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INVERTIBLE OPERATORS ON CERTAIN BANACH SPACES

BY
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Introduction. It has long been the practice in the theory of Hilbert spaces to use the Fourier series expansion (i.e. the Levy inversion formula) for the resolution of the identity associated with a unitary operator to obtain results for this operator, and hence for any power bounded invertible operator on such spaces since they are necessarily isomorphic to unitary operators [5, p. 1945]. Though many important power bounded operators on Banach spaces are not spectral [6, p. 1045–1051] the approach of this paper permits us to deduce for such operators results similar to those known for spectral operators. That which permits this and which makes this paper different from the usual approach as found in Dunford and Schwartz [5, p. 1941, 2007–2010], Sz.-Nagy and Foiaş [12, p. 109–118, 153–163] and Colojoară and Foiaş [3, p. 154] is the application of the Levy inversion formula to arbitrary invertible power bounded operators. Starting directly from this formula we obtain for an arbitrary invertible power bounded operator on $L^p(I)$, I a finite measure space and p fixed $(1 \le p \le \infty)$, a family of operators with projection properties analogous to those for a spectral measure associated with a spectral operator (theorem 1) and which reduces to the spectral measure for those operators which are spectral. Furthermore, by Lamperti's theorem we are able to establish a relationship between invertible isometries on $L^p(I)$, $p \ne 2, \infty$, which also need not be spectral [6, p. 1045–1051], and unitary operators with the multiplicative property on $L^2(I)$ (theorem 3). The inversion formula also permits us to establish an operational calculus on a somewhat more general class of functions than usual (theorem 4) and a convergence result for certain contractions (theorem 5).

Invertible Operators on L^p: For a fixed p, $1 \le p \le \infty$, let U be an invertible operator whose iterates U^j, $j = 0, \pm 1, \pm 2, \ldots$, are uniformly bounded in norm on $L^p(I)$, I denoting a finite measure space. Given f in $L^p(I)$ and g in $L^q(I)$, $1/p + 1/q = 1$, define the square integrable function of t (on $[0, 2\pi)$)

$$(G_t f, g) = \sum_{j \ne 0} e^{ijt}((U^{-j} - 1)f, g)/2\pi ij - \tfrac{1}{2}(G(\{e^{it}\})f, g) \tag{1}$$

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where

$$(G(\{e^{it}\})f, g) = \lim_{n\to\infty} \sum_{-n}^{n} e^{ijt}(U^{-j}f, g)/(2n+1),$$

(f, g) represents the integral over I of the product $f\bar{g}$ of f with the complex conjugate of g, and for the sum $\sum_{j\neq 0}$ we take its principal value. Though $(G(\{e^{it}\})f, g)$ vanishes as a square integrable function of t we shall see later that when $p=2$ it converges pointwise as does $(G_t f, g)$.

THEOREM 1. *For a uniformly bounded sequence $\{U^j : j = 0, \pm 1, \pm 2, \ldots\}$ of iterates of an invertible operator on $L^p(I)$ we have for almost all (s', s, t', t) in $[0, 2\pi)^4$ with $0 \le s, s' \le t, t' < 2\pi$*

$$((G_s - G_{s'})(G_t - G_{t'})f, g) = ((G_{s\wedge t} - G_{s'\vee t'})f, g)$$

where f and g are any given functions in $L^p(I)$ and $L^q(I)$ respectively, $1/p + 1/q = 1$.

Proof. Since $((G_s - G_{s'})(G_t - G_{t'})f, g)$ is square integrable on $[0, 2\pi)^4$. we will be able to change the order of the limits in what follows to obtain equality almost everywhere. For convenience of notation we will omit the inner product and take $0 = s' = t' \le s \le t < 2\pi$. Hence, introducing E_t for $G_t - G_0$ and $E(\{e^{it}\})$ for $G(\{e^{it}\})$ (as we shall do later), we get

$$\begin{aligned}
E(\{e^{is}\})E(\{e^{it}\}) &= \lim_{m,n\to\infty} \frac{1}{(2m+1)(2n+1)} \sum_{-m}^{m}\sum_{-n}^{n} e^{i(j+k)s} e^{ik(t-s)} U^{-j-k} \\
&= \lim_{n\to\infty} \frac{1}{2n+1} \sum_{-n}^{n} e^{ik(t-s)} E(\{e^{is}\}) \\
&= \begin{cases} 0 & \text{if } t \neq s \\ E(\{e^{is}\}) & \text{if } t = s \end{cases}
\end{aligned}$$

Also

$$\begin{aligned}
E(\{e^{is}\}) \sum_{j\neq 0} \frac{(e^{ijt}-1)}{2\pi i j} U^{-j} &= \lim_{m\to\infty} \sum_{-m}^{m} \sum_{j\neq 0} \frac{(e^{i(k+j)s}e^{ij(t-s)} - e^{i(k+j)s}e^{-ijs})}{(2m+1)2\pi i j} U^{-j-k} \\
&= \sum_{j\neq 0} \frac{(e^{ij(t-s)} - e^{-ijs})}{2\pi i j} E(\{e^{is}\}) \\
&= \begin{cases} \dfrac{(2\pi - t)}{2\pi} E(\{e^{is}\}) & \text{if } 0 < s < t \\[2ex] -\dfrac{t}{2\pi} E(\{e^{is}\}) & \text{if } 0 < t < s \\[2ex] \dfrac{(\pi - s)}{2\pi} E(\{e^{is}\}) & \text{if } s = t \neq 0 \end{cases}
\end{aligned}$$

