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SPECIAL PAIRS OF SEMI-BILOGIC AND BILOGIC TETRAHEDRA

SAHIB RAM MANDAN

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Abstract

Let (A), (B) be a special pair of perspective tetrahedra such that the 4 joins of their corresponding vertices are perpendicular to their plane of perspectivity. It has been already established (Mandan (1977a), p. 573) that they are, in general, *skew-orthologic* such that the perpendiculars from the vertices of one to the corresponding opposite faces of the other lie in a regulus. Here we give a construction of a special pair of *semi-orthologic* and *orthologic* tetrahedra, the said regulus degenerating into 2 pairs of intersecting lines and 4 concurrent lines respectively. Following Thébault ((1952), p. 25; (1955), p. 67), we call them *semi-bilogic* and *bilogic* respectively.

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1. Introduction

Let the 4 joins $p_i = A_i B_i$ of the corresponding vertices of a special pair of perspective tetrahedra $(X) = X_1 X_2 X_3 X_4$ $(X = A, B)$ be all perpendicular to the plane p of their perspectivity at the points $0i$ $(= i0)$ and hence parallel to one another to concur at a point P *at infinity* such that their corresponding faces $x_m = X_i X_j X_k$ $(i, j, k, m = 1, 2, 3, 4)$ meet p in the same 4 lines ijk and the plane p' at infinity in 4 pairs of lines x_{ijk} $(x = a, b)$ meeting ijk in the same 4 points (ijk) on the line $(p) = p.p'$ common to p, p'. Their corresponding edges $X_i X_j$ meet p in the same 6 points ij $(= ji)$ and p' in 6 pairs of points X_{ij} forming vertices of 2 quadrilaterals q_x perspective from the point P and line (p) as section of the tetrahedra (X) by the plane p'.

Obviously enough, the 4 points $0i$ and the 6 points ij form a quadrangle and a quadrilateral respectively so interwoven that the sides of the former pass through the vertices of the other to form a Desargues' figure 10_3 (Baker (1930), p. 251; Coxeter (1975), p. 231) as shown in Figure 1 (same as in Mandan (1979)).

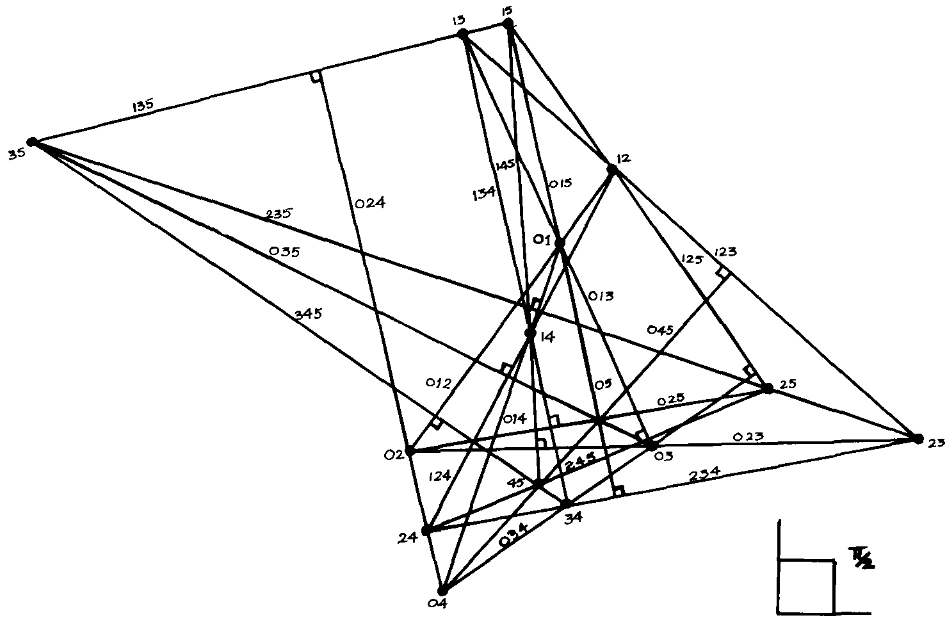

FIGURE 1. An orthologic Veronese configuration.

