Nagoya Math. J., **222**(1) (2016), 186–209
DOI 10.1017/nmj.2016.16

ON THE MODIFIED FUTAKI INVARIANT OF COMPLETE INTERSECTIONS IN PROJECTIVE SPACES

RYOSUKE TAKAHASHI

Abstract. Let M be a Fano manifold. We call a Kähler metric $\omega \in c_1(M)$ a Kähler–Ricci soliton if it satisfies the equation $\mathrm{Ric}(\omega) - \omega = L_V \omega$ for some holomorphic vector field V on M. It is known that a necessary condition for the existence of Kähler–Ricci solitons is the vanishing of the modified Futaki invariant introduced by Tian and Zhu. In a recent work of Berman and Nyström, it was generalized for (possibly singular) Fano varieties, and the notion of algebrogeometric stability of the pair (M, V) was introduced. In this paper, we propose a method of computing the modified Futaki invariant for Fano complete intersections in projective spaces.

§1. Introduction

Let M be an n-dimensional Fano manifold, that is, M is a compact complex manifold and $c_1(M)$ is represented by some Kähler form ω on M. If we take holomorphic coordinates $(z^1, \ldots, z^n)$ of M, ω and its Ricci form $\mathrm{Ric}(\omega)$ are locally written as

$$\begin{cases} g_{i\bar{j}} = g\left(\dfrac{\partial}{\partial z^i}, \dfrac{\partial}{\partial z^{\bar{j}}}\right), \\ \omega = \dfrac{\sqrt{-1}}{2\pi} \displaystyle\sum_{i,j} g_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}} \end{cases}$$

and

$$\begin{cases} r_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log(\det(g_{k\bar{l}})), \\ \mathrm{Ric}(\omega) = \dfrac{\sqrt{-1}}{2\pi} \displaystyle\sum_{i,j} r_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}. \end{cases}$$

Received October 31, 2014. Revised June 17, 2015. Accepted June 21, 2015.
2010 Mathematics subject classification. Primary 53C25.
This work was supported by Grant-in-Aid for JSPS Fellows Number 25-3077.

Since both ω and $\mathrm{Ric}(\omega)$ are in $c_1(M)$, $\mathrm{Ric}(\omega)-\omega$ is an exact $(1,1)$-form. Therefore, there exists a real-valued smooth function κ on M such that

$$\mathrm{Ric}(\omega)-\omega=\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\kappa.$$

Let $\mathfrak{g}$ be the Lie algebra consisting of all holomorphic vector fields on M. Then, any $V\in\mathfrak{g}$ can be lifted to the anticanonical bundle $-K_M$ of M, and naturally acts on the space of Hermitian metrics on $-K_M$. Let h be a Hermitian metric on $-K_M$ such that $\omega=-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$, and let $\mu_{h,V}$ be the holomorphy potential of the pair (h,V) defined by this action (cf. Definition 2.2). Then, we can easily check that

$$\begin{cases} i_V\omega=\dfrac{\sqrt{-1}}{2\pi}\bar{\partial}\mu_{h,V}, \\ -\Delta_{\partial}\mu_{h,V}+\mu_{h,V}+V(\kappa)=0, \end{cases}$$

where $\Delta_\partial=-g^{i\bar{j}}\frac{\partial^2}{\partial z^i\partial z^{\bar{j}}}$ denotes the ∂-Laplacian with respect to ω. A metric ω is called a Kähler–Ricci soliton if it satisfies the equation

$$\mathrm{Ric}(\omega)-\omega=L_V\omega$$

for some $V\in\mathfrak{g}$, where L_V denotes the Lie derivative with respect to V. This is equivalent to the condition $\kappa=\mu_{h,V}$ (up to an additive constant). In particular, in the case when $V\equiv 0$, this metric is a well-known Kähler–Einstein metric. An obstruction to the existence of Kähler–Ricci solitons was first discovered by Tian and Zhu [TZ02]. Let $\mathcal{F}$ be a function on $\mathfrak{g}$ defined by

$$\mathcal{F}(V)=-\frac{1}{c_1(M)^n}\int_M e^{\mu_{h,V}}\omega^n,$$

and define the modified Futaki invariant $\mathrm{Fut}_V(W)$ as the Gâteaux differential of $\mathcal{F}$ at V in the direction W, that is,

$$\begin{aligned}\mathrm{Fut}_V(W)&=\frac{d}{dt}\mathcal{F}(V+tW)\bigg|_{t=0}=-\frac{1}{c_1(M)^n}\int_M\mu_{h,W}e^{\mu_{h,V}}\omega^n\\ &=\frac{1}{c_1(M)^n}\int_M W(\kappa-\mu_{h,V})e^{\mu_{h,V}}\omega^n.\end{aligned}$$

Hence, if there exists a Kähler–Ricci soliton ω with respect to V, then we have $\kappa=\mu_{h,V}$ (up to an additive constant), and $\mathrm{Fut}_V(W)$ must vanish.

Tian and Zhu showed that $\mathrm{Fut}_V(W)$ is independent of the choice of $\omega \in c_1(M)$. (In the case when $V \equiv 0$, this function coincides with the original Futaki invariant, and its independence was shown in [Fut83].) Recently, Berman and Nyström [BN14] generalized this obstruction to arbitrary Fano varieties (i.e., projective normal varieties with log terminal singularities and satisfying the property that $-K_M$ is an ample $\mathbb{Q}$-line bundle), and introduced the notion of K-stability for the pair (M, V). (Wang, Zhou and Zhu [WZZ14] also defined the slightly modified notion of K-stability inspired by the algebraic formula for the modified Futaki invariant in [BN14].) It is important to examine the sign of the modified Futaki invariant, since we can know whether $c_1(M)$ contains a Kähler–Ricci soliton or not if we examine the sign of the modified Futaki invariant on the central fiber for any special test configuration, that is, check the K-polystability.

Chen, Donaldson and Sun [CDS15] and Tian [Tian15] proved that if M is K-polystable, there exists a Kähler–Einstein metric. In the case of Kähler–Ricci solitons, Berman and Nyström [BN14] showed that if M admits a Kähler–Ricci soliton with respect to V, then (M, V) is K-polystable. They also showed that if M is strongly analytically K-polystable and all the higher-order modified Futaki invariants of (X, V) vanish, then there exists a Kähler–Ricci soliton with respect to V, where *strongly analytically K-polystable* means that the modified K-energy is coercive modulo automorphisms. However, it is still an open question whether the K-polystability of (M, V) leads to the existence of a Kähler–Ricci soliton with respect to V.

Motivated by the above reasons, we propose a method of calculating the function $\mathcal{F}$ (therefore, the modified Futaki invariant Fut_V as well) for Fano complete intersections in projective spaces. The main theorem of this paper is as follows.

THEOREM 1.1. *Let M be a Fano complete intersection in $\mathbb{C}P^N$, that is, M is an $(N-s)$-dimensional Fano variety in $\mathbb{C}P^N$ defined by homogeneous polynomials $F_1, \ldots, F_s$ of degree $d_1, \ldots, d_s$ respectively, and*

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=0}^{N}|z^i|^2\right)$$

is the Fubini–Study metric of $\mathbb{C}P^N$. We suppose that there exists a constant $m > 0$ such that $m\omega \in c_1(M)$. Let $V \in \mathfrak{sl}(N+1, \mathbb{C})$ be a holomorphic vector field on $\mathbb{C}P^N$ such that $VF_i = \alpha_i F_i$ for some constants α_i $(i = 1, \ldots, s)$.

Then, we have $m = N + 1 - d_1 - \cdots - d_s$, *and the function* $\mathcal{F}$ *can be written as*

$$\mathcal{F}(V) = -\frac{(N-s)!}{d_1 \cdots d_s m^{N-s}} \times \exp\left(\sum_{i=1}^{s} \alpha_i\right) \int_{\mathbb{C}P^N} \prod_{i=1}^{s} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}, \tag{1.1}$$

where $\theta_V := V \log\left(\sum_{i=0}^{N} |z^i|^2\right)$.

