, but is modified by, for example, the exchange of an η' pole, which at zero momentum produces an additionnal power of N_c ($M^2_{\eta'} \sim 1/N_c$ in the chiral limit). This counting rule can be understood in the following way: the leading contributions to the Green's functions containing quark currents ($q\neq 0$) arise from graphs with a single quark loop (planar diagrams with the quark loop running at the edge of the diagram). These graphs are given by the functional integral over the gluon field of the product of the form $\mathbf{Tr}\left(\Gamma_{i_1}S\Gamma_{i_2}S\ldots\Gamma_{i_q}S\right)$, where $i_1,\ldots i_q$ is some permutation of $1,\ldots,q$ and where S denotes the quark propagator in the presence of the gluon field. In the chiral limit, the propagator is flavour independent, and the leading contribution to the Green's function depends on the flavour of the current only through the trace $\mathbf{Tr}\left(\lambda_{i_1},\ldots,\lambda_{i_q}\right)$ where λ_i is the flavour factor in the matrix Γ_i. From Eq. (43.7), one can deduce the large-N_c behaviour of the generating functional:

$$Z(v,a,s,p,\theta) = N_c^2 f_0(\theta/N_c) + N_c f_1(v,a,s,p,\theta/N_c) + \mathcal{O}(1)\ , \tag{43.8}$$

where the functional $f_0(\alpha)$ and $f_1(v,a,s,p,\alpha)$ are independent of N_c. One can deduce the

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counting rule for one particle matrix elements [522]:

$$\begin{aligned} \langle 0|J|\text{meson}\rangle &= \mathcal{O}\big(N_c^{1/2}\big)\,, &\qquad \langle 0|J|\text{glueball}\rangle &= \mathcal{O}(1)\,, \\ \langle 0|Q|\text{meson}\rangle &= \mathcal{O}\big(N_c^{-1/2}\big)\,, &\qquad \langle 0|Q|\text{glueball}\rangle &= \mathcal{O}(1)\,. \end{aligned} \tag{43.9}$$

Every additional meson in a vertex brings a suppression factor $1/N_c^{-1/2}$. Therefore, three-meson amplitudes are of order $1/N_c^{-1/2}$, four-meson amplitudes are of order $1/N_c, \ldots$. Loop corrections in the meson sector are suppressed by powers of $1/N_c$, and are consistent with a semiclassical expansion in powers of $\hbar$.

43.2.2 Chiral Lagrangian in the large N_c-limit

It would be a major breakthrough, if one could derive the low-energy effective Lagrangian of the interactions between Nambu–Goldstone modes in the large-N_c limit of QCD. To analyse the large N_c behaviour of the effective Lagrangian, it suffices to expand the matrix in terms of the meson fields and to look at the terms independent of these fields. The desired results are obtained by comparing these terms with those in Eq. (43.8). As examples, one obtains:

$$f = \mathcal{O}\big(N_c^{1/2}\big)\,, \qquad B = \gamma = \mathcal{O}(1)\,, \tag{43.10}$$

where γ quantifies the $\eta - \eta'$ mixing. For the non-vanishing coupling constants, one has obtained the large N_c behaviour [499]:

$$\begin{aligned} \mathcal{O}\big(N_c^2\big) &: L_7\,, \\ \mathcal{O}(N_c) &: L_1, L_2, L_3, L_5, L_8, L_9, L_{10}, H_1, H_2\,, \\ \mathcal{O}(1) &: 2L_1 - L_2, L_4, L_6\,. \end{aligned} \tag{43.11}$$

So far, it has only been possible to obtain *constraints* among various coupling constants in this limit; but not their *values* in terms, say, of Λ_{QCD}. A typical example is the relation:

$$2L_1 = L_2\,, \tag{43.12}$$

which, as first noticed by Gasser and Leutwyler [499], follows in the large-N_c limit of QCD. Unfortunately, nobody can claim as yet to be able to *compute*, say L_2, in that limit. Often in the literature, there appear statements about 'large-N_c predictions' but, in fact, they have been all derived with some extra *ad hoc* assumptions.

An interesting approach to do approximate calculations within the framework of the $1/N_c$-expansion is the one proposed by Bardeen, Buras and Gérard [524], which they have applied extensively to the calculation of non-leptonic weak matrix elements. The basic idea is to start with the factorized form of the four-quark operators in the effective weak Hamiltonian, and to do one-loop chiral perturbation theory, keeping track of the quadratic divergences which appear. If one was able to work with the *full* hadronic low-energy effective Lagrangian, it would be possible to obtain a smooth matching between the scale dependence of the Wilson coefficients, calculated at short distances, and the hadronic matrix elements calculated with the *full* hadronic low-energy effective Lagrangian. The hope with the approach proposed by [524] is that the *numerical* matching of the *quadratic* long-distance scale with the *logarithmic* short-distance scale, may turn out to be already a good first approximation to the problem one would like to solve. The technology of their approach is explained with detail in their papers.

43.2.3 Minimal hadronic ansatz to large N_c QCD

The hadronic spectrum predicted by large N_c–QCD seems a priori different from the *real world*, as one expects here the presence of an infinite sum of narrow resonances with specific quantum numbers [520]. This feature can be better understood from the Coleman–Witten theorem [525] which states that if QCD at $N_c = 3$ is confined, and if confinement persists for large N_c, then, in this limit, the chiral $U(n_f) \times U(n_f)$ invariance of the QCD Lagrangian with n_f massless flavours is spontaneously broken down to the diagonal $U(n_f)$ subgroup. Though, the real world has a much more complicated structure, one expects that the hadronic world predicted by large N_c can give an approximate good prediction of this real world, when observables in terms of spectral functions are involved, as in this case, one needs only to know the *global properties of the hadronic spectrum.*

The left–right correlation function

Of particular interest for our purposes is the correlation function ($Q^2 \equiv -q^2 \geq 0$ for q^2 space-like):

$$\Pi^{\mu\nu}_{LR}(q) = 2i \int d^4x \, e^{iq\cdot x} \langle 0|\mathrm{T}(L^\mu(x) R^\nu(0)^\dagger)|0\rangle \,, \tag{43.13}$$

with colour singlet currents:

$$R^\mu \, (L^\mu) = \bar{d}(x)\gamma^\mu \frac{1}{2}(1 \pm \gamma_5)u(x) \,. \tag{43.14}$$