We define *perpendicularity as conjugacy* with respect to a *fixed conic* (q) in the plane p', called *absolute polarity* or the circle at infinity (Mandan (1958), p. 492; (1965), p. 409). Thus the point P is the pole of the line (p) as well as of the plane p for (q) to make the 4 lines p_i perpendicular to p.

2. Special pair of semi-bilogic tetrahedra

EXISTENCE. The perpendiculars from the vertices A_1, A_4 of the tetrahedron (A) respectively to the faces b_1, b_4 of (B) meet at a point A_5 (say) if, and only if, A_1A_4 is orthogonal to B_2B_3, or the point A_{14} is conjugate to B_{23} for the conic (q) in the plane p' (Figure 2). Now the perspectivity of the quadrilaterals q_x implies that of their diagonal triangles also from the *same* centre P and axis (p) of perspectivity. That is, the 3 pairs of their diagonals: $X_{23}X_{14}$, $X_{31}X_{24}$, $X_{12}X_{34}$ meet on (p) respectively at A, B, C (say). Thus A, P and A_{14}, B_{23} are 2 pairs of opposite vertices of a quadrilateral conjugate for (q) to make the third pair A_{23}, B_{14} also conjugate for (q) by Hesse's theorem (Baker (1930), p. 31; Coxeter (1964), p. 68; (1969), p. 251; Cremona (1960), p. 238). Hence A_2A_3 is perpendicular to B_1B_4 that implies that the perpendiculars from the vertices A_2, A_3 of (A) respectively to the faces b_2, b_3 of (B) meet at a point A'_5 (say). Consequently the perpendiculars from the vertices B_1, B_4 of (B) respectively to the faces a_1, a_4 of (A) meet at B_5 (say), and those from B_2, B_3 respectively to a_2, a_3 at B'_5 (say).

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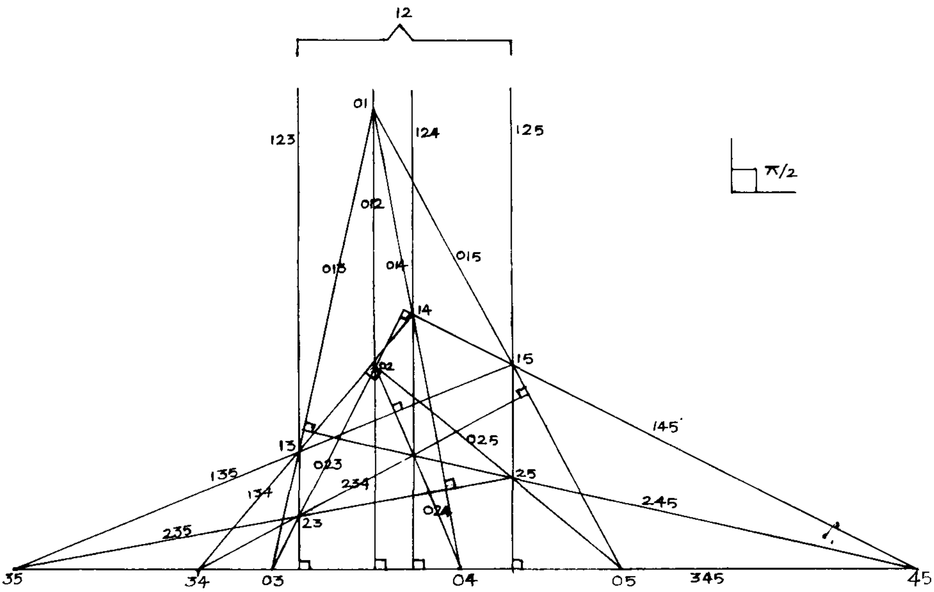

FIGURE 2. Two quadrilaterals q_x $(x = a, b)$ perspective from P and (p).