From the above theorem, we know that $\mathcal{F}(V)$ can be written as a linear combination of the integrals $I_{0,l} := m^l \int_{\mathbb{C}P^N} (\theta_V)^l e^{m\theta_V} \omega^N$ $(0 \leqslant l \leqslant s)$.

Although we can easily get a method of computing $\mathcal{F}$ using the localization formula for orbifolds in [DT92], our formula (1.1) is still valuable since we need not assume that M has at worst orbifold singularities. Moreover, we also do not require the explicit geometric knowledge of M, V and ω (local coordinates (uniformization), the zero set of V, curvature, etc.). More concretely, in order to apply the localization formula in [DT92] directly to our case, we have to know the following.

(1) The zero set Zero(V) of V, where we assume that Zero(V) consists of disjoint nondegenerate submanifolds $\{Z_i\}$.
(2) The values of integrals

$$\int_{Z_i} \frac{e^{m(\omega+\theta_V)}}{\det(L_{i,V} + K_i)},$$

where $L_{i,V}(W) := [V, W]$ denotes an endomorphism, and K_i is the curvature matrix of the normal bundle of Z_i.

If $s(= \operatorname{codim}(M)) = 1$ and $\dim(Z_i) = 0$, the above integral can be computed by taking local coordinates (or uniformization) around Z_i. However, it is very hard to compute in general.

The Futaki invariant of complete intersection was first computed by Lu [Lu99] using the adjunction formula and the Poincaré–Lelong formula. Then, it was also computed by many mathematicians using different techniques (see [PS04, Hou08, AV11]). Lu [Lu03] also computed the modified Futaki invariant for smooth hypersurfaces in projective spaces. Our formula (Theorem 1.1) extends Lu's result [Lu03] for (possibly singular) Fano complete intersections of arbitrary codimension. Compared with the

Kähler–Einstein case [Lu99], our formula has in common that $\mathcal{F}(V)$ is expressed by the degree $d_1, \ldots, d_s$ of defining polynomials of M and the weights $\alpha_1, \ldots, \alpha_s$ of the actions induced by the vector field V. However, we need more knowledge of V to compute the integrals $I_{0,l}$ $(0 \leqslant l \leqslant s)$ (see Section 5 for more details).

In this paper, we prove the main theorem (Theorem 1.1) based on the calculations in [Lu99, AV11]. In Section 2, we review some fundamental materials and results for Kähler–Ricci solitons. The standard references for (holomorphic) equivariant cohomology theory are [BGV92, Hou08, Liu95]. We introduce an algebraic formula for $\mathcal{F}$ with reference to the quantization of the modified Futaki invariant studied in [BN14]. In Section 3, we give a proof of Theorem 1.1 by the Poincaré–Lelong formula. Then, in Section 4, we also give another proof of Theorem 1.1 using the algebraic formula for $\mathcal{F}$ (cf. Proposition 2.8). Finally, we give examples of computation of $\mathcal{F}$ in Section 5.

§2. Preminaries

2.1 Holomorphic equivariant coholomogy

Let M be a complex manifold, and let G be a Lie group acting holomorphically on M. Denote $\mathfrak{g} := \mathrm{Lie}(G)$ the Lie algebra of G. Then, for each $\xi \in \mathfrak{g}$, we denote by $\xi_M^{\mathbb{R}}$ the real holomorphic vector field on M given by

$$\xi_M^{\mathbb{R}}(f)(p) = \frac{d}{dt} f(\exp(-t\xi) \cdot p)\bigg|_{t=0}, \quad f \in C^\infty(M), \quad p \in M,$$

and by $\xi_M := \frac{1}{2}(\xi_M^{\mathbb{R}} - \sqrt{-1} J \xi_M^{\mathbb{R}})$ the complex holomorphic vector field on M. Let $\mathbb{C}[\mathfrak{g}]$ be the algebra of a complex-valued polynomial function on $\mathfrak{g}$. We regard each element in $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ as a polynomial function which takes values in differential forms. The group G acts on an element $\sigma \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ by

$$(g \cdot \sigma)(\xi) = g \cdot (\sigma(g^{-1} \cdot \xi)), \quad g \in G \text{ and } \xi \in \mathfrak{g}.$$

Let $\mathcal{A}_G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))^G$ be the space of G-invariant elements in $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$. For $\sigma \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$, we define the bidegree of σ by

$$\mathrm{bideg}(\sigma) = (\deg(\mathrm{P}) + p, \deg(P) + q),$$

where $\sigma = P \otimes \varphi$ $(P \in \mathbb{C}[\mathfrak{g}]$ and $\varphi \in \mathcal{A}^{p,q}(M))$. For instance, $\mathrm{bideg}(\xi) = (1, 1)$. Thus, $\mathcal{A}_G(M) = \bigoplus \mathcal{A}_G^{p,q}(M)$ has the structure of a bigraded algebra.

We define the equivariant exterior differential $\bar{\partial}_{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ as

$$(\bar{\partial}_{\mathfrak{g}}\sigma)(\xi) = \bar{\partial}(\sigma(\xi)) + 2\pi\sqrt{-1}i_{\xi_M}(\sigma(\xi)), \quad \sigma \in \mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M).$$

Then, $\bar{\partial}_{\mathfrak{g}}$ increases by $(0,1)$, the total bidegree on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$, and preserves $\mathcal{A}_G(M)$. Hence, we have a complex $(\mathcal{A}_G(M), \bar{\partial}_{\mathfrak{g}})$.

DEFINITION 2.1. The holomorphic equivariant cohomology $H_{\mathfrak{g}}(M)$ of the pair (M, G) is the cohomology of the complex $(\mathcal{A}_G(M), \bar{\partial}_{\mathfrak{g}})$.

Let E be a G-linearized holomorphic vector bundle over M, and let $\mathrm{Herm}(E)$ be the space of Hermitian metrics on E. The group G acts on $\mathrm{Herm}(E)$ by the formula

$$(g \cdot h)(u, v) = h(g^{-1} \cdot u, g^{-1} \cdot v), \quad g \in G \text{ and } u, v \in E.$$

Hence, for $\xi \in \mathfrak{g}$, we define the real Lie derivative of $\mathfrak{g}$ on $\mathrm{Herm}(E)$ by

$$L^{\mathbb{R}}_{\xi} h = \frac{d}{dt} \exp(t\xi) \cdot h \Big|_{t=0}$$

and the complex Lie derivative of $\mathfrak{g}$ on $\mathrm{Herm}(M)$ by

$$L_{\xi} h = \tfrac{1}{2}(L^{\mathbb{R}}_{\xi} h - \sqrt{-1} L^{\mathbb{R}}_{J\xi} h).$$

We can also define the representation of $\mathfrak{g}$ on the space of sections $\Gamma(E)$ in a similar way. Let ∇ be the Chern connection with respect to h, and put

$$\mu_{h,\xi} = L_{\xi} - \nabla_{\xi_M}.$$

Since $\mu_{h,\xi}(fs) = \xi_M f \cdot s + f \cdot L_{\xi} s - \xi_M f \cdot s - f \cdot \nabla_{\xi_M} s = f \cdot \mu_{h,\xi}(s)$ for any $f \in C^{\infty}(M)$ and $s \in \Gamma(E)$, we have $\mu_{h,\xi} \in \Gamma(\mathrm{End}(E))$. Moreover, one can show that

$$L_{\xi} h = -\mu_{h,\xi} \cdot h, \qquad i_{\xi_M}\theta(h) = -\mu_{h,\xi} \qquad \text{and} \qquad i_{\xi_M}\Theta(h) = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\mu_{h,\xi},$$

where $\theta(h) = \partial h \cdot h^{-1}$ is the connection form and $\Theta(h) = \frac{\sqrt{-1}}{2\pi}\bar{\partial}(\partial h \cdot h)$ is the curvature form with respect to h. Define the equivariant curvature form $\Theta_{\mathfrak{g}}(h)$ by

$$\Theta_{\mathfrak{g}}(h) = \Theta(h) + \mu_{h,\xi}.$$

Then, $\Theta_{\mathfrak{g}}(h)$ is $\bar{\partial}_{\mathfrak{g}}$-closed and defines an element in $H^{1,1}_{\mathfrak{g}}(M)$.

