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HIGHER ORDER SCHEMES AND RICHARDSON EXTRAPOLATION FOR SINGULAR PERTURBATION PROBLEMS

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Semilinear singular perturbation problems are solved numerically by using finite-difference schemes on non-equidistant meshes which are dense in the layers. The fourth order uniform accuracy of the Hermitian approximation is improved by the Richardson extrapolation.

1. Introduction

We consider the following singularly perturbed boundary value problem:

$$-\varepsilon^2 u'' + c(x,u) = 0,\ x \in I = [0,1],\ u(0) = u(1) = 0, \tag{1}$$

with a small parameter ε, $\varepsilon \in (0, \varepsilon_0)$. Our assumptions are

$$c \in C^8(I \times \mathbb{R}), \tag{2.1}$$

$$g(x) \leqslant c_u(x,u) \leqslant G(x),\ (x,u) \in I \times \mathbb{R}, \tag{2.2}$$

$$\delta := \min\{5g(x) - 2G(x) \colon x \in I\} > 0, \tag{2.3}$$

$$0 < \gamma^2 < g(x),\ |g'(x)| \leqslant L,\ |G'(x)| \leqslant L,\ x \in I. \tag{2.4}$$

It is well-known that under the given conditions there exists a unique solution, $u \in C^{10}(I \times \mathbb{R})$, to the problem (1), and that the following representation holds:

$$u(x) = v_0(x) + v_1(x) + y(x), \tag{3.1}$$

where

$$v_0(x) = \exp(-\gamma x/\varepsilon), \tag{3.2}$$

$$v_1(x) = \exp(\gamma(x-1)/\varepsilon), \tag{3.3}$$

and

$$\left|y^{(s)}(x)\right| \leqslant M,\ s = 0, 1, \dots, 8,\ x \in I, \tag{3.4}$$

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(see [6, 7]). Here and throughout the paper M denotes any positive constant independent of ε.

As well as in [6] and [7], problem (1) was solved numerically in [1, 5, 3, 4] - just to mention some of the papers. For other references see [3, 4].

Our aim is to solve (1) numerically by using classical finite difference schemes on special non-equidistant meshes which are dense in the layers of u, located at $x = 0$ and $x = 1$. The same approach can be found in the papers we have mentioned. In this paper we combine the methods from [7] and [4] to obtain high order convergence uniform in ε. In [7] Richardson extrapolation was applied to the central difference scheme and high accuracy uniform in ε was proved. In [4] (see [3] as well) the Hermite scheme was used and fourth order uniform convergence was proved. Here we shall apply Richardson extrapolation to the Hermite scheme. We shall give a proof of sixth order convergence uniform in ε. We believe that a general theory can be developed in the same way as in [7] and that even higher order uniform convergence can be obtained. Numerical experiments confirm this.

Conditions (2.2) and (2.3) are the same as in [3, 4] and they guarantee stability uniform in ε. In a forthcoming paper we shall avoid these constraints on the function c.

In Section 2 the discretisation is given and stability uniform in ε is proved. In Section 3 we give a representation of the consistency error, which justifies the use of Richardson extrapolation. We end the paper by giving some numerical results in Section 4.

The constants M will be independent of the discretisation mesh as well.

2. Discretisation

Let I_h be the discretisation mesh with the points:

$$x_i = \lambda(t_i),\ t_i = ih,\ i = 0, 1, \dots, n, h = \frac{1}{n},\ n = 2m,\ m \in \mathbb{N}, \tag{4.1}$$

$$\lambda(t) = \begin{cases} \omega(t) = \frac{a\varepsilon t}{q-t}, & t \in [0, \alpha], \\ \pi(t), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1], \end{cases} \tag{4.2}$$

where

$$\pi(t) = A(t-\alpha)^4 + \omega'''(\alpha)(t-\alpha)^3/6 + \omega''(\alpha)(t-\alpha)^2/2 + \omega'(\alpha)(t-\alpha) + \omega(\alpha). \tag{4.3}$$