A PROPERTY. Calling X_5, X'_5 as pairs of corresponding *semi-orthologic centres* (s.o.c.) of the 2 tetrahedra (X), we note at once *that the joins of the 2 pairs of corresponding s.o.c. of any special pair of semi-bilogic tetrahedra are parallel to those of their corresponding vertices.* For the triad of lines: $A_1 B_1$, $A_1 A_5$, $B_1 B_5$ are all perpendicular to the common line 234 in the plane p of the 2 corresponding faces x_1 of (X) and therefore lie in a plane, and similarly lie the 3 lines: $A_4 B_4$, $A_4 A_5$, $B_4 B_5$ in a plane perpendicular to the line 123 in p, and hence the common line $A_5 B_5$ of these 2 planes is perpendicular to p; similar is the story of the join $A'_5 B'_5$ common to the 2 planes $A_2 B_2 A'_5 B'_5$, $A_3 B_3 A'_5 B'_5$ perpendicular respectively to the 2 lines: 134, 124 in p and therefore to p.

CONSTRUCTION. To construct such a pair of tetrahedra (X), we have the following rule if one of them, say (A), is given to fix a Desargues' figure 10_3, in a plane p, formed of the 4 orthogonal projections $0i$ in p of its vertices A_i and the 6 meets ij of its edges $A_i A_j$ with p such that these 10 points lie in triads on 10 lines ijk as ij, jk, ki $(i, j, k = 0, 1, 2, 3, 4)$.

Draw the plane perpendicular to its edge $A_2 A_3$ from the point 14 to meet the lines p_1, p_4 (perpendicular to p at 01, 04 and passing through A_1, A_4 respectively) in the 2 vertices B_1, B_4 of the second tetrahedron (B) respectively such that their join $B_1 B_4$ passes through 14. Now the 2 planes joining $B_1 B_4$ to the 2 lines 124, 134 meet the lines p_2, p_3 (perpendicular to p at 02, 03 and passing through A_2, A_3 respectively) in the other 2 vertices B_2, B_3 of (B) respectively as may be easily verified to satisfy the necessary conditions. This construction fails to give us the second tetrahedron (B) if $A_2 A_3$ be orthogonal to $A_1 A_4$. For in that case (B) coincides with (A) to become a semi-orthocentric tetrahedron (Court (1953); Mandan (1957)).

3. Special pair of bilogic tetrahedra

EXISTENCE. The special pair of semi-bilogic tetrahedra (X), as considered above, become bilogic if, besides an edge $A_1 A_4$ of the tetrahedron (A) being orthogonal to the corresponding opposite edge $B_2 B_3$ of (B), another edge $A_1 A_2$ of (A) is also so related to $B_3 B_4$. For in that case the 2 vertices A_{14}, A_{12} of the quadrilateral q_a in Figure 2 are conjugate to the respectively opposite vertices B_{23}, B_{34} of q_b for the conic (q) in the plane p', and that implies, by an extension of Hesse's theorem (Mandan and Sanyal (1977)), the conjugacy for (q) of the remaining 4 pairs of corresponding opposite vertices of the 2 quadrilaterals q. Hence every pair of corresponding opposite edges of the 2 tetrahedra (X) are orthogonal such that the perpendiculars from the vertices of (A) to the corresponding opposite faces of (B) meet one another and therefore concur at a point A_5 (say), and those from the vertices of (B) to the corresponding opposite faces of (A) concur at a point B_5 (say).