Now, let us consider the case when $E = L$ is a G-linearized ample line bundle. Then, $\mu_{h,\xi}$ is a complex-valued smooth function on M.

https://doi.org/10.1017/nmj.2016.16 Published online by Cambridge University Press

Definition 2.2. The function $\mu_{h,\xi}$ is said to be the holomorphy potential of the pair (h, ξ).

2.2 Kähler–Ricci soliton

Let M be an n-dimensional Fano manifold.

Definition 2.3. A Kähler metric ω on M is a Kähler–Ricci soliton if the metric ω solves the equation

$$\mathrm{Ric}(\omega) - \omega = L_V \omega \tag{2.1}$$

for some holomorphic vector field V on M.

If the pair (ω, V) is a Kähler–Ricci soliton, taking the imaginary part of (2.1) yields $L_{\mathrm{Im}(V)}\omega = 0$, so ω is invariant under the group action generated by $\mathrm{Im}(V)$. More generally, we have the following proposition.

Proposition 2.4. [BN14, Lemma 2.13] *Let M be a Fano manifold, and let V be a holomorphic vector field on M. If there exists a Kähler metric ω that is invariant under the action of* $\mathrm{Im}(V)$*, then there exists a complex torus T_c acting holomorphically on M such that* $\mathrm{Im}(V)$ *may be identified with an element in the Lie algebra of the corresponding real torus $T \subset T_c$.*

Proof. First, we check that the isometry group K of ω is a compact Lie group. This is shown by considering the canonical embedding $M \hookrightarrow H^0(M, -kK_M)$ and the K-invariant Hilbert norm

$$\|s\|^2 := \int_M |s|_k^2 \omega^n \quad (s \in H^0(M, -kK_M)).$$

Actually, K is identified with a subgroup of the group consisting of unitary transformations on $H^0(M, -kK_M)$ with respect to $\|\cdot\|$, which yields that K is compact. Taking the topological closure of the 1-parameter subgroup generated by $\mathrm{Im}(V)$ in K, we get a real torus T as desired. In general, any holomorphic action of a real torus on M can be naturally extended to the corresponding complex torus action on M. ∎

2.3 Modified Futaki invariant

Let M be an n-dimensional Fano variety. For simplicity, let us make the following assumptions.

(1) M is a compact subvariety of a projective manifold N.
(2) L is an ample line bundle on N such that on the regular part M_{reg} of M the isomorphism

$$L|_{M_{\mathrm{reg}}} \simeq -kK_{M_{\mathrm{reg}}} \tag{2.2}$$

holds for some integer k.
(3) The Lie group $G := \mathrm{Aut}(M)$ acts on (N, L) such that the isomorphism (2.2) is G-equivariant.

REMARK 2.5. In fact, M can be embedded into

$$\mathbb{C}P^N \simeq \mathbb{P}H^0(M, -kK_M)^*$$

for a sufficient large k, and $(\mathbb{C}P^N, \mathcal{O}(1))$ satisfies the requirement above.

We say that V is a holomorphic vector field on a Fano variety M if V is a holomorphic vector field defined only on its regular part M_{reg}. Then, V induces a local one-parameter family of automorphisms, which extends to a family of G since $\mathrm{codim}(M \backslash M_{\mathrm{reg}}) \geqslant 2$ by the normality of M (cf. [BBEGZ11, Lemma 5.2]). Thus, by the assumption (3), V is given as the restriction of some holomorphic vector field on N to M.[1]

DEFINITION 2.6. A Hermitian metric h on $-K_{M_{\mathrm{reg}}}$ is said to be admissible if h^k can be extended to a Hermitian metric $\tilde{h}_L$ on L over N under the isomorphisms (2.2).

Let h be an admissible Hermitian metric on $-K_{M_{\mathrm{reg}}}$, and put $\omega := -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$. For holomorphic vector fields V, W, we define the function $\mathcal{F}$ as

$$\mathcal{F}(V) = -\frac{1}{c_1(M)^n}\int_{M_{\mathrm{reg}}} e^{\mu_{h,V}}\omega^n \tag{2.3}$$

and the modified Futaki invariant Fut_V by

$$\mathrm{Fut}_V(W) = \frac{d}{dt}\mathcal{F}(V+tW)\bigg|_{t=0} = -\frac{1}{c_1(M)^n}\int_{M_{\mathrm{reg}}} \mu_{h,W}e^{\mu_{h,V}}\omega^n, \tag{2.4}$$

where $\mu_{h,V}$ denotes the holomorphy potential of (h, V) defined on M_{reg}. Since the construction of equivariant Chern curvature form is local,

[1]Such a vector field was called an *admissible vector field* in [DT92, Definition 1.2]. However, the above argument implies that every holomorphic vector field on M_{reg} is automatically admissible (see also [BBEGZ11, Remark 5.3]).

if $i\colon M_{\mathrm{reg}} \hookrightarrow N$ is the embedding, we obtain

$$\begin{aligned}\mathcal{F}(V) &= -\frac{1}{c_1(M)^n}\int_{M_{\mathrm{reg}}} P(\Theta_{\mathfrak{g}}(h, -K_{M_{\mathrm{reg}}}))\\ &= -\frac{1}{c_1(M)^n}\int_{M_{\mathrm{reg}}} P\left(i^*\frac{\Theta_{\mathfrak{g}}(\tilde{h}_L, L)}{k}\right)\\ &= -\frac{1}{c_1(M)^n}\int_{M_{\mathrm{reg}}} P\left(\frac{\Theta_{\mathfrak{g}}(\tilde{h}_L, L)}{k}\right),\end{aligned}$$

where $P(z) := n!e^z$, and this shows that the integral (2.3) is finite. Moreover, using the equivariant Chern–Weil theorem, we can show the following.

THEOREM 2.7. [Hou08, Section 2.3] *The functions $\mathcal{F}$ and Fut_V are independent of the embedding $M \hookrightarrow N$ and the choice of an admissible Hermitian metric h on $-K_{M_{\mathrm{reg}}}$.*

On the other hand, a pluripotential theoretical formulation of Fut_V was introduced by Berman and Nyström [BN14]. They also introduced the quantized version of the modified Futaki invariant, which is defined more algebraically in terms of the commuting action on the cohomology $H^0(M, -kK_M)$. Let V be a holomorphic vector field on M generating a torus action, and put

$$N_k := \dim(H^0(M, -kK_M)).$$

We define the quantization of the function $\mathcal{F}$ at level k as

$$\mathcal{F}_k(V) := -k\mathrm{Trace}(e^{V/k})_{H^0(M,-kK_M)} = -k\sum_{i=1}^{N_k}\exp(v_i^{(k)}/k), \tag{2.5}$$

where $(v_i^{(k)})$ are the joint eigenvalues for the action of $\mathrm{Re}(V)$ on $H^0(M, -kK_M)$ defined by the canonical lift of V to $-K_M$. Additionally, let W be a holomorphic vector field on M generating a $\mathbb{C}^*$-action and commuting with V. We define the quantization of $\mathrm{Fut}_V(W)$ at level k as

$$\mathrm{Fut}_{V,k}(W) := \frac{d}{dt}\mathcal{F}_k(V + tW)\bigg|_{t=0} = -\sum_{i=1}^{N_k}\exp(v_i^{(k)}/k)w_i^{(k)}, \tag{2.6}$$

where $(v_i^{(k)}, w_i^{(k)})$ are the joint eigenvalues for the commuting action of $\mathrm{Re}(V)$ and $\mathrm{Re}(W)$. Then, we have the following.