In the chiral limit, ($m_{u,d,s} \to 0$,) this correlation function has only a transverse component

$$\Pi^{\mu\nu}_{LR}(Q^2) = (q^\mu q^\nu - g^{\mu\nu}q^2)\Pi_{LR}(Q^2) \,. \tag{43.15}$$

The self-energy like function $\Pi_{LR}(Q^2)$ vanishes order by order in perturbative QCD (pQCD) and is an order parameter of ChSB for all values of Q^2; therefore it obeys an unsubtracted dispersion relation

$$\Pi_{LR}(Q^2) = \int_0^\infty dt \frac{1}{t+Q^2} \frac{1}{\pi} \mathrm{Im}\Pi_{LR}(t) \,. \tag{43.16}$$

In large N_c–QCD, the spectral function $\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t)$ consists of the difference of an infinite number of narrow vector and axial-vector states, together with the Goldstone pole of the pion:

$$\frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = \sum_V f_V^2 M_V^2 \delta\big(t - M_V^2\big) - F_0^2\delta(t) - \sum_A f_A^2 M_A^2 \delta\big(t - M_A^2\big)\,. \quad (43.17)$$

The low Q^2 behaviour of $\Pi_{LR}(Q^2)$, namely the long-distance behaviour of the correlation function in Eq. (43.13), is governed by chiral perturbation theory:

$$-Q^2\Pi_{LR}(Q^2)|_{Q^2\to 0} = f_0^2 + 4L_{10}Q^2 + \mathcal{O}(Q^4)\,, \quad (43.18)$$

where f_0 is the pion coupling constant in the chiral limit, and L_{10} is one of the coupling constants of the $\mathcal{O}(p^4)$ effective chiral Lagrangian. The high Q^2 behaviour of $\Pi_{LR}(Q^2)$, that is, the short-distance behaviour of the correlation function in Eq. (43.13), is governed by the operator product expansion (OPE) of the two local currents in Eq. (43.13) [1],

$$\lim_{Q^2\to\infty} Q^6\Pi_{LR}(Q^2) = \left[-4\pi^2\frac{\alpha_s}{\pi} + \mathcal{O}\big(\alpha_s^2\big)\right]\langle\bar\psi\psi\rangle^2\,, \quad (43.19)$$

which implies the two Weinberg sum rules:

$$\int_0^\infty dt\,\mathrm{Im}\Pi_{LR}(t) = \sum_V f_V^2 M_V^2 - \sum_A f_A^2 M_A^2 - F_0^2 = 0\,, \quad (43.20)$$

and:

$$\int_0^\infty dt\,t\,\mathrm{Im}\Pi_{LR}(t) = \sum_V f_V^2 M_V^4 - \sum_A f_A^2 M_A^4 = 0\,. \quad (43.21)$$

In fact, as pointed out in [526], in large N_c QCD, there exist an infinite number of Weinberg-like sum rules. In full generality, the moments of the Π_{LR} spectral function with $n = 3, 4, \ldots,$

$$\int_0^\infty dt\,t^{n-1}\left[\frac{1}{\pi}\mathrm{Im}\Pi_V(t) - \frac{1}{\pi}\mathrm{Im}\Pi_A(t)\right] = \sum_V f_V^2 M_V^{2n} - \sum_A f_A^2 M_A^{2n}\,, \quad (43.22)$$

govern the short-distance expansion of the $\Pi_{LR}(Q^2)$ function;

$$\Pi_{LR}(Q^2)|_{Q^2\rangle\infty} = \left(\sum_V f_V^2 M_V^6 - \sum_A f_A^2 M_A^6\right)\frac{1}{Q^6} + \left(\sum_V f_V^2 M_V^8 - \sum_A f_A^2 M_A^8\right)\frac{1}{Q^8} + \cdots\,. \quad (43.23)$$

On the other hand, inverse moments of the Π_{LR} spectral function, with the pion pole removed, (which we denote by $\mathrm{Im}\tilde\Pi_A(t)$,) determine a class of coupling constants of the low-energy effective chiral Lagrangian.

For example:

$$\int_0^\infty dt\frac{1}{t}\left[\frac{1}{\pi}\mathrm{Im}\Pi_V(t) - \frac{1}{\pi}\mathrm{Im}\tilde\Pi_A(t)\right] = \sum_V f_V^2 - \sum_A f_A^2 = -4L_{10}\,. \quad (43.24)$$

Moments with higher inverse powers of t are associated with couplings of composite operators of higher dimension in the chiral Lagrangian. Tests of the two Weinberg sum rules in Eqs. (43.20) and (43.21) and of the L_{10} sum rule in Eq. (43.24), albeit in a different context to the one we are interested in here, have also been discussed in the literature, (see e.g. [527,528], [33,34]).

The minimal ansatz

We shall now consider the approximation which we call the *minimal hadronic ansatz* to large N_c–QCD. In the case of the left–right two-point function in Eq. (43.13), this is the approximation where the hadronic spectrum consists of one vector state V, one axial-vector state A and the Goldstone pion, with the ordering [526] $M_V < M_A$. This is the *minimal spectrum* which is required to satisfy the two Weinberg sum rules in Eqs. (43.20) and (43.21.) In this approximation, $\Pi_{LR}(Q^2)$ has a very simple form:

$$\begin{aligned} -Q^2\Pi_{LR}(Q^2) &= \frac{f_0^2}{\left(1+\frac{Q^2}{M_V^2}\right)\left(1+\frac{Q^2}{M_A^2}\right)} \\ &= \frac{M_A^2 M_V^2}{Q^4}\frac{f_0^2}{\left(1+\frac{M_V^2}{Q^2}\right)\left(1+\frac{M_A^2}{Q^2}\right)}\,. \end{aligned} \tag{43.25}$$