The parameter α is

$$\alpha = t_k, \tag{4.4}$$

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for some $k \in \{1, 2, \dots, m-1\}$,

$$q = \alpha + \sqrt[4]{\varepsilon}. \tag{4.5}$$

and the coefficient A is determined from

$$\pi(0.5) = 0.5. \tag{4.6}$$

Moreover, the coefficient a should satisfy

$$\left(B + 2\varepsilon^{-1/4} q(0.5-\alpha)^4\right)^{-1} \leqslant a \leqslant B^{-1}, \tag{4.7}$$

where

$$B = 2\left(\varepsilon^{3/4}\alpha + \varepsilon^{1/2} q(0.5-\alpha) + \varepsilon^{1/4} q(0.5-\alpha)^2 + q(0.5-\alpha)^3\right). \tag{4.8}$$

We have $\lambda\colon I \to I$, $\lambda \in C^1(I)$, $\lambda \in C^3[0, 0.5]$, $\lambda \in C^\infty[0, \alpha]$, $\lambda \in C^\infty[\alpha, 0.5]$. The second inequality in (4.7) implies $A \geqslant 0$, so π''' is nondecreasing, and

$$\pi^{(s)}(t) \geqslant \pi^{(s)}(\alpha) = \omega^{(s)}(\alpha) > 0, \; s = 3, 2, 1, \; t \in [\alpha, 0.5].$$

At the same time

$$\omega^{(s)}(t) > 0, \; s = 1, 2, \dots, \quad t \in [0, q),$$

and taking (4.5) into account we get

$$0 < \lambda^{(s)}(t) \leqslant M, \; s = 1, 2, 3, \; t \in [0, 0.5]. \tag{5.1}$$

Moreover,

$$0 < \lambda^{(4)}(t) \leqslant M\varepsilon^{-1/4}, \quad t \in [0, 0.5] \setminus \{\alpha\}. \tag{5.2}$$

The first inequality in (4.7) means that

$$(\omega(t) - \pi(t))^{(4)} \geqslant 0, \quad t \in [\alpha, q),$$

and it follows that

$$\omega'(t) \geqslant \pi'(t), \quad t \in [\alpha, q). \tag{6}$$

The inequalities (5.1) and (6) will be used later on, as well as

$$\exp(-\gamma\lambda(t)/\varepsilon) \leqslant M \exp(-M/(q-t)), \quad t \in (0, q). \tag{7}$$

Let

$$Q = \frac{(S-1)}{S}, \; S = \sqrt{1+\sqrt{3}}.$$

If

$$P = \alpha + Q\sqrt[4]{\varepsilon} - \frac{2h}{Q} > 0,$$

we denote by j an index such that

$$t_{j-1} \leqslant P < t_j.$$

Let

$$I_h' = \{x_i \in I_h : j+1 \leqslant i \leqslant k+6 \text{ or } n-k-6 \leqslant i \leqslant n-j-1\}$$

(if $j > k+5$ then $I_h' = \emptyset$).

Now we discretise problem (1) on the mesh I_h by using the same scheme as in [**3**, **4**]:

(8.1) $$T_h w_0 = w_0 = 0,$$

(8.2)
$$T_h w_i = \varepsilon^2(a_1(i)w_{i-1} + a_0(i)w_i + a_2(i)w_{i+1}) + b_1(i)c_{i-1} + b_0(i)c_i + b_2(i)c_{i+1} = 0$$
$$i = 1, 2, \ldots, n-1,$$

(8.3) $$T_h w_n = w_n = 0,$$

where $T_h w_i = (T_h w_h)_i$, $w_h = [w_0, w_1, \ldots, w_n]^T \in \mathbb{R}^{n+1} (w_i = w_{h,i})$ is a mesh function on I_h,

$$c_s = c(x_s, w_s), \; s = i-1, i, i+1$$
$$a_1(i) = \frac{-2}{h_i(h_i+h_{i+1})}, \; a_2(i) = \frac{-2}{h_{i+1}(h_i+h_{i+1})}, \; a_0(i) = \frac{2}{h_i h_{i+1}},$$