PROPOSITION 2.8. *In the case when M is smooth, we have the following.*

(1) *We have the asymptotic expansion of $\mathcal{F}_k(V)$ as $k \to \infty$:*

$$\mathcal{F}_k(V) = \mathcal{F}^{(0)}(V) \cdot k^{n+1} + \mathcal{F}^{(1)}(V) \cdot k^n + \cdots ,$$

where $\mathcal{F}^{(0)}(V)$ is proportional to $\mathcal{F}(V)$.

(2) *We have the asymptotic expansion of $\mathrm{Fut}_{V,k}(W)$ as $k \to \infty$:*

$$\mathrm{Fut}_{V,k}(W) = \mathrm{Fut}_V^{(0)}(W) \cdot k^{n+1} + \mathrm{Fut}_V^{(1)}(W) \cdot k^n + \cdots ,$$

where $\mathrm{Fut}_V^{(i)}(W)$ is the ith-order modified Futaki invariant defined in [BN14, Section 4.4], and $\mathrm{Fut}_V^{(0)}(W)$ is proportional to $\mathrm{Fut}_V(W)$.

(3) *The ith-order modified Futaki invariant $\mathrm{Fut}_V^{(i)}(W)$ is the Gâteaux differential of $\mathcal{F}^{(i)}$ at V in the direction W, that is,*

$$\frac{d}{dt}\mathcal{F}_k^{(i)}(V + tW)\Big|_{t=0} = \mathrm{Fut}_V^{(i)}(W).$$

In general, when M is a (possibly singular) Fano variety, we have the following.

(4)
$$\mathcal{F}(V) = \lim_{k\to\infty} \frac{1}{kN_k}\mathcal{F}_k(V).$$

(5)
$$\mathrm{Fut}_V(W) = \lim_{k\to\infty} \frac{1}{kN_k}\mathrm{Fut}_{V,k}(W).$$

Proof. The statements (2) and (5) were shown in [BN14, Section 4.4]. The statement (3) is trivial from the definition of $\mathrm{Fut}_{k,V}(W)$.

(1) As with the proof of (2) (cf. [BN14, Section 4.4]) or [WZZ14, Lemma 1.2], $\mathcal{F}_k(V)$ can be calculated by the equivariant Riemann–Roch formula as

$$\begin{aligned}
\mathcal{F}_k(V) &= -k\mathrm{Trace}(e^{V/k})_{H^0(M,-kK_M)} \\
&= -k\int_M \mathrm{ch}^{\mathfrak{g}}(-kK_M)\mathrm{td}^{\mathfrak{g}}(M) \\
&= -k\int_M e^{\mu_{h,V}} \cdot e^{k\omega}\mathrm{td}^{\mathfrak{g}}(M) \\
&= -\frac{1}{n!}\int_M e^{\mu_{h,V}}\omega^n \cdot k^{n+1} + O(k^n),
\end{aligned}$$

https://doi.org/10.1017/nmj.2016.16 Published online by Cambridge University Press

where $\mathrm{ch}^{\mathfrak{g}}$ (respectively $\mathrm{td}^{\mathfrak{g}}$) denotes the equivariant Chern character (respectively the equivariant Todd class). Thus, $\mathcal{F}^{(0)}(V) = \frac{c_1(M)^n}{n!} \cdot \mathcal{F}(V)$.

(4) By definition, $\mathcal{F}(V)$ can be written as

$$\mathcal{F}(V) = -\frac{1}{c_1(M)^n} \int_M e^{\mu_{h,V}} \omega^n = -\int_{\mathbb{R}} e^v \nu^V,$$

where ν^V is the push-forward measure of the Monge–Ampère measure $\frac{\omega^n}{c_1(M)^n}$ under $\mu_{h,V}$. Let ν_k^V be the spectral measure on $\mathbb{R}$ attached to the infinitesimal action of $\mathrm{Re}(V)$ on $H^0(M, -kK_M)$:

$$\nu_k^V = \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{v_i^{(k)}/k},$$

where $\delta_{v_i^{(k)}/k}$ denotes the Dirac measure at $v_i^{(k)}/k$. Then, by [BN14, Proposition 4.1], ν_k^V converges to ν^V as $k \to \infty$ in a weak topology. Hence, we have

$$\frac{1}{kN_k}\mathcal{F}_k(V) = -\frac{1}{N_k} \sum_{i=1}^{N_k} \exp(v_i^{(k)}/k) = -\int_M e^v \nu_k^V \to -\int_{\mathbb{R}} e^v \nu^V = \mathcal{F}(V)$$

as $k \to \infty$. ∎

REMARK 2.9. When M is smooth, by the equivariant Riemann–Roch formula, we have an asymptotic expansion as $k \to \infty$:

$$N_k = \frac{1}{n!} c_1(M)^n \cdot k^n + O(k^{n-1}). \tag{2.7}$$

Combining with Proposition 2.8(1), we have

$$\frac{1}{kN_k}\mathcal{F}_k(V) = \mathcal{F}(V) + O(k^{-1}) \tag{2.8}$$

as $k \to \infty$. In general, when M is a (possibly singular) Fano variety, we do not know whether we can obtain the expansion (2.8). However, Proposition 2.8(4) allows us to use the equivariant Riemann–Roch formula formally to compute the leading term of (2.8) (i.e., the limit $\lim_{k\to\infty} \frac{1}{kN_k}\mathcal{F}_k(V)$) even if M has singularities.

§3. The calculation of the function $\mathcal{F}$

Let M be an n-dimensional variety in $\mathbb{C}P^N$, and let X be a holomorphic vector field on $\mathbb{C}P^N$. Then, X can be identified with a linear vector field $\sum_{i,j=0}^N a_{ij} z^i \frac{\partial}{\partial z^j}$ on $\mathbb{C}^{N+1}$, and the traceless matrix $(a_{ij})_{0\leqslant i,j\leqslant N} \in \mathfrak{sl}(N+1,\mathbb{C})$, such that the push-forward of $\sum_{i,j=0}^N a_{ij} z^i \frac{\partial}{\partial z^j}$ with the standard projection $\pi : \mathbb{C}^{N+1} - \{0\} \to \mathbb{C}P^N$ is equal to X.

For a holomorphic vector field X, we define a complex-valued smooth function on $\mathbb{C}^{N+1} - 0$ by

$$\theta_X := X\left(\log\left(\sum_{i=0}^N |z^i|^2\right)\right), \tag{3.1}$$

which descends to a smooth function on $\mathbb{C}P^N$. Let

$$\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{i=1}^N |z^i|^2\right) \in c_1(\mathcal{O}(1))$$

be the Fubini–Study metric of $\mathbb{C}P^N$. Then, we have

$$i_X\omega = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\theta_X. \tag{3.2}$$

We say that X is tangent to M if $\mathrm{Re}(X)$ leaves M invariant. If M is a hypersurface defined by a homogeneous polynomial F of degree d, X is tangent to M if and only if X fixes $[F] \in \mathbb{P}(H^0(M,\mathcal{O}(d)))$, or, equivalently, $XF = \gamma F$ for some constant γ. For any X that is tangent to M, equation (3.2) can be written as

$$X^i = g^{i\bar{j}}\frac{\partial\theta_X}{\partial x^{\bar{j}}} \quad (i = 1,\ldots,n), \qquad X = \sum_{i=1}^n X^i\frac{\partial}{\partial x^i} \tag{3.3}$$

at some smooth point in local holomorphic coordinates $(x^1,\ldots,x^n)$ of M, where $(g_{i\bar{j}})$ is the matrix of ω.

Now, let M be a Fano complete intersection in $\mathbb{C}P^N$ defined by the homogeneous polynomials $F_1,\ldots,F_s$ of degree $d_1,\ldots,d_s$ respectively, and suppose that $m\omega \in c_1(M)$ for some constant $m > 0$. Let X be a holomorphic vector field tangent to M, and let G be the Lie group generated by X. Using the adjunction formula, we know that $m = N+1-d_1-\cdots-d_s$ and

$$-K_{M_{\mathrm{reg}}} \simeq \mathcal{O}(m)|_{M_{\mathrm{reg}}}, \tag{3.4}$$

where we remark that this isomorphism is not G-equivariant. However, studying the G-action on the normal bundle of M, Hou [Hou08, Section 3] (also refer to [Lu99, Theorem 4.1]) showed the following.