This equation shows, explicitly, a remarkable short-distance $\rightleftharpoons$ long-distance duality [529]. Indeed, with g_A defined as:

$$M_V^2 = g_A M_A^2 \quad \text{and} \quad z \equiv \frac{Q^2}{M_V^2}\,, \tag{43.26}$$

the non-local order parameters corresponding to the long-distance expansion for $z\rangle 0$, which are couplings of the effective chiral Lagrangian i.e.:

$$-Q^2\Pi_{LR}(Q^2)|_{z\rangle 0} = f_0^2\left\{1-(1+g_A)z+\left(1+g_A+g_A^2\right)z^2+\cdots\right\}, \tag{43.27}$$

are correlated to the local-order parameters of the short-distance OPE for $z\rangle\infty$ in a very simple way:

$$-Q^2\Pi_{LR}(Q^2)|_{z\rangle\infty} = f_0^2\frac{1}{g_A}\frac{1}{z^2}\left\{1-\left(1+\frac{1}{g_A}\right)\frac{1}{z}+\left(1+\frac{1}{g_A}+\frac{1}{g_A^2}\right)\frac{1}{z^2}+\cdots\right\}; \tag{43.28}$$

in other words, there is a one-to-one correspondence between the two expansions by changing

$$g_A \rightleftharpoons \frac{1}{g_A} \quad \text{and} \quad z^n \rightleftharpoons \frac{1}{g_A}\frac{1}{z^{n+2}}\,. \tag{43.29}$$

The moments of the Π_{LR} spectral function, when evaluated in the *minimal hadronic ansatz* approximation, can be converted into a very simple set of finite energy sum rules

(FESR's), corresponding to the OPE in Eq. (43.28):

$$\int_0^{s_0} dt\, t^2 \frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -f_0^2 M_V^4 \frac{1}{g_A}\,, \tag{43.30}$$

$$\int_0^{s_0} dt\, t^3 \frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -f_0^2 M_V^6 \frac{1+\frac{1}{g_A}}{g_A}\,, \tag{43.31}$$

$$\int_0^{s_0} dt\, t^4 \frac{1}{\pi}\mathrm{Im}\Pi_{LR}(t) = -f_0^2 M_V^8 \frac{1+\frac{1}{g_A}+\frac{1}{g_A^2}}{g_A}\,, \tag{43.32}$$

$$\cdots \qquad\qquad \cdots\,.$$

where the upper limit of integration s_0 denotes the onset of the pQCD continuum which, in the chiral limit, is common to the vector and axial-vector spectral functions. It is important to realize that s_0 is not a free parameter. Its value is fixed by the requirement that the OPE of the correlation function of two vector currents, (or two axial-vector currents,) in the chiral limit, have no $1/Q^2$ term, which results in an implicit equation for s_0 [405,530]. In the *minimal hadronic ansatz* approximation the onset of the pQCD continuum, which we shall call s_0^*, is then fixed by the equation

$$\frac{N_c}{16\pi^2}\frac{2}{3}s_0^*\,(1+\mathcal{O}(\alpha_s)) = f_0^2\frac{1}{1-g_A}\,. \tag{43.33}$$

Also, the moments which correspond to the chiral expansion in Eq. (43.27) are given by another simple set of FESR's:

$$\int_0^{s_0} dt\, \frac{1}{\pi}\mathrm{Im}\tilde{\Pi}_{LR}(t) \;=\; f_0^2\,, \tag{43.34}$$

$$\int_0^{s_0} \frac{dt}{t}\, \frac{1}{\pi}\mathrm{Im}\tilde{\Pi}_{LR}(t) \;=\; \frac{f_0^2}{M_V^2}(1+g_A)\,, \tag{43.35}$$

$$\int_0^{s_0} \frac{dt}{t^2}\, \frac{1}{\pi}\mathrm{Im}\tilde{\Pi}_{LR}(t) \;=\; \frac{f_0^2}{M_V^4}\left(1+g_A+g_A^2\right)\,, \tag{43.36}$$

$$\cdots \qquad\qquad \cdots\,.$$

These duality relations have been tested by comparing moments of the *physical spectral function* $1/\pi\ \mathrm{Im}\Pi_{LR}^{\mathrm{exp}}(t)$ determined from experiment (tau-decay data) to the predictions of the *minimal hadronic ansatz* as shown in the RHS of Eqs. (43.30) to (43.32) and Eqs. (43.34) to (43.36), where one finds that the tau-decay data is consistent with the simple pattern of duality properties between short and long distances which follow from the minimal hadronic ansatz of a narrow resonances in the large N_c limit of QCD.

43.2.4 Baryons in the large N_c limit

Many features of the baryon sector have been also understood using the $1/N_c$ expansion, where a new $SU(4)$ symmetry connects the $u\uparrow, u\downarrow, d\uparrow, u\downarrow$ states in the baryon (see

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e.g. the review of Manohar in [502]). A systematic computation of the $1/N_c$ corrections then becomes possible, and some results obtained previously from the quark and Skyrme models [3] can be proved to order $1/N_c$ or $1/N_c^2$. However, in the large N_c limit, baryons are more difficult to study than mesons, as the number of quarks in the baryon is N_c. Large N_c counting rules for baryons were given by Witten [522]. In particular, if one assumes that the baryon mass and axial coupling g_A are of order N_c, one can deduce using a non-relativistic quark model:

$$g_A = \frac{N_c + 2}{3} , \tag{43.37}$$

which is equal to the well-known quark model prediction $5/3$ for $N_c = 3$. Phenomenology of multicolour QCD in the baryon sector using QCD spectral sum rules has been studied in [531] for N_c flavour, and in [532] for two flavours. In the latter case (see the details of the derivation in [532,3]), the Skyrme parameter has been obtained to be:

$$e \simeq 9/N_c^{1/2} , \tag{43.38}$$

in agreement with large N_c-expectations.

43.3 Lowest meson dominance models

There has been quite a lot of progress during the last few years in understanding the rôle of resonances in ChPT. At the phenomenological level [533,534], it turns out that the observed values of the L_i-constants are practically saturated by the contribution from the lowest resonance exchanges between the pseudoscalar particles; and particularly by vector-exchange, whenever vector mesons can contribute. The specific form of an effective chiral invariant Lagrangian describing the couplings of vector and axial-vector particles to the (pseudo) Nambu–Goldstone modes is not uniquely fixed by chiral symmetry requirements alone. When the vector fields describing heavy vector particles are integrated out, different field theory descriptions may lead to different predictions for the L_i-couplings. It has been shown however that, if a few QCD short-distance constraints are imposed, the ambiguities of different formulations are then removed [535]. The most compact effective Lagrangian formulation, compatible with the short-distance constraints, has two free parameters: f_π and M_V. When the vector and axial-vector fields are integrated out, it leads to specific predictions for *five* of the L_i constants:

$$L_1^{(V)} = L_2^{(V)}/2 = -L_3^{(V)}/6 = L_9^{(V)}/8 = -L_{10}^{(V+A)}/6 = \frac{f_\pi^2}{16 M_V^2} \simeq 0.6 \times 10^{-3} , \tag{43.39}$$

in good agreement, within errors, with experiment. [See Table 42.1]

It is fair to conclude that the old phenomenological concept of *vector meson dominance* (VMD) [14] can now be formulated in a way that is compatible with the chiral symmetry properties and the short-distance behaviour of QCD.