$h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$, and

$$b_1(i) = \frac{-a_1(i)}{12}(h_i^2 - h_{i+1}^2 + h_i h_{i+1}), \; b_2(i) = \frac{-a_2(i)}{12}(h_{i+1}^2 - h_i^2 + h_i h_{i+1}),$$
$$b_0(i) = \frac{a_0(i)}{12}(h_i^2 + h_{i+1}^2 + 3h_i h_{i+1}), \; \text{ if } x_i \in I_h \setminus I_h';$$
$$b_0(i) = 1, \; b_1(i) = 0, \; b_2(i) = 0, \; \text{ if } x_i \in I_h'.$$

From now on we shall consider the mesh points in $(0, 0.5)$ only (that is x_i, $i = 1, 2, \ldots, m-1$) since the interval $[0.5, 1)$ can be treated analogously (note that $h_i = h_{n-i+1}$, $i = 1, 2, \ldots, m$).

As in [**3**, **4**] we have, for $x_i \in (I_h \setminus I'_h) \cap (0, 0.5)$,

$$\frac{1}{12} \leqslant b_2(i) \leqslant \frac{1}{6}, \quad b_2(i) \geqslant b_1(i), \quad b_0(i) \geqslant \frac{5}{6},$$

(9)
$$b_1(i) \geqslant -\frac{1}{6}$$

We shall prove (9) here, since this proof differs from the one in [**3**, **4**]. It is easy to see that (9) is equivalent to

$$\frac{h_{i+1}}{h_i} \leqslant S^2$$

and this inequality will follow from

(10)
$$\frac{\lambda'(t_{i+1})}{\lambda'(t_{i-1})} \leqslant S^2,$$

(see (5.1)). Now if $k + 6 < i(< m)$ we have

$$t_{i-1} \geqslant \alpha + 6h.$$

Let

$$p(t) = S^2\pi'(t) - \pi'(t + 2h), \, (t = t_{i-1}).$$

It follows that

$$p^{(s)}(t) \geqslant 0, \; s = 3, 2, 1, \quad t \in [\alpha + 6h, 0.5],$$

so (10) holds in this case. Let us now consider the case $i < j + 1$, that is,

(11)
$$t_{i-1} \leqslant P,$$

and $t_{i+1} < q$. If $t_{i-1} \leqslant \alpha$ from (11) we have

$$\frac{\omega'(t_{i+1})}{\omega'(t_{i-1})} = \left(\frac{q - t_{i-1}}{q - t_{i+1}}\right)^2 \leqslant S^2,$$

and (10) follows because of (6). If $t_{i-1} > \alpha$, the inequality

$$\frac{\omega'(t_{i+1})}{\omega'(\alpha)} \leqslant S^2$$

holds because of (11) and (10) is proved again.

Thus, in the same way as in [**4**] we can prove

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THEOREM 1. *Let (2.1)-(2.4) hold and let the discrete problem (8.1)-(8.3) be given on the mesh (4.1)-(4.8) with*

$$n > \frac{2L\pi'(1)}{\delta}. \tag{12}$$

Then the problem (8.1)-(8.3) has a unique solution w_h, which is a point of attraction of SOR-Newton and Newton-SOR methods with the relaxation parameter in $(0, 1]$. Moreover, for any $v_h^1, v_h^2 \in \mathbf{R}^{n+1}$, the following stability inequality holds:

$$\left\|v_h^1 - v_h^2\right\|_\infty \leqslant \sigma^{-1} \left\|T_h v_h^1 - T_h v_h^2\right\|_\infty, \tag{13}$$

where σ is a positive constant, independent of ε.

Remark. Note that the right-hand-side of (12) is bounded uniformly in ε.