LEMMA 3.1. *Let h be the Hermitian metric on $\mathcal{O}(1)$ such that $\omega = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ is a Fubini–Study metric of $\mathbb{C}P^N$, and let V be a holomorphic vector field such that*

$$VF_i = \alpha_i F_i$$

for some constants α_i $(i = 1, \dots, s)$. Then, we have

$$\mu_{h^m,V} = \sum_{i=1}^{s} \alpha_i + m\theta_V, \tag{3.5}$$

where h^m is the Hermitian metric on $-K_{M_{\mathrm{reg}}}$ defined via the isomorphism (3.4).

Let V be a holomorphic vector field defined in Lemma 3.1. We set

$$N_i := \{F_i = 0\} \subset \mathbb{C}P^N \quad (i = 1, \dots, s)$$

and $M_i := N_1 \cap \cdots \cap N_i$ $(i = 1, \dots, s)$. Then, we have

$$M = M_s \subset M_{s-1} \subset \cdots \subset M_1 \subset M_0 := \mathbb{C}P^N.$$

We define the integrals $I_{k,l} = I_{k,l}(V)$ $(k = 0, 1, \dots, s;\ l \geqslant 0)$ by

$$I_{k,l} = m^l \int_{M_k} (\theta_V)^l e^{m\theta_V} \omega^{N-k}. \tag{3.6}$$

LEMMA 3.2. *For $k = 1, \dots, s$, $I_{k,0}$ satisfies*

$$I_{k,0} = \left(d_k - \frac{m\alpha_k}{N-k+1}\right) I_{k-1,0} + \frac{d_k}{N-k+1} I_{k-1,1}. \tag{3.7}$$

Proof. We can prove (3.7) in the same way as [Lu99, Lemma 5.1]. Define a smooth function ξ_i $(i = 1, \dots, s)$ on $\mathbb{C}P^N$ by

$$\xi_i = \frac{|F_i|^2}{\left(\sum_{i=0}^{N} |z^i|^2\right)^{d_i}}.$$

Using the Poincaré–Lelong formula, we obtain

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log \xi_k = [N_k] - d_k\omega,$$

https://doi.org/10.1017/nmj.2016.16 Published online by Cambridge University Press

where $[N_k]$ is the divisor of the zero locus of F_k. Then, we have

$$\begin{aligned} I_{k,0} &= \int_{M_k} e^{m\theta_V}\omega^{N-k} \\ &= \int_{M_{k-1}} \left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k + d_k\omega\right)\wedge e^{m\theta_V}\omega^{N-k} \\ &= \int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k \wedge e^{m\theta_V}\omega^{N-k} + d_k I_{k-1,0}. \end{aligned}$$

On the other hand, using the relation

$$V\log\xi_k = \alpha_k - d_k\theta_V$$

and integrating by parts, we obtain

$$\begin{aligned} &\int_{M_{k-1}} \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\xi_k \wedge e^{m\theta_V}\omega^{N-k} \\ &\quad = -\frac{m}{N-k+1}\int_{M_{k-1}} V(\log\xi_k)e^{m\theta_V}\omega^{N-k+1} \\ &\quad = -\frac{m\alpha_k}{N-k+1}I_{k-1,0} + \frac{d_k}{N-k+1}I_{k-1,1}. \end{aligned}$$

Thus, we get the desired result. ∎

If we set $V \equiv 0$ and $l = 0$, then we obtain the following.

Corollary 3.3.

$$c_1(M)^{N-s}\left(= m^{N-s}\int_M \omega^{N-s}\right) = d_1\cdots d_s m^{N-s}. \tag{3.8}$$

In order to get the explicit expression of $I_{k,0}$, we show the next lemma.

Lemma 3.4. *For $k = 1, \ldots, s$, the equation*

$$\begin{aligned} &\frac{(N-k)!}{m^{N-k}}\int_{\mathbb{C}P^N}\prod_{i=1}^{k}(d_i\omega + d_i\theta_V - \alpha_i)e^{m\theta_V}\cdot e^{m\omega} \\ &\quad + \frac{(N-k-1)!}{m^{N-k}}\sum_{i=1}^{k}\int_{\mathbb{C}P^N}(d_i\theta_V - \alpha_i) \end{aligned}$$

https://doi.org/10.1017/nmj.2016.16 Published online by Cambridge University Press

$$\cdot \prod_{p\in\{1,\ldots,k\}-\{i\}} (d_p\omega + d_p\theta_V - \alpha_p)e^{m\theta_V} \cdot e^{m\omega}$$

$$(3.9) \qquad = \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) \cdot \omega \cdot e^{m\theta_V} \cdot e^{m\omega}$$

holds.

Proof. For $i = 0, \ldots, k$, we define integrals J_i by

$$J_i := \begin{cases} \displaystyle\int_{\mathbb{C}P^N} \prod_{i=1}^{k} (d_i\theta_V - \alpha_i)e^{m\theta_V}\omega^N & \text{(when } i = 0\text{)}, \\ \displaystyle d_1 \cdots d_k \int_{\mathbb{C}P^N} e^{m\theta_V}\omega^N & \text{(when } i = k\text{)}, \\ \displaystyle\sum_{1\leqslant p_1<\cdots<p_i\leqslant k} d_{p_1} \cdots d_{p_i} \int_{\mathbb{C}P^N} (d_{q_1}\theta_V - \alpha_{q_1}) \\ \quad \times \cdots (d_{q_{k-i}}\theta_V - \alpha_{q_{k-i}})e^{m\theta_V}\omega^N & \text{(otherwise)}, \end{cases}$$

where $q_1 < \cdots < q_{k-i}$ and $\{q_1, \ldots, q_{k-i}\} = \{1, \ldots, k\} - \{p_1, \ldots, p_i\}$. Then, the direct computation shows that

$$\frac{(N-k)!}{m^{N-k}} \int_{\mathbb{C}P^N} \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i)e^{m\theta_V} \cdot e^{m\omega} = \sum_{i=0}^{k} \frac{(N-k)!m^{k-i}}{(N-i)!} J_i$$

and

$$\frac{(N-k-1)!}{m^{N-k}} \sum_{i=1}^{k} \int_{\mathbb{C}P^N} (d_i\theta_V - \alpha_i) \cdot \prod_{p\in\{1,\ldots,k\}-\{i\}} (d_p\omega + d_p\theta_V - \alpha_p)e^{m\theta_V} \cdot e^{m\omega}$$
$$= \sum_{i=0}^{k} \frac{(N-k-1)!(k-i)m^{k-i}}{(N-i)!} J_i.$$

Hence, the left-hand side of (3.9) is

$$\sum_{i=0}^{k} \frac{(N-k-1)!m^{k-i}}{(N-i-1)!} J_i,$$

which is equal to the right-hand side of (3.9). ∎

LEMMA 3.5. *For $k = 1, \ldots, s$, $I_{k,0}$ can be written as*

$$I_{k,0} = \frac{(N-k)!}{m^{N-k}} \int_{\mathbb{C}P^N} \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}. \tag{3.10}$$

Proof. We will prove (3.10) by induction for k. When $k = 1$, equation (3.10) coincides exactly with (3.7), so the statement holds.