43.4 The constituent chiral quark model

This model was introduced by Georgi and Manohar [512], in an attempt to reconcile the successful features of the constituent quark model [81], with the chiral symmetry requirements of QCD. The basic assumption of the model is the idea that between the scale of chiral symmetry breaking Λ_χ and the confinement scale $\sim \Lambda_{\mathrm{QCD}}$ the underlying QCD theory, may admit a useful effective Lagrangian realization in terms of *constituent quark fields* Q; *pseudoscalar particles*; and, perhaps, *'gluons'*. The Lagrangian in question has the form:

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}}^{\mathrm{GM}} &= i\bar{Q}\gamma_\mu(\partial_\mu + ig_s G_\mu + \Gamma_\mu)Q \\ &+ \frac{i}{2} g_A \bar{Q}\gamma_5\gamma^\mu \xi_\mu Q - M_Q \bar{Q}Q \\ &+ \frac{1}{4} f_\pi^2\, tr\, D_\mu U D^\mu U^\dagger - \frac{1}{4}\, \vec{G}_{\mu\nu}\vec{G}^{\mu\nu}\ . \end{aligned} \tag{43.40}$$

Some explanations about the notation here are in order. Remember that under chiral rotations (V_L, V_R), U transforms like: $U \to V_R U V_L$. The unitary matrix U is the product of the so-called left and right coset representatives: $U = \xi_R \xi_L^\dagger$ and, without lost of generality, one can always choose the gauge where $\xi_L^\dagger = \xi_R \equiv \xi$. The coset representative ξ, $(U = \xi\xi^\dagger,)$ transforms as:

$$\xi \to V_R \xi h^\dagger = h \xi V_L^\dagger \qquad h \in SU(3)_V\ , \tag{43.41}$$

where h denotes the rotation induced by the chiral transformation (V_L, V_R) in the diagonal $SU(3)_V$. In Eq. (43.40) the constituent quark fields Q transform as:

$$Q \to hQ, \qquad h \in SU(3)_V\ . \tag{43.42}$$

In the presence of external sources:[1]

$$\Gamma_\mu = \frac{1}{2}\{\xi^\dagger[\partial_\mu - i(v_\mu + a_\mu)]\xi + \xi[\partial_\mu - i(v_\mu - a_\mu)]\xi^\dagger\} \tag{43.43}$$

and:

$$\xi_\mu = i\xi^\dagger D_\mu U \xi^\dagger\ . \tag{43.44}$$

The free parameters of the theory are f_π, M_Q, and g_A. The QCD coupling constant is assumed to have entered a regime (below Λ_χ,) where its running is frozen and is taken to be constant.

The merit of this model is that it automatically incorporates the phenomenological successes of the constituent quark model, in a way compatible with chiral symmetry. This model indeed appears in practically all QCD low-energy models where quarks are not confined. The weak point of the model is its 'vagueness' about the gluonic sector. In the absence of a dynamic justification for the 'freezing' of the QCD running coupling constant, it is very unclear what the 'left out' gluonic interactions mean; and in fact, in most applications, they are simply ignored.

[1] The original formulation of the model of Georgi and Manohar [512] was in fact made without external fields.

43.5 Effective action approach models

The basic idea in this class of models is to make some kind of drastic approximation to compute the non–anomalous part of the QCD-fermionic determinant in the presence of external v_μ and a_μ fields, but with the external s and p fields frozen to the quark matrix:

$$s + ip = \mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)\,. \tag{43.45}$$

Although the integral over the quark fields in Eq. (42.16) can be done explicitly, we do not know how to perform analytically the remaining integration over the gluon fields. A perturbative evaluation of the gluonic contribution would obviously fail in reproducing the correct dynamics of Spontaneous Chiral Symmetry Breaking (SCSB). A possible way out is to parametrize phenomenologically the SCSB and make a weak gluon-field expansion around the resulting physical vacuum. The simplest parametrization [413] is obtained by adding to the QCD Lagrangian the chiral invariant term:

$$\Delta\mathcal{L}_{\mathrm{QCD}} = -M_Q(\bar{q}_R U q_L + \bar{q}_L U^\dagger q_R)\,, \tag{43.46}$$

which serves to introduce the U field, and a mass parameter M_Q, which regulates the IR behaviour of the low-energy effective action. In the presence of this term the operator $\bar{q}q$ acquires a vacuum expectation value; therefore, Eq. (43.46) is an effective way to generate the order parameter due to SCSB. Making a chiral rotation of the quark fields, $Q_L \equiv u(\phi)q_L$, $Q_R \equiv u(\phi)^\dagger q_R$, with $U = u^2$, the interaction Eq. (43.46) reduces to a mass-term for the *dressed* quarks Q; the parameter M_Q can then be interpreted as a *constituent-quark mass*.

The derivation of the low-energy effective chiral Lagrangian within this framework has been extensively discussed by [413]. In the chiral and large-N_C limits, and including the leading gluonic contributions, one gets:

$$\begin{aligned} 8L_1 = 4L_2 = L_9 &= \frac{N_C}{48\pi^2}\left[1 + \mathcal{O}\big(1/M_Q^6\big)\right]\,, \\ L_3 = L_{10} &= -\frac{N_C}{96\pi^2}\left[1 + \frac{\pi^2}{5N_C}\frac{\left\langle\frac{\alpha_s}{\pi}GG\right\rangle}{M_Q^4} + \mathcal{O}\big(1/M_Q^6\big)\right]\,, \end{aligned} \tag{43.47}$$

where the positive sign of the corrections helps for a better agreement with experiments. Due to dimensional reasons, the leading contributions to the $\mathcal{O}(p^4)$ couplings only depend on N_C and geometrical factors. It is remarkable that L_1, L_2 and L_9 do not get any gluonic correction at this order; this result is independent of the way SCSB has been parametrized (M_Q can be taken to be infinite). Table 43.1 compares the predictions obtained with only the leading term in Eq. (43.47) (i.e. neglecting the gluonic correction) with the phenomenological determination of the L_i couplings. The numerical agreement is quite impressive; both the order of magnitude and the sign are correctly reproduced (notice that this is just a free-quark result!). Moreover, the gluonic corrections shift the values of L_3 and L_{10} in the right direction, making them more negative.