3. Richardson extrapolation

Let us consider the consistency error

$$r_h = T_h u_h - T_h w_h = T_h u_h$$

where $u_h = [u(x_0), u(x_1), \ldots, u(x_n)]^T \in \mathbf{R}^{n+1}$ is the restriction of the solution u to the problem (1) on the mesh I_h. The components of the vector r_h are

$$r_0 = r_n = 0,$$

and for $i = 1, 2, \ldots, n$, if $x_i \in I_h \setminus I_h'$

(14.1)

$$\begin{aligned} r_i = \varepsilon^2 \Bigg[& \frac{u^{(5)}(x_i)}{180}(h_{i+1} - h_i)\left(2h_i^2 + 2h_{i+1}^2 + 5h_i h_{i+1}\right) \\ & + \frac{u^{(6)}(x_i)}{720}\left(3h_i^4 + 3h_{i+1}^4 - 7h_i^2 h_{i+1}^2 + 2h_i^3 h_{i+1} + 2h_i h_{i+1}^3\right) \\ & + \frac{u^{(7)}(x_i)}{7!}(h_{i+1} - h_i)\left(7h_i h_{i+1}\left(h_i^2 + h_{i+1}^2\right) + 5\left(h_i^4 + h_{i+1}^4\right) - 2h_i^2 h_{i+1}^2\right) \\ & - 2\frac{u^{(8)}(\vartheta_i^1)\left(h_i^7 + h_{i+1}^7\right)}{8!(h_i + h_{i+1})} + \frac{u^{(8)}(\vartheta_i^2) h_i^5 \left(h_i^2 - h_{i+1}^2 + h_i h_{i+1}\right)}{6 \cdot 6!(h_i + h_{i+1})} \\ & + \frac{u^{(8)}(\vartheta_i^3) h_{i+1}^5 \left(h_{i+1}^2 - h_i^2 + h_i h_{i+1}\right)}{6 \cdot 6!(h_i + h_{i+1})} \Bigg], \end{aligned}$$

and for $x_i \in I_h'$

$$r_i = \varepsilon^2\left(u''(x_i) - u''(\vartheta_i^4)\right), \tag{14.2}$$

where $\vartheta_i^s \in (x_{i-1}, x_{i+1})$, $s = 1, 2, 3, 4$.

THEOREM 2. *Let (2.1)–(2.4) hold. On the mesh (4.1)–(4.8) we have, for* $i = 1, 2, \ldots, n$,

$$r_i = K_i h^4 + R_i,\ |R_i| \leqslant Mh^6,\ \text{for } x_i \in I_h \setminus I'_h, \tag{15.1}$$

$$|r_i| \leqslant Mh^8,\ \text{for } x_i \in I'_h, \tag{15.2}$$

where K_i *is independent of* h.

PROOF: We shall use (14.1)–(14.2) and the representation (3.1)–(3.4). Again, we shall give the proof for $i = 1, 2, \ldots, m-1$ only. Note that for $x \in [0, 0.5]$

$$\left|z^{(s)}(x)\right| \leqslant M,\ s = 0, 1, \ldots, 8, \quad z = v_1 + y. \tag{16}$$

Let us prove (15.1). Let q_s denote the coefficient at $u^{(s)}(x_i)$ in (14.1), $s = 5, 6, 7$, and let q_8^p be the coefficient at $u^{(8)}(\vartheta_i^p)$, $p = 1, 2, 3$. By expanding λ (note that $\lambda \in C^\infty(x_i, x_{i+1})$) we get

$$q_5 = \varepsilon^2 \lambda'(t_i)^2 \lambda''(t_i) h^4/20 + r_i^1,$$
$$q_6 = \varepsilon^2 \lambda'(t_i)^4 h^4/240 + r_i^2,$$

(see [7] for the technique). Thus we have (15.1) with

$$R_i = r_i^1 u^{(5)}(x_i) + r_i^2 u^{(6)}(x_i) + q_7 u^{(7)}(x_i) + \sum_{p=1}^{3} q_8^p u^{(8)}(\vartheta_i^p).$$