Next, we assume that (3.10) holds for a fixed k. Then, by Lemma 3.2, we have

$$I_{k+1,0} = \left(d_{k+1} - \frac{m\alpha_{k+1}}{N-k} \right) I_{k,0} + \frac{d_{k+1}}{N-k} I_{k,1}.$$

Since $\theta_{V+tV} = \theta_V + t\theta_V$, $(V + tV)F_i = (\alpha_i + t\alpha_i)F_i$ and

$$\frac{d}{dt}(d_i\omega + d_i\theta_{V+tV} - \alpha_i - t\alpha_i)\Big|_{t=0} = d_i\theta_V - \alpha_i,$$

using the induction hypothesis, we have

$$\frac{m\alpha_{k+1}}{N-k} I_{k,0} = \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} \alpha_{k+1} \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}$$

and

$$\begin{aligned} \frac{d_{k+1}}{N-k} I_{k,1} &= \frac{d_{k+1}}{N-k} \cdot \frac{d}{dt} I_{k,0}(V + tV)\Big|_{t=0} \\ &= d_{k+1} \frac{(N-k-1)!}{m^{N-k}} \sum_{i=1}^{k} \int_{\mathbb{C}P^N} (d_i\theta_V - \alpha_i) \\ &\quad \times \prod_{p \in \{1,\ldots,k\} - \{i\}} (d_p\omega + d_p\theta_V - \alpha_p) e^{m\theta_V} \cdot e^{m\omega} \\ &\quad + \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} d_{k+1}\theta_V \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}. \end{aligned}$$

Hence, combining with Lemma 3.4, we obtain

$$\begin{aligned} I_{k+1,0} = d_{k+1} \text{ (the LHS of (3.9))} + \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} (d_{k+1}\theta_V - \alpha_{k+1}) \\ \times \prod_{i=1}^{k} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega} \end{aligned}$$

$$= \frac{(N-k-1)!}{m^{N-k-1}} \int_{\mathbb{C}P^N} \prod_{i=1}^{k+1} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega}.$$

Hence, the statement holds for $k+1$. ∎

Proof of Theorem 1.1. By Lemma 3.1, $\mathcal{F}$ can be written as

$$\begin{aligned} \mathcal{F}(V) &= -\frac{1}{c_1(M)^{N-s}} \int_M \exp\left(\sum_{i=1}^s \alpha_i + m\theta_V\right) (m\omega)^{N-s} \\ &= -\frac{m^{N-s}}{c_1(M)^{N-s}} \cdot \exp\left(\sum_{i=1}^s \alpha_i\right) I_{s,0}. \end{aligned}$$

Thus, combining with Corollary 3.3 and Lemma 3.5, we get the desired formula for $\mathcal{F}$. ∎

§4. Another proof of Theorem 1.1

In this section, we give another proof of Theorem 1.1 using the algebraic formula for $\mathcal{F}$ (cf. Proposition 2.8).

LEMMA 4.1. [AV11, Lemma 5.1] *Let B be a holomorphic vector bundle of rank b on a manifold M, then*

$$\sum_{i=0}^b (-1)^i \mathrm{ch}(\wedge^i B) = c_b(B)\mathrm{td}(B)^{-1}.$$

Proof. Let $r_1, \dots, r_b$ be the Chern roots of B. Since $\mathrm{ch}(\wedge^i B^*) = \sum_{1 \leqslant p_1 < \cdots < p_i \leqslant b} e^{-(r_{p_1} + \cdots + r_{p_i})}$, we obtain

$$\begin{aligned} \sum_{i=0}^b (-1)^i \mathrm{ch}(\wedge^i B^*) &= \sum_{i=0}^b (-1)^i \sum_{1 \leqslant p_1 < \cdots < p_i \leqslant b} e^{-(r_{p_1} + \cdots + r_{p_i})} \\ &= \prod_{p=1}^b (1 - e^{-r_p}) \\ &= \prod_{p=1}^b r_p \prod_{p=1}^b \frac{1 - e^{-r_p}}{r_p} \\ &= c_b(B)\mathrm{td}(B)^{-1}. \end{aligned}$$

∎

Now, let M be an $(N-s)$-dimensional Fano complete intersection in $\mathbb{C}P^N$, that is, M is a Fano variety in $\mathbb{C}P^N$ defined by homogeneous polynomials $F_1, \ldots, F_s$, and V is a holomorphic vector field on $\mathbb{C}P^N$ tangent to M. We adopt the notation of Section 3. We further assume that $V \in \mathfrak{sl}(N+1, \mathbb{C})$ is a Hermitian matrix, so that $\mathrm{Im}(V)$ is Killing with respect to the Fubini–Study metric ω.

LEMMA 4.2. [AV11, Lemma 5.2] *We have the following asymptotic expansion of N_k as $k \to \infty$:*

$$N_k = \frac{d_1 \cdots d_s m^{N-s}}{(N-s)!} \cdot k^{N-s} + O(k^{N-s-1}). \tag{4.1}$$

LEMMA 4.3. *We have the following asymptotic expansion of $\mathcal{F}_k(V)$ as $k \to \infty$:*

$$\mathcal{F}_k(V) = -\exp\left(\sum_{i=1}^{s} \alpha_i\right) \int_{\mathbb{C}P^N} \prod_{i=1}^{s} (d_i\omega + d_i\theta_V - \alpha_i) e^{m\theta_V} \cdot e^{m\omega} \cdot k^{N-s+1} + O(k^{N-s}). \tag{4.2}$$

Proof. This proof is essentially based on the argument in [AV11, Lemma 5.3]. The only difference between Lemma 4.3 and [AV11, Lemma 5.3] is the linearization of $-K_M$, to which we have only to pay attention. In order to avoid confusion, let $L(\simeq O(m))$ be a linearized line bundle on $\mathbb{C}P^N$ such that $L|_M$ is isomorphic to $-K_M$ as a linearized line bundle whose linearization is determined by the canonical lift of V/k to $-K_M$.

Let $\mathbb{C}_{-\alpha_i/k}$ be a trivial bundle on $\mathbb{C}P^N$ with linearization $t \cdot u = t^{-\alpha_i/k} \cdot u$. Set $L_i := \mathcal{O}(d_i) \otimes \mathbb{C}_{-\alpha_i/k}$ and $B := L_1 \oplus \cdots \oplus L_s$. Then, rank $B = s$, and the section $F := (F_1, \ldots, F_s) \in H^0(\mathbb{C}P^N, B)$ is invariant. Since M is complete, the Koszul complex

$$0 \to \wedge^s B^* \to \wedge^{s-1} B^* \to \cdots \to B^* \to \mathcal{O}_{\mathbb{C}P^N} \to \mathcal{O}_M \to 0$$

is exact and equivariant, where $\mathcal{O}_M$ denotes the structure sheaf of M. Tensoring by L^k preserves the exactness and equivariance, so we obtain

$$\chi^{\mathfrak{g}}(M, L^k|_M) = \sum_{i=0}^{s} (-1)^i \chi^{\mathfrak{g}}(\mathbb{C}P^N, L^k \otimes \wedge^i B^*),$$

where $\chi^{\mathfrak{g}}$ denotes the Lefschetz number. By the equivariant Riemann–Roch formula and Lemma 4.1, we get

$$\begin{aligned}\mathcal{F}_k(V) &= -k\sum_{i=0}^{s}(-1)^i\chi^{\mathfrak{g}}(\mathbb{C}P^N, L^k\otimes\wedge^i B^*)\\ &= -k\sum_{i=0}^{s}(-1)^i\int_{\mathbb{C}P^N}\mathrm{ch}^{\mathfrak{g}}(\wedge^i B^*)e^{kc_1^{\mathfrak{g}}(L)}\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N)\\ &= -k\int_{\mathbb{C}P^N}\left(\sum_{i=0}^{s}(-1)^i\mathrm{ch}^{\mathfrak{g}}(\wedge^i B^*)\right)e^{kc_1^{\mathfrak{g}}(L)}\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N)\\ &= -k\int_{\mathbb{C}P^N}c_s^{\mathfrak{g}}(B)\mathrm{td}^{\mathfrak{g}}(B)^{-1}e^{kc_1^{\mathfrak{g}}(L)}\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N)\\ &= -k\int_{\mathbb{C}P^N}\prod_{i=1}^{s}\left(d_ic_1^{\mathfrak{g}}(\mathcal{O}(1))-\frac{\alpha_i}{k}\right)\cdot\mathrm{td}^{\mathfrak{g}}(B)^{-1}e^{kc_1^{\mathfrak{g}}(L)}\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N).\end{aligned}$$