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Table 43.1. *Leading-order* ($\alpha_s = 0$) *predictions for the* L_i*'s, within the QCD-inspired model in Eq. (43.46). The phenomenological values are shown in the second row for comparison. All numbers are given in units of* 10^{-3}

	L_1	L_2	L_3	L_9	L_{10}
$L_i^{\rm th}(\alpha_s = 0)$	0.79	1.58	−3.17	6.33	−3.17
$L_i^r(M_\rho)$	0.4 ± 0.3	1.4 ± 0.3	-3.5 ± 1.1	6.9 ± 0.7	-5.5 ± 0.7

The results in Eq. (43.47) obey almost all relations in (43.39). In the same way, one also obtains a relation between the quark constituent mass and the pion decay constant [413]:

$$f_\pi^2 = \frac{N_c}{16\pi^2} 4M_Q^2 \left[\log\frac{\Lambda^2}{M_Q^2} + \frac{\pi^2}{6N_c}\frac{< \frac{\alpha_s}{\pi} GG >}{M_Q^4} + \frac{1}{360 N_c}\frac{< g^3 GGG >}{M_Q^6} + \cdots\right]. \tag{43.48}$$

The authors mention that the gluon condensate appearing here has nothing to do with the one from QCD spectral sum rules phenomenology, which is hard to digest as the quantity $\langle \alpha_s G^2 \rangle$ has a very weak scale dependence. This approach has been also extended to the estimate of four-fermion non-leptonic weak operators, which the interested readers can find in [537]. Analogous result has been derived in [538] using a variational mass expansion.

43.6 The Extended Nambu–Jona-Lasinio Model

There have been many suggestions in the literature proposing that Nambu and Jona-Lasinio [539]-like models are relevant models for low-energy hadron dynamics. In e.g. [540,541], one assumes that at intermediate energies below or of the order of the spontaneous chiral symmetry breaking scale Λ_χ, the leading operators of higher dimension which, after integration of the high-frequency modes of the quark and gluon fields down to the scale Λ_χ, become relevant in the QCD Lagrangian, are those which can be cast in the form of four-fermion operators, i.e.:

$$\mathcal{L}_{\rm QCD} \Longrightarrow \mathcal{L}^\chi_{\rm QCD} + \mathcal{L}_{\rm S,P} + \mathcal{L}_{\rm V,A} + \cdots, \tag{43.49}$$

where:

$$\mathcal{L}_{\rm S,P} = \frac{1}{N_c}\frac{8\pi^2}{\Lambda_\chi^2}\mathbf{G}_S \sum_{i,j}\left(\bar{q}^i_R q_{Lj}\right)\left(\bar{q}^j_L q_{Ri}\right), \tag{43.50}$$

and:

$$\mathcal{L}_{\rm V,A} = -\frac{1}{N_c}\frac{8\pi^2}{\Lambda_\chi^2}\mathbf{G}_V \sum_{i,j}\left[\left(\bar{q}^i_L \gamma^\mu q_{Lj}\right)\left(\bar{q}^j_L \gamma_\mu q_{Li}\right) + L \leftrightarrow R\right]. \tag{43.51}$$

Here i, j denote u, d, and s flavour indices and summation over colour degrees of freedom within each bracket is understood; $q_{L,R} \equiv \frac{1}{2}(1 \pm \gamma_5)q$. The couplings $\mathbf{G}_{S,V}$ are

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dimensionless functions of the UV integration cut-off Λ. They are expected to grow as Λ approaches the critical value Λ_χ, where spontaneous chiral symmetry breaking occurs. (This is the reason why the operators $\mathcal{L}_{\rm S,P}$ and $\mathcal{L}_{\rm V,A}$ become relevant). In QCD, and with the factor N_c^{-1} pulled out, both couplings $\mathbf{G}_S$ and $\mathbf{G}_V$ are $\mathcal{O}(1)$ in the large-N_c limit. These constants are in principle calculable functions of the ratio $\Lambda/\Lambda_{\rm QCD}$. In practice however, the calculation requires non-perturbative knowledge of QCD in the region where $\Lambda \simeq \Lambda_\chi$, and we shall take $\mathbf{G}_S$ and $\mathbf{G}_V$, as well as Λ_χ, as independent unknown parameters. The χ index in $\mathcal{L}^\chi_{\rm QCD}$ means that only the low-frequency modes $\Lambda \leq \Lambda_\chi$ of the quark and gluon fields are to be considered from now onwards.

Notice that in QCD, couplings of the type $\mathcal{L}_{\rm S,P}$ and $\mathcal{L}_{\rm V,A}$ appear naturally from gluon exchange between two QCD colour currents. Using Fierz rearrangement, one has in the large-N_c limit:

$$g_s^2 \sum_a \left(\bar{q}\gamma^\mu \frac{\lambda^a}{2} q\right)\left(\bar{q}\gamma_\mu \frac{\lambda^a}{2} q\right) \Rightarrow \frac{1}{N_c}\frac{8\pi^2}{\Lambda_\chi^2} 4\frac{\alpha_s N_c}{\pi} \sum_{i,j} \left(\bar{q}^i_R q_{Lj}\right)\left(\bar{q}^j_L q_{Ri}\right)$$
$$- \frac{1}{N_c}\frac{8\pi^2}{\Lambda_\chi^2}\frac{\alpha_s N_c}{\pi} \sum_{i,j} \left[\left(\bar{q}^i_L \gamma^\mu q_{Lj}\right)\left(\bar{q}^j_L \gamma_\mu q_{Li}\right)\right.$$
$$\left. + L \leftrightarrow R\right]; \tag{43.52}$$

i.e.; $\mathbf{G}_V = \mathbf{G}_S/4 = \alpha_s N_c/\pi$ in this case. The two operators $\mathcal{L}_{\rm S,P}$ and $\mathcal{L}_{\rm V,A}$ have, however, different anomalous dimensions, and it is therefore not surprising that $\mathbf{G}_S \neq 4\mathbf{G}_V$ for the corresponding physical values.