By using (3.1)–(3.4), (5.1), (7) and the technique from [6] (see [7, 3, 4] as well) we can prove

$$|R_i| \leqslant Mh^6, \quad x_i \in I_h \setminus I'_h. \tag{17}$$

Note that $x_i \in I_h \setminus I'_h$ corresponds to the cases 1^0 and 2^0 of the proof of Theorem 2 from [6] (Theorem 1 from [7]). On the other hand, $x_i \in I'_h$ corresponds to the case 3^0. Let us illustrate the proof of (17) by showing

$$D_1 = \left|r_i^1 u^{(5)}(x_i)\right| \leqslant Mh^6, \tag{18.1}$$

$$D_2 = \left|q_7 u^{(7)}(x_i)\right| \leqslant Mh^6. \tag{18.2}$$

We have

$$r_i^1 = \frac{\varepsilon^2}{180}\left[\frac{h^4}{24}\left(\lambda^{(4)}(\tau_i^-) + \lambda^{(4)}(\tau_i^+)\right)\left(9h^2\lambda'(t_i)^2 + Z_i\right) + h^2\lambda_i'' Z_i\right],$$

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where

$$Z_i = 3h^4\lambda'(t_i)\lambda'''(\eta_i) + \frac{2}{9}h^6\lambda'''(\eta_i)^2 + \frac{h^6}{36}\lambda'''(\eta_i^-)\lambda'''(\eta_i^+)$$
$$+ \frac{h^5}{12}\lambda''(t_i)\big(\lambda'''(\eta_i^-) - \lambda'''(\eta_i^+)\big),$$
$$\tau_i^-,\, \eta_i^- \in (x_{i-1},\, x_{i+1}),\ \tau_i^+,\, \eta_i^+ \in (x_i,\, x_{i+1}),\ \eta_i \in (x_{i-1},\, x_{i+1}).$$

Now, if $i > k+6$, that is, $t_{i-1} \geqslant \alpha + 6h$, from (5.1), (5.2), (7), (3.1)–(3.4) and (16) we have

$$D_1 \leqslant Mh^6\varepsilon^2\left(\varepsilon^{-1/4} + 1\right)[1 + \varepsilon^{-5}\exp\left(-\gamma\lambda(\alpha + 6h)/\varepsilon\right)],$$
$$D_1 \leqslant Mh^6[1 + \varepsilon^{-13/4}\exp\left(-\gamma\omega(\alpha)/\varepsilon\right)] \leqslant Mh^6,$$

and

$$D_2 \leqslant Mh^6\varepsilon^2[1 + \varepsilon^{-7}\exp\left(-\gamma\lambda(\alpha + 6h)/\varepsilon\right)],$$
$$D_2 \leqslant Mh^6[1 + \varepsilon^{-5}\exp\left(-\gamma\omega(\alpha)/\varepsilon\right)] \leqslant Mh^6.$$

If $i < j+1$ that is, $t_{i-1} \leqslant P$ we get

$$D_1 \leqslant Mh^6[\varepsilon^2\left(\varepsilon^{-1/4} + 1\right) + \varepsilon^5(q - t_{i+1})^{-13}\varepsilon^{-5}\exp\left(-\gamma\lambda(t_{i-1})/\varepsilon\right)], \tag{19.1}$$
$$D_2 \leqslant Mh^6[1 + \varepsilon^7(q - t_{i+1})^{-11}\varepsilon^{-7}\exp\left(-\gamma\lambda(t_{i-1})/\varepsilon\right)]. \tag{19.2}$$

Since, from $t_{i-1} \leqslant P < q - 3h$ it follows that

$$q - t_{i+1} \geqslant \frac{q - t_{i-1}}{3},$$

from (19.1), (19.2) and (7) we get (18.1), (18.2) again.