Let h be a Hermitian metric on $\mathcal{O}(1)$ such that $\omega=-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log h$ is the Fubini–Study metric of the $\mathbb{C}P^N$. Then, by Lemma 3.1, the equivariant 1st Chern forms for $(h, V/k)$ and $(h^m, V/k)$ are written as

$$\omega+\frac{1}{k}\theta_V\in c_1^{\mathfrak{g}}(\mathcal{O}(1))\qquad\text{and}\qquad m\omega+\frac{m}{k}\theta_V+\frac{1}{k}\sum_{i=1}^{s}\alpha_i\in c_1^{\mathfrak{g}}(L)$$

respectively. Both $\mathrm{td}^{\mathfrak{g}}(B)^{-1}$ and $\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N)$ can be written as the form

$$1+A+\sum_{i\geqslant 1}\frac{1}{k^i}B_i,$$

where A (respectively B_i) denotes $2l$-forms ($l\geqslant 1$ (respectively $l\geqslant 0$)) not depending on k. Hence, we have

$$\begin{aligned}\mathcal{F}_k(V) &= -k\exp\left(\sum_{i=1}^{s}\alpha_i\right)\int_{\mathbb{C}P^N}\prod_{i=1}^{s}\left(d_i\omega+\frac{1}{k}(d_i\theta_V-\alpha_i)\right)\\ &\qquad\times\mathrm{td}^{\mathfrak{g}}(B)^{-1}e^{m\theta_V}\cdot e^{km\omega}\mathrm{td}^{\mathfrak{g}}(\mathbb{C}P^N)\\ &= -\exp\left(\sum_{i=1}^{s}\alpha_i\right)\int_{\mathbb{C}P^N}\prod_{i=1}^{s}(d_i\omega+d_i\theta_V-\alpha_i)e^{m\theta_V}\\ &\qquad\cdot e^{m\omega}\cdot k^{N-s+1}+O(k^{N-s}).\end{aligned}$$

∎

https://doi.org/10.1017/nmj.2016.16 Published online by Cambridge University Press

Proof of Theorem 1.1. By Lemmas 4.2 and 4.3, we have an asymptotic expansion as $k \to \infty$:

$$\frac{1}{kN_k}\mathcal{F}_k(V) = -\frac{(N-s)!}{d_1 \cdots d_s m^{N-s}} \exp\left(\sum_{i=1}^{s} \alpha_i\right) \int_{\mathbb{C}P^N} \prod_{i=1}^{s}(d_i\omega + d_i\theta_V - \alpha_i) \times e^{m\theta_V} \cdot e^{m\omega} + O(k^{-1}).$$

On the other hand, by Proposition 2.8(4), $\frac{1}{kN_k}\mathcal{F}_k(V)$ converges to $\mathcal{F}(V)$ as $k \to \infty$. Hence, we have the desired formula. □

§5. Examples

In this section, we compute $\mathcal{F}$ for several examples in [Lu99, Section 6]. Let M be a Fano complete intersection in $\mathbb{C}P^N$. We adopt the notation of Section 3. First, we mention some results obtained as a corollary of the localization formula in holomorphic equivariant cohomology theory (cf. [Liu95, Theorem 1.6]).

LEMMA 5.1. *If $V = \mathrm{diag}(\lambda_0, \ldots, \lambda_N)$ is a diagonal matrix with different eigenvalues $\lambda_0, \ldots, \lambda_N$, then we have*

$$I_{0,0} = N! \sum_{i=0}^{N} \frac{e^{m\lambda_i}}{\prod_{p\in\{0,\ldots,N\}-\{i\}}(\lambda_i - \lambda_p)}. \tag{5.1}$$

Since the $I_{0,l}$ are given by the derivatives of $I_{0,0}$, we can compute $I_{0,l}$ for any integer l. On the other hand, by Theorem 1.1, $\mathcal{F}(V)$ can be written as a linear combination of $I_{0,l}$ $(0 \leqslant l \leqslant s)$. Hence, we can express $\mathcal{F}(V)$ in terms of the eigenvalues of V.

However, we can calculate $\mathcal{F}(V)$ without using Theorem 1.1 in a special case. We assume that M has at worst orbifold singularities, and V satisfies the following conditions.

(1) V has isolated zero points $\{p_i\}$.
(2) V is nondegenerate at each zero point p_i, that is, for each local uniformization $\pi : U \to U/\Gamma_i \subset M$ with $\pi(U) \cap p_i \neq \emptyset$, π^*V vanishes along $\pi^{-1}(p_i)$ and the matrix $B_i = \left(-\frac{\partial v^i_j}{\partial z^k}\right)_{1\leqslant j,k\leqslant N-s}$ is nondegenerate near $\pi^{-1}(p_i)$, where $(z^1, \ldots, z^{N-s})$ are local holomorphic coordinates around $\pi^{-1}(p_i)$ and $V = \sum_{j=1}^{N-s} v^i_j \frac{\partial}{\partial z^j}$.

In the same way as [DT92, Proposition 1.2], we have the following lemma.

LEMMA 5.2. *Let M and V be as above. Then, we have*

$$\mathcal{F}(V) = -\frac{(N-s)!}{d_1 \cdots d_s} \exp\left(\sum_{i=1}^{s} \alpha_i\right) \cdot \sum_i \frac{1}{|\Gamma_i|} \cdot \frac{e^{m\theta_V(p_i)}}{\det B_i}, \tag{5.2}$$

where $|\Gamma_i|$ is the order of the local uniformization group Γ_i at a point p_i.

REMARK 5.3. One can extend Lemmas 5.1 and 5.2 to the case when the zero set of V is the sum of nondegenerate submanifolds, where the word *nondegenerate* means that the induced actions of V to the normal bundle of submanifolds are nondegenerate. However, since $I_{0,0}(V)$ and $\mathcal{F}(V)$ are clearly continuous with respect to V, we may think that equations (5.1) and (5.2) hold in the sense of the limit $V_\epsilon \to V$ of any expression. For instance, we have the following lemma.

LEMMA 5.4. *Let $m = 1$, and let $V = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_2) \in \mathfrak{sl}(4, \mathbb{C})$ be a holomorphic vector field on $\mathbb{C}P^3$, where λ_0, λ_1 and λ_2 are different numbers. Then, we have*

$$\begin{aligned} I_{0,0} = 6 \Bigg[&\frac{e^{\lambda_0}}{(\lambda_0-\lambda_1)(\lambda_0-\lambda_2)^2} + \frac{e^{\lambda_1}}{(\lambda_1-\lambda_0)(\lambda_1-\lambda_2)^2} \\ &+ \frac{\{\lambda_0+\lambda_1-2\lambda_2+(\lambda_2-\lambda_0)(\lambda_2-\lambda_1)\}e^{\lambda_2}}{(\lambda_2-\lambda_0)^2(\lambda_2-\lambda_1)^2} \Bigg]. \end{aligned} \tag{5.3}$$

Proof. Let $\epsilon \neq 0$ be a small number. If we set $V_\epsilon := \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2 + \epsilon, \lambda_2 - \epsilon)$, then V_ϵ has different eigenvalues. Hence, we can compute $I_{0,0}(V) = \lim_{\epsilon\to 0} I_{0,0}(V_\epsilon)$ directly using (5.1). □

EXAMPLE 5.5. Let $M \subset \mathbb{C}P^3$ be the zero set of a cubic polynomial $F := z_0 z_1^2 + z_2 z_3(z_2 - z_3)$, where (z_0, z_1, z_2, z_3) are homogeneous coordinates of $\mathbb{C}P^3$, and let $V = \mathrm{diag}(-7t, 5t, t, t)$ $(t \neq 0)$ be a holomorphic vector field tangent to M. We compute $\mathcal{F}$ by two methods.