If furthermore, one assumes that the relevant gluonic effects for low-energy physics are those already absorbed in the new couplings $\mathbf{G}_S$ and $\mathbf{G}_V$, then:

$$\mathcal{L}^\chi_{\rm QCD} \Rightarrow i\bar{q}\,\not{D} q \tag{43.53}$$

in Eq. (43.49) with $\not{D}$ the Dirac operator given in Eq. (43.2), where now the gluon field G_μ plays the rôle of an external colour field source. There is no gluonic kinetic term any longer.

As is well known from the early work of Nambu and Jona-Lasinio [539], the operator $\mathcal{L}_{\rm S,P}$, for values of $\mathbf{G}_S > 1$, is at the origin of the spontaneous chiral symmetry breaking. This can best be seen following the standard procedure of introducing auxiliary field variables to convert the four-fermion coupling operators into bilinear quark operators. For this purpose, one introduces a 3×3 auxiliary field matrix $M(x)$ in flavour space; the so called collective field variables, which under chiral-$SU(3)$ transform as:

$$M \to V_R M V_L^\dagger\,; \tag{43.54}$$

and uses the functional integral identity:

$$\exp\left[i\int d^4x\, \frac{1}{N_c}\frac{8\pi^2}{\Lambda_\chi}\mathbf{G}_S \sum_{i,j}\left(\bar{q}^i_R q_{Lj}\right)\left(\bar{q}^j_L q_{Ri}\right)\right]$$
$$= \int \mathcal{D}M \,\exp\left[i\int d^4x\left\{-(\bar{q}_L M^\dagger q_R + h.c.) - N_c\frac{\Lambda_\chi^2}{8\pi^2}\frac{1}{\mathbf{G}_S} tr M M^\dagger\right\}\right]. \tag{43.55}$$

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By polar decomposition:

$$M = \xi H \xi^\dagger , \tag{43.56}$$

with $\xi\xi^\dagger = U$ unitary and H hermitian.

Next, we look for translational-invariant solutions, which minimize the effective action;

$$\left.\frac{\partial \Gamma_{\rm eff}}{\partial M}\right|_{H=<H>=M_Q,\xi=1;v=a=s=p=0.} = 0 \, .$$

The minimum is reached when all the eigenvalues of $< H >$ are equal, i.e., $< H >= M_Q 1$; and the minimum condition leads to

$$\mathrm{Tr}\left(x\left|\frac{1}{\not{D}}\right|x\right) = -2M_Q N_c \frac{\Lambda_\chi^2}{8\pi^2}\frac{1}{\mathbf{G}_S}\int d^4x \, . \tag{43.57}$$

The trace in the LHS of this equation is formally proportional to $< \bar\psi\psi >$. The calculation, however, requires a regularization, with Λ_χ the UV cut-off. We choose the proper time regularization. [See e.g. [540] for technical details.] Then:

$$< \bar\psi\psi >= -\frac{N_c}{16\pi^2}4M_Q^3\Gamma\left(-1, \frac{M_Q^2}{\Lambda_\chi}\right) ; \tag{43.58}$$

and the minimum condition in Eq. (43.57) leads to the so-called *gap equation*:

$$\frac{M_Q}{\mathbf{G}_S} = M_Q\left\{\exp\left(-\frac{M_Q^2}{\Lambda_\chi^2}\right) - \frac{M_Q^2}{\Lambda_\chi^2}\Gamma\left(0, \frac{M_Q^2}{\Lambda_\chi^2}\right)\right\} . \tag{43.59}$$

The functions:

$$\Gamma\left(n-2, x \equiv \frac{M_Q^2}{\Lambda_\chi^2}\right) = \int_x^\infty \frac{dz}{z}e^{-z}z^{n-2} ; \quad n = 1, 2, 3, \ldots , \tag{43.60}$$

are incomplete gamma functions. Equations (43.58) and (43.59) show the existence of two phases with regards to chiral symmetry. The unbroken phase corresponds to the trivial solution $M_Q = 0$, which implies $< \bar\psi\psi > = 0$. The broken phase corresponds to the possibility that the coupling $\mathbf{G}_S$ increases as we decrease the UV cut-off Λ down to Λ_χ, allowing for solutions to Eq. (43.59) with $M_Q > 0$ and therefore $< \bar\psi\psi > \neq 0$ and negative. In this phase the Hermitian auxiliary field $H(x)$ develops a non-vanishing vacuum expectation value, which is at the origin of a constituent chiral quark mass term [see the RHS of Eq. (43.55)]:

$$-M_Q(\bar q_L U^\dagger q_R + \bar q_R U q_L) = -M_Q \bar Q Q \, , \tag{43.61}$$

like the one which appears in the Georgi–Manohar model [512]; and like the one proposed in the effective action approach of [413]. In the presence of the operator $\mathcal{L}_{\rm V,A}$, we need two more auxiliary 3×3 complex field matrices $L_\mu(x)$ and $R_\mu(x)$ to rearrange the Lagrangian in

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Eq. (43.49) into an equivalent Lagrangian which is only quadratic in the quark fields. Under chiral (V_L, V_R) transformations these collective field variables are chosen to transform as follows:

$$L_\mu \rangle V_L L_\mu V_L^\dagger \,, \qquad R_\mu \rangle V_R R_\mu V_R^\dagger \,.$$

Then, the following functional identity follows:

$$\exp\left(-i\int d^4x\,\frac{1}{N_c}\frac{8\pi^2}{\Lambda_\chi^2}\mathbf{G}_V\sum_{i,j}\left[(\bar q^i_L\gamma^\mu q_{Lj})(\bar q^j_L\gamma_\mu q_{Li}) + L\leftrightarrow R\right]\right)$$
$$= \int \mathcal{D}L_\mu\,\mathcal{D}R_\mu\,\exp\left[i\int d^4x\left\{\bar q_L\gamma^\mu L_\mu q_L + N_c\frac{\Lambda_\chi^2}{8\pi^2}\frac{1}{\mathbf{G}_V}\frac{1}{4}tr L^\mu L_\mu + L\leftrightarrow R\right\}\right]. \tag{43.62}$$