Let us now prove (15.2). From (14.2) we have

$$|r_i| \leqslant M\varepsilon^2\max\{|u''(x)| : x_{i-1} \leqslant x \leqslant x_{i+1}\}.$$

From $x \in I_h'$ it follows that $\varepsilon \leqslant Mh^4$, thus by using (3.1)–(3.4), (16) and (7) we get

$$|r_i| \leqslant M[h^8 + \exp\left(-\gamma\lambda(P)/\varepsilon\right)] \leqslant M[h^8 + \exp\left(-Mn\right)] \leqslant Mh^8.$$

∎

By using Richardson extrapolation we can eliminate the $0(h^4)$-term from (15.1), and , having in mind stability (13), we can prove

THEOREM 3. *Let the conditions of Theorem 1 hold and let w_h and $w_{h/2}$ be the solutions to the problem (8.1)–(8.3) with n and $2n$ mesh steps, respectively. Then we have*

$$\|u_h - \bar{w}_h\|_\infty \leqslant Mh^6,$$

where $\bar{w}_h$ is the vector with components

$$\bar{w}_i = \frac{16w_{h/2,2i} - w_{h,i}}{15}, \; i = 0, 1, \ldots, n.$$

4. NUMERICAL RESULTS

We shall use the following test example:

$$-\varepsilon^2 u'' + u + cos^2 \pi x + 2(\varepsilon\pi)^2 \cos 2\pi x = 0, \; x \in I,$$
$$u(0) = u(1) = 0,$$

whose solution is known:

$$u(x) = \frac{\exp(-x/\varepsilon) + \exp((x-1)/\varepsilon)}{1 + \exp(-1/\varepsilon)} - \cos^2 \pi x.$$

This problem was considered in [**2**, **3**, **4**, **5**, **6**, **7**] as well.

In Table 1 we present the error

$$E_h = \|u_h - \bar{w}_h\|_\infty,$$

(where $\bar{w}_h$ is the same as in Theorem 3), and the experimental order of convergence, (see [**2**])

$$\text{Ord} = \frac{\log E_h - \log E_{h/2}}{\log 2}.$$

Different values of ε and n are considered. The corresponding values of α are given in Table 2. They are determined in such a way that the percentage of the mesh steps lying within the layers is the highest possible. We take the interval $[0, \varepsilon]$ to represent the left-hand layer. The percentage, $p = (i_0/n) * 100$, where i_0 is an index such that $x_{i_0} \leqslant \varepsilon < x_{i_0+1}$, is shown in Table 2 as well. For a given ε, we take the smallest value of the parameter a (see (4.7))

$$a = \left(B + 2\varepsilon^{-1/4} q(0.5 - \alpha)^4\right)^{-1}.$$

Then we consider the condition

$$t \leqslant \frac{q}{a+1} = K,$$

which is equivalent to

$$\omega(t) \leqslant \varepsilon,$$

and determine α as the point satisfying (4.4), for which K is maximal (note that K is a function of α for ε fixed).

All computations have been carried out on the ATARI 1040 ST with 48 bits accuracy in floating point.

Table 1

n ε	2^{-15}	2^{-20}	2^{-25}	2^{-30}	2^{-35}	2^{-40}	
10	1.565(−3)	–	4.448(−2)	2.779(−2)	3.108(−2)	1.370(−1)	E_l
	–	–	–	–	–	–	Or
20	1.236(−3)	1.914(−3)	9.826(−4)	1.032(−2)	1.591(−2)	1.651(−2)	
	3.402	4.446	5.501	1.429	0.966	3.053	
40	3.745(−5)	3.165(−5)	2.581(−5)	3.632(−4)	6.578(−3)	2.170(−3)	
	5.045	5.918	5.251	4.829	1.275	2.928	
80	5.849(−7)	4.707(−7)	3.243(−7)	7.852(−6)	3.894(−4)	6.181(−4)	
	6.001	6.071	6.314	5.532	4.079	1.812	
160	9.115(−9)	7.338(−9)	5.012(−9)	1.074(−7)	6.717(−6)	2.090(−7)	
	6.004	6.004	6.016	6.192	5.857	5.702	
320	3.942(−10)	4.295(−10)	3.121(−10)	1.615(−9)	1.123(−7)	3.133(−9)	
	4.531	4.095	4.005	6.056	5.902	6.060	

Table 2

ε	2^{-15}	2^{-20}	2^{-25}	2^{-30}	2^{-35}	2^{-40}
α	0.10	0.15	0.20	0.20	0.20	0.25
p	2.18	2.81	4.38	7.81	11.88	16.56

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