(1) The variety M has a unique quotient singularity at $p_0 := [1, 0, 0, 0]$. If we restrict V to M, V has five zeros, $p_0 = [1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $[0, 0, 0, 1]$ and $[0, 0, 1, 1]$. Let $\zeta_i := \frac{z_i}{z_0}$ $(i = 1, 2, 3)$ be Euclidean coordinates defined near p_0. Then, we can rewrite F near p_0 in the standard form

$$f = \frac{F}{z_0^3} = \zeta_1^2 - \zeta_3(\zeta_2^2 - 4\zeta_3^2).$$

https://doi.org/10.1017/nmj.2016.16 Published online by Cambridge University Press

According to [Lu99, Example 1], we see that there is a uniformization $\phi : \mathbb{C}^2 \to \mathbb{C}^2/\Gamma \subset M$ defined by

$$\phi : \begin{cases} \zeta_1 = uv(u^4 - v^4), \\ \zeta_2 = u^4 + v^4, \\ \zeta_3 = u^2 v^2, \end{cases}$$

where Γ is the dihedral subgroup in $SU(2)$ of type D_4. Thus, we have $\phi^*(V) = 2tu\frac{\partial}{\partial u} + 2tv\frac{\partial}{\partial v}$. Since the order of the group D_4 is 8, applying Lemma 5.2, we obtain

$$\begin{aligned} \mathcal{F}(V) &= -\frac{2}{3}e^{3t}\left(\frac{1}{8}\cdot\frac{e^{-7t}}{4t^2} + \frac{e^{5t}}{16t^2} + 3\cdot\frac{e^t}{-32t^2}\right) \\ &= -\frac{e^{-4t}}{48t^2} - \frac{e^{8t}}{24t^2} + \frac{e^{4t}}{16t^2}. \end{aligned}$$

(2) By Theorem 1.1, we obtain

$$\begin{aligned} \mathcal{F}(V) &= -\frac{2}{3}e^{3t}\int_{\mathbb{C}P^3}(3\omega + 3\theta_V - 3t)e^{\theta_V}e^{\omega} \\ &= -e^{3t}\left\{\left(1 - \frac{t}{3}\right)I_{0,0} + \frac{1}{3}I_{0,1}\right\}. \end{aligned}$$

By Lemma 5.4, we have

$$I_{0,0} = -\frac{e^{-7t}}{128t^3} + \frac{e^{5t}}{32t^3} - \frac{3(1+8t)e^t}{128t^3}$$

and

$$I_{0,1} = \frac{(7t+3)e^{-7t}}{128t^3} + \frac{(5t-3)e^{5t}}{32t^3} - \frac{3(8t^2-15t-3)e^t}{128t^3}.$$

Hence, we have

$$\mathcal{F}(V) = -\frac{e^{-4t}}{48t^2} - \frac{e^{8t}}{24t^2} + \frac{e^{4t}}{16t^2}.$$

EXAMPLE 5.6. Let $M \subset \mathbb{C}P^4$ be the zero locus defined by

$$\begin{cases} F_1 = z_0 z_1 + z_2^2, \\ F_2 = z_1^2 + z_3 z_4, \end{cases}$$

and let $V = \mathrm{diag}(-7t, 3t, -2t, 5t, t)$ $(t \neq 0)$ be a holomorphic vector field tangent to M. In the same way as (2) in Example 5.5, we get

$$\mathcal{F}(V) = -e^{2t}\left\{\left(1 - \frac{t}{3} - \frac{t^2}{2}\right)I_{0,0} + \left(\frac{2}{3} - \frac{t}{12}\right)I_{0,1} + \frac{1}{12}I_{0,2}\right\},$$

$$I_{0,0} = \frac{e^{-7t}}{200t^4} - \frac{3e^{3t}}{25t^4} - \frac{24e^{-2t}}{525t^4} + \frac{e^{5t}}{28t^4} + \frac{e^t}{8t^4},$$

$$I_{0,1} = -\frac{(7t+4)e^{-7t}}{200t^4} + \frac{3(4-3t)e^{3t}}{25t^4} + \frac{48(t+2)e^{-2t}}{525t^4} + \frac{(5t-4)e^{5t}}{28t^4} + \frac{(t-4)e^t}{8t^4}$$

and

$$I_{0,2} = \frac{(49t^2+56t+20)e^{-7t}}{200t^4} - \frac{3(9t^2-24t+20)e^{3t}}{25t^4} - \frac{96(t^2+4t+5)e^{-2t}}{525t^4} + \frac{5(5t^2-8t+4)e^{5t}}{28t^4} + \frac{(t^2-8t+20)e^t}{8t^4}.$$

Hence, we have

$$\mathcal{F}(V) = -\frac{e^{-5t}}{48t^2} - \frac{e^{7t}}{24t^2} + \frac{e^{3t}}{16t^2}.$$

Here, we remark that V has only three zero points, $p_1 = [1, 0, 0, 0, 0]$, $p_2 = [0, 0, 0, 1, 0]$ and $p_3 = [0, 0, 0, 0, 1]$, in M. Actually, the exponents appearing in the above expression of $\mathcal{F}(V)$ are $-5t = \theta_V(p_1) + 2t$, $7t = \theta_V(p_2) + 2t$ and $3t = \theta_V(p_3) + 2t$, and hence correspond to the three zero points of V.

Acknowledgements. The author would like to express his gratitude to Professor Ryoichi Kobayashi for his advice on this article, and to the referee for useful suggestions that helped him to improve the original manuscript.

References

[AV11] C. Arezzo and A. D. Vedova, *On the K-stability of complete intersections in polarized manifolds*, Adv. Math. **226** (2011), 4796–4815.

[BBEGZ11] R. J. Berman, S. Boucksom, P. Eyssidieux, V. Guedj and A. Zeriahi, *Kähler–Einstein metrics and the Kähler–Ricci flow on log Fano varieties*, Preprint, 2011, arXiv:1111.7158.

[BGV92] N. Berline, E. Getzler and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences) 298 Springer, Berlin, 1992.

[BN14] R. J. Berman and D. W. Nyström, *Complex optimal transport and the pluripotential theory of Kähler–Ricci solitons*, Preprint, 2014, arXiv:1401.8264.

[CDS15] X. Chen, K. Donaldson and S. Sun, *Kähler–Einstein metrics on Fano manifolds, III: limit as cone angle approach 2π and completion of the main proof*, J. Amer. Math. Soc. **28** (2015), 235–278.

[DT92] W. Y. Ding and G. Tian, *Kähler–Einstein metrics and the generalized Futaki invariant*, Invent. Math. **110** (1992), 315–335.

[Fut83] A. Futaki, *An obstruction to the existence of Kähler–Einstein metrics*, Invent. Math. **73** (1983), 437–443.

[Hou08] Z. Hou, *Equivariant cohomology and holomorphic invariant*, Commun. Contemp. Math. **100** (2008), 433–447.

[Liu95] K. Liu, *Holomorphic equivariant cohomology*, Math. Ann. **303** (1995), 125–148.

[Lu99] Z. Lu, *On the Futaki invariants of complete intersections*, Duke Math. J. **100** (1999), 359–372.

[Lu03] Z. Lu, *A note on the holomorphic invariants of Tian–Zhu*, Port Math. **60** (2003), 263–268.

[PS04] D. H. Phong and J. Sturm, *The Futaki invariant and the Mabuchi energy of a complete intersection*, Comm. Anal. Geom. **12** (2004), 321–343.

[Tian15] G. Tian, *K-stability and Kähler–Einstein metrics*, Comm. Pure Appl. Math. **68** (2015), 1085–1156.

[TZ02] G. Tian and X. H. Zhu, *A new holomorphic invariant and uniqueness of Kähler–Ricci solitons*, Comment. Math. Helv. **77** (2002), 297–325.

[WZZ14] F. Wang, B. Zhou and X. Zhu, *Modified Futaki invariant and equivariant Riemann–Roch formula*, Preprint, 2014, arXiv:1408.3784.

Mathematical Institute
Tohoku University
6-3, Aoba, Aramaki, Aoba-ku
Sendai 980-8578
Japan

ryosuke.takahashi.a7@tohoku.ac.jp