It is convenient to trade the auxiliary field matrices $L_\mu(x)$ and $R_\mu(x)$ by new vector field matrices:

$$W_\mu^{(\pm)} = \xi L_\mu \xi^\dagger \pm \xi^\dagger R_\mu \xi \,,$$

which transform homogeneously under chiral transformations (V_L, V_R); i.e.:

$$W_\mu^{(\pm)}\rangle h W_\mu^{(\pm)} h^\dagger \,,$$

with h the $SU(3)_V$ rotation induced by (V_L, V_R). The fermionic determinant can then be obtained using standard techniques, like for example the *heat kernel* expansion we described earlier. When computing the resulting effective action, there appears a mixing term between the fields $W_\mu^{(-)}$ and ξ_μ. One needs a new redefinition of the auxiliary field $W_\mu^{(-)}$:

$$W_\mu^{(-)}\rangle \hat W_\mu^{(-)} + (1-g_A)\xi_\mu \,,$$

in order to diagonalize the quadratic form in the variables $W_\mu^{(-)}$ and ξ_μ. It is this mixing which is at the origin of an effective axial coupling of the constituent quarks with the Nambu–Goldstone modes:

$$\frac{1}{2} i g_A \bar Q \gamma^\mu \gamma_5 \xi_\mu Q \,,$$

a term like the axial coupling which appears in the Georgi–Manohar model. but with a specific form for the axial coupling constant g_A:

$$g_A = \frac{1}{1 + \mathbf{G}_V \frac{4M_Q^2}{\Lambda_\chi^2}\Gamma\left(0, \frac{M_Q}{\Lambda_\chi^2}\right)} \,. \tag{43.63}$$

In terms of Feynman diagrams this result can be understood as an *infinite* sum of constituent quark bubbles, with a coupling at the end to the pion field. These are the diagrams generated by the $\mathbf{G}_V$ four-fermion coupling to leading order in the $1/N_c$ expansion. The quark propagators in these diagrams are constituent quark propagators, solution of the

Schwinger–Dyson which is at the origin of the *gap equation* in Eq. (43.59). In the limit where $\mathbf{G}_V = 0$, $g_A = 1$; but in general [542], $g_A \neq 1$ to leading order in the $1/N_c$ expansion.

Kinetic terms for the auxiliary field variables are also generated by the functional integral over the quark fields Q and $\bar{Q}$. The resulting Lagrangian, after wave-function rescaling of the auxiliary fields, has the form of a constituent chiral quark model, with scalar $S(x)$, vector $V(x)$, and axial-vector $A(x)$ field couplings:

$$\begin{aligned}
\mathcal{L}_{\rm eff}^{ENJL} = & \, i\bar{Q}\gamma^\mu\left(\partial_\mu + \Gamma_\mu - \frac{i}{\sqrt{2}f_V}V_\mu\right)Q - M_Q\bar{Q}Q \\
& + \frac{i}{2}g_A\bar{Q}\gamma_5\gamma^\mu\left(\xi_\mu - \frac{\sqrt{2}}{f_A}A_\mu\right)Q - \frac{1}{\lambda_S}\bar{Q}S(x)Q \\
& + \frac{1}{2}{\rm tr}\big[\partial_\mu S\partial^\mu S - M_S^2 SS\big] \\
& - \frac{1}{4}{\rm tr}[(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) - 2M_V V_\mu V^\mu] \\
& - \frac{1}{4}{\rm tr}\big[(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - 2M_A^2 A_\mu A^\mu\big] \\
& + \frac{1}{4}f_\pi^2{\rm tr}D_\mu U D^\mu U^\dagger + \mathcal{O}(p^4){\rm terms}\,,
\end{aligned} \tag{43.64}$$

where Γ_μ and ξ_μ are the same as those defined in Eqs.(43.43) and (43.44), and the coupling constants and masses are now expressed in terms of only three input parameters. As input parameters, we can either fix: $\mathbf{G}_S$, $\mathbf{G}_V$, and Λ_χ; or the more physical parameters:

$$M_Q\,, \qquad \Lambda\chi\,, \qquad g_A\,. \tag{43.65}$$

The coupling constants are then:

$$\begin{aligned}
f_\pi^2 &= \frac{N_c}{16\pi^2}4M_Q^2 g_A\Gamma\big(0, M_Q^2/\Lambda_\chi^2\big)\,, \\
f_V^2 &= \frac{N_c}{16\pi^2}\frac{2}{3}\Gamma\big(0, M_Q^2/\Lambda_\chi^2\big)\,, \\
f_A^2 &= \frac{N_c}{16\pi^2}\frac{2}{3}g_A^2\big[\Gamma\big(0, M_Q^2/\Lambda_\chi^2\big) - \Gamma\big(1, M_Q^2/\Lambda_\chi^2\big)\big]\,, \\
[3pt]\lambda_S^2 &= \frac{N_c}{16\pi^2}\frac{2}{3}\big[3\Gamma\big(0, M_Q^2/\Lambda_\chi^2\big) - 2\Gamma\big(1, M_Q^2/\Lambda_\chi^2\big)\big]\,;
\end{aligned} \tag{43.66}$$

and the masses:

$$\begin{aligned}
M_V^2 &= 6M_Q^2\frac{g_A}{1-g_A}\,, \\
M_A^2 &= 6M_Q^2\frac{1}{1-g_A}\frac{1}{1-\frac{\Gamma\left(1,M_Q^2/\Lambda_\chi^2\right)}{\Gamma\left(0,M_Q^2/\Lambda_\chi^2\right)}}\,, \\
M_S^2 &= 4M_Q^2\frac{1}{1-\frac{2}{3}\frac{\Gamma\left(1,M_Q^2/\Lambda_\chi^2\right)}{\Gamma\left(0,M_Q^2/\Lambda_\chi^2\right)}}\,.
\end{aligned} \tag{43.67}$$

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Table 43.2. *The L_i-coupling constants in the ENJL model of [540], with g_A defined in Eq.(43.63), and $\Gamma_n \equiv \Gamma(n, M_Q^2/\Lambda_\chi^2)$. The second column gives the results corresponding to the input parameter values in Eq. (43.68). The third column gives the experimental values of Table 42.1.*