CONSTRUCTION. But the construction of such a pair of tetrahedra is not that straight and simple as has been the case for a special pair of semi-bilogic tetrahedra. For, given (A), it requires that the 3 planes $B_1 B_4 B_5$, $B_2 B_4 B_5$, $B_3 B_4 B_5$ perpendicular to its 3 edges $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ respectively from the vertices 14, 24, 34 of a triangle T' in the plane p are *coaxal* to meet the lines p_1, p_2, p_3 (perpendicular to p at the vertices 01, 02, 03 of a triangle T) in the points B_1, B_2, B_3 respectively and the line p_4 (perpendicular to p at 04) in a *unique* point B_4 to form vertices of the required second tetrahedron (B). That implies that they meet p in the lines 145, 245, 345 concurrent at a point 45 (say) as perpendiculars from the vertices of T' to the corresponding opposite sides of T (*perspective* to T' from the point 04 and the line 123) to make T, T' *orthologic* with the further property that the join of 04, 45 is perpendicular to 123. Thus it requires the existence of a special Desargues' figure of T, T' to satisfy these conditions, and the same is then established in a separate paper (Mandan (1979)) with many more properties, as visible in Figure 1, such that an *arbitrary* plane through the line 123 of this figure cuts the 3 lines p_i $(i = 1, 2, 3)$ in 3 points A_i respectively to form a triangle perspective to T' from 123 and therefore from a point A_4 that completes the construction of a tetrahedron (A). For the triangles $(02, 03, 04)$, $A_2 A_3 A_4$ are then perspective from the line 234 and therefore from a point $P = p_2 . p_3$ at infinity such that A_4 lies on p_4, and the other 2 faces a_2, a_3 of (A) meet p in the lines 134, 124 respectively as required.

Now the construction for the tetrahedron (B) as given above is valid to couple with (A) as desired.

PROPERTIES. (i) Calling the 2 points X_5 $(X = A, B)$ in (i) as the *orthologic centres* (o.c.) of the 2 tetrahedra (X), we observe that *the join of the* 2 *o.c. of any special pair of bilogic tetrahedra is parallel to those of their corresponding vertices* by an argument similar to that for the join of a pair of corresponding s.o.c. of any special pair of semi-bilogic tetrahedra such that the join $p_5 = A_5 B_5$ is perpendicular to the plane p at the point 05 (say) in consistency with Sondat's (1894); p. 10) theorem as stated by Servais ((1921), p. 57) and established by Thébault ((1952), p. 26).

(ii) X_i $(i = 1, 2, 3, 4, 5)$ now form a *special pair of bilogic sets* (compare with orthologic dupoints of Gerber (1977), p. 54, and Mandan (1977b), p. 414) such that *any* 2 *corresponding tetrads of them form a special pair of bilogic tetrahedra with the remaining* 2 *points as* 2 *o.c., the join of any* 2 *points of one set being perpendicular to the plane of the* 3 *non-corresponding points of the other.*

(iii) The lines 145, 245, 345 are also seen as the meets of p with the planes of the 3 triangles: $A_1 A_4 A_5$, $A_2 A_4 A_5$, $A_3 A_4 A_5$ perpendicular to $B_2 B_3$, $B_3 B_1$, $B_1 B_2$ from the points 14, 24, 34 respectively as their axes of perspectivity with the triangles: $B_1 B_4 B_5$, $B_2 B_4 B_5$, $B_3 B_4 B_5$ such that *p is the plane of perspectivity of the* 2 *sets X_i in* (ii) *and the* 2 *tetrads of points:*

$$(01, 02, 03, 04), \quad (15 = A_1 A_5 . B_1 B_5, 25 = ., 35 = ., 45 = .)$$

form 2 *bilogic quadrangles with* 05 *as their centre of perspectivity, any two of their corresponding triads forming* 2 *bilogic triangles with the remaining* 2 *points as* 2 *o.c.* (every side of each quadrangle being perpendicular to the corresponding opposite side of the other) *whose join is perpendicular to the axis of perspectivity of the* 2 *triangles* leading to an interesting *orthologic Veronese configuration* (Mandan (1979)) of 15 points and 20 lines. The third tetrahedron of Thébault ((1952), p. 27), bilogic with both the tetrahedra (X), collapses here into the second quadrangle.

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Flat 19, "Vijaya", Chhedanagar, Bombay–400089.
India