The L_i couplings of $\mathcal{O}(p^4)$ in the ENJL–model	Fit 1	Experiment
$L_1 = \frac{N_c}{16\pi^2}\frac{1}{48}\left[\left(1-g_A^2\right)^2\Gamma_0 + 4g_A^2\left(1-g_A^2\right)\Gamma_1 + 2g_A^4\Gamma_2\right]$	0.85	0.7 ± 0.5
$L_2 = 2L_1$	1.7	1.2 ± 0.4
$L_3 = -\frac{N_c}{16\pi^2}\frac{1}{8}\left\{\left[\left(1-g_A^2\right)^2\Gamma_0 + 4g_A^2\left(1-g_A^2\right)\Gamma_1 + \right.\right.$ $\left.\left. -\frac{2}{3}g_A^4\left[2\Gamma_1 - 4\Gamma_2 + 3\frac{1}{\Gamma_0}(\Gamma_0-\Gamma_1)^2\right]\right]\right\}$	-4.2	-3.6 ± 1.3
$L_5 = \frac{N_c}{16\pi^2}\frac{1}{4}g_A^3[\Gamma_0 - \Gamma_1]$	1.6	1.4 ± 0.5
$L_8 = \frac{N_c}{16\pi^2}\frac{1}{16}g_A^2\left[\Gamma_0 - \frac{2}{3}\Gamma_1\right]$	0.8	0.9 ± 0.3
$L_9 = \frac{N_c}{16\pi^2}\frac{1}{6}\left[\left(1-g_A^2\right)\Gamma_0 + 2g_A^2\Gamma_1\right]$	7.1	6.9 ± 0.7
$L_{10} = -\frac{N_c}{16\pi^2}\frac{1}{6}\left[\left(1-g_A^2\right)\Gamma_0 + g_A^2\Gamma_1\right]$	-5.9	-5.5 0.7

In the absence of the vector and axial-vector four-fermion-like coupling i.e., when $\mathbf{G}_V = 0$: $g_A = 1$, $M_V\rangle\infty$ and $M_A\rangle\infty$. Then the vector and axial-vector interactions decouple, and the model becomes equivalent to the *Constituent Chiral Quark Model* of [512], with $g_A = 1$ and a non-trivial coupling to a scalar field.

The functional integration over the quark fields and the auxiliary $S(x)$, $V(x)$, and $A(x)$ fields results in an effective action among the Nambu–Goldstone boson particles, with all the couplings fixed by the three parameters M_Q, Λ_χ, and g_A. The explicit results one gets for the L_i constants which appear in the large-N_c limit at $\mathcal{O}(p^4)$ in the chiral expansion are shown in Table 43.2. The reason why the constant L_7 does not appear in this table is that, phenomenologically, this constant gets a large contribution from the integration of the *heavy* singlet η' particle. However, in the chiral limit, the mass of the η' is induced by the axial-$U(1)$ anomaly, which only appears to next-to-leading order in the $1/N_c$ expansion. By definition, the ENJL model, as formulated here, ignores this effect. In order to take these next-to-leading effects in $1/N_c$ systematically, together with the chiral expansion, one has to resort to a $U(3) \times U(3)$ formulation of the effective theory [543]. The constants L_4 and L_6 are of next-to-leading order in the $1/N_c$ expansion; this is the reason why they do not appear in Table 43.2 either. We also show in Table 43.2 the numerical results of the fit 1 discussed in [540]. These results correspond to the set of input parameter values:

$$M_Q = 265\ MeV\,, \qquad \Lambda_\chi = 1165\ MeV\,, \qquad g_A = 0.61\,. \tag{43.68}$$

The overall picture which emerges from this simple model is quite remarkable. The main improvement with respect to the results obtained in the *effective action approach model*

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discussed in the previous section is on the constants L_5 and L_8, where the combined effect of the vector and scalar degrees of freedom leads to rather simple results modulated by powers of the g_A-constant, which agree very well with the phenomenological determinations. One of the characteristic features of the ENJL model, is that it interpolates successfully between pure VMD-type predictions and those of the constituent chiral quark model. A nice illustration is the result for L_9 in Table 43.2, where the first term is the one coming from vector–exchange, whereas the second one comes from the chiral quark loop integral.

There is no difficulty to reproduce the anomalous Wess–Zumino–Witten functional within the ENJL model [544] QCD two-point functions, beyond the low-energy expansion, have also been evaluated in the ENJL model [541]. This involves calculations to leading order in the $1/N_c$ expansion (i.e., an infinite number of chains of fermion bubbles; but no loops of chains) and to all orders in powers of momenta Q^2/Λ_χ^2. As a result, vector and axial-vector correlation functions have a VMD-like form, but with slowly varying couplings and masses. For the transverse invariant functions for example, the results are:

$$\Pi_V^{(1)}(Q^2) = \frac{2f_V^2(Q^2)M_V^2(Q^2)}{M_V^2(Q^2) - Q^2}\,, \tag{43.69}$$

and:

$$\Pi_A^{(1)}(Q^2) = \frac{2f_\pi^2(Q^2)}{Q^2} + \frac{2f_A^2(Q^2)M_A^2(Q^2)}{M_A^2(Q^2) - Q^2}\,, \tag{43.70}$$

where:

$$f_V^2(Q^2) = 4\frac{N_c}{16\pi^2}\int_0^1 dx x(1-x)\Gamma\big(0, x_Q \equiv [M_{Q^2} + x(1-x)Q^2]/\Lambda_\chi^2\big)\,. \tag{43.71}$$

The product:

$$2f_V^2(Q^2)M_V^2(Q^2) = N_c\frac{\Lambda_\chi^2}{8\pi^2}\frac{1}{\mathbf{G}_V} \tag{43.72}$$

is scale invariant. With:

$$g_A(Q^2) = \frac{1}{1 + \mathbf{G}_V\frac{4M_{Q^2}}{\Lambda_\chi^2}\int_0^1 dx\Gamma(0, x_Q)}\,, \tag{43.73}$$

the other couplings are fixed by:

$$f_A^2(Q^2) = g_A^2(Q^2)f_V^2(Q^2)\,, \tag{43.74}$$

and the relations

$$\begin{aligned} f_V^2(Q^2)M_V^2(Q^2) &= f_A^2(Q^2)M_A^2(Q^2) + f_\pi^2(Q^2)\,, \\ f_V^2(Q^2)M_V^4(Q^2) &= f_A^2(Q^2)M_A^4(Q^2)\,, \end{aligned} \tag{43.75}$$

where the last two equalities are the Q^2-dependent version of the first- and second-Weinberg sum rules [26]. In the case of the scalar two-point function there appears a pole in the

Q^2-summed expression at:

$$M_S = 2M_Q \ . \tag{43.76}$$

The case of the other two-point functions is somewhat more involved because they mix through the four-fermion interaction terms. In principle, the ENJL model can be applied to obtain a systematic calculation of the low-energy constants of the weak non-leptonic Lagrangian (B_K-parameter, ...). These applications can be found in some more dedicated reviews.