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and

$$E(\{1\})\sum_{j\neq 0}\frac{(e^{ijt}-1)}{2\pi ij}U^{-j}=\frac{(\pi-t)}{2\pi}E(\{1\})$$

since [4, p. 98]

$$\sum_{j\neq 0}\frac{e^{iju}}{2\pi ij}=\frac{\pi-u}{2\pi},\qquad 0<u<2\pi$$

Also, since $(j+k)/jk=1/j+1/k$ if $j\neq 0$ and $k\neq 0$, then

$$\begin{aligned}
&\sum_{j\neq 0}\frac{(e^{ijt}-1)}{2\pi ij}U^{-j}\sum_{k\neq 0}\frac{(e^{iks}-1)}{2\pi ik}U^{-k}\\
&=\sum_{\substack{k\neq 0\\ k\neq -j}}\frac{(e^{i(j+k)t}e^{ik(s-t)}-e^{iks}-e^{i(j+k)t}e^{-ikt}+1)}{4\pi^2 iki(k+j)}U^{-j-k}\\
&\quad+\sum_{\substack{j\neq 0\\ k\neq -j}}\frac{(e^{i(j+k)s}e^{ij(t-s)}-e^{i(j+k)s}e^{-ijs}-e^{ijt}+1)}{4\pi^2 iji(k+j)}U^{-j-k}\\
&\quad-\sum_{j\neq 0}\frac{(e^{ij(t-s)}-e^{-ijs}-e^{ijt}+1)}{4\pi^2(ij)^2}e\\
&=\sum_{\substack{k\neq 0\\ k\neq -j}}\frac{(e^{i(j+k)t}-1)}{2\pi i(j+k)}\frac{(e^{ik(s-t)}-e^{-ikt})}{2\pi ik}U^{-j-k}\\
&\quad+\sum_{\substack{k\neq 0\\ k\neq -j}}\frac{(e^{ik(s-t)}-e^{-ikt})}{2\pi i(j+k)2\pi ik}U^{-j-k}-\sum_{\substack{k\neq 0\\ k\neq -j}}\frac{(e^{iks}-1)}{2\pi ik}\frac{U^{-j-k}}{2\pi i(j+k)}\\
&\quad+\sum_{\substack{j\neq 0\\ j\neq -k}}\frac{(e^{i(j+k)s}-1)}{2\pi i(j+k)}\frac{(e^{ij(t-s)}-e^{-ijs})}{2\pi ij}U^{-j-k}\\
&\quad+\sum_{\substack{j\neq 0\\ j\neq -k}}\frac{(e^{ij(t-s)}-e^{-ijs})}{2\pi ij2\pi i(j+k)}U^{-j-k}-\sum_{\substack{j\neq 0\\ j\neq -k}}\frac{(e^{ijt}-1)}{2\pi ij}\frac{U^{-j-k}}{2\pi i(j+k)}\\
&\quad-\sum_{j\neq 0}\frac{(e^{ij(t-s)}-e^{-ijs}-e^{ijt}+1)}{4\pi^2(ij)^2}\\
&=-\frac{s}{2\pi}E_t+\frac{st}{4\pi^2}+\frac{s}{2\pi}(E(\{1\})-E(\{e^{it}\}))\\
&\quad-\frac{t}{2\pi}E_s+\frac{st}{4\pi^2}+\frac{t}{2\pi}(E(\{1\})-E(\{e^{is}\}))\\
&\quad+E_s-\frac{s}{2\pi}-\tfrac{1}{2}(E(\{1\})-E(\{e^{is}\}))+\frac{s}{2\pi}-\frac{st}{4\pi^2}
\end{aligned}$$

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since [4, p. 99]

$$\sum_{j\neq 0} (\exp(iju))/j^2 = (3u^2 - 6\pi u + 2\pi^2)/6$$

for all u in $[0, 2\pi)$.

Thus, if $0 \leq s \leq t < 2\pi$, we get $E_tE_s = E_s$ as can be verified by substituting equation (1) for E_t and E_s and using the relations just derived. Q.E.D.

COROLLARY. *For almost all (s', s, t', t) in $[0, 2\pi)^4$ for which $[s', s)$ and $[t', t)$ are disjoint, $((G_s - G_{s'})(G_t - G_{t'})f, g) = 0$ for any given functions f in $L^p(I)$ and g in $L^q(I)$, $1/p + 1/q = 1$.*

Proof. Choosing $s'' < s' < s < t' < t$, by the theorem we have almost everywhere

$$\begin{aligned}(G_s - G_{s'}) = (G_s - G_{s'})(G_t - G_{s''}) &= (G_s - G_{s'})(G_t - G_{t'}) + (G_s - G_{s'})(G_{t'} - G_{s''}) \\ &= (G_s - G_{s'})(G_t - G_{t'}) + (G_s - G_{s'})\end{aligned}$$

from which we deduce the corollary. Q.E.D.

REMARK. In case $p = 2$ the idempotents $E_t = G_t - G_0$ are spectral projections of U which are known to be similar to a unitary operator; see [12, proof of proposition 5.3 (page 79)] or see B. Sz.-Nagy [11, p. 152–157]. In both these references it is also shown that the associated spectral measure is no greater in norm than the uniform bound for the iterates of the operator U.

If $U_t, -\infty < t < \infty$, is a group of operators on $L^p(I)$ then for all f in $L^p(I)$ and all g in $L^q(I)$, $1/p + 1/q = 1$, define

$$(F_tf, g) = \int_{-\infty}^{\infty} e^{-irt}(U_rf, g)/2\pi ir\, dr - \tfrac{1}{2}(F(\{t\})f, g) + \tfrac{1}{2}(f, g) \tag{2}$$

where

$$(F(\{t\})f, g) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-irt}(U_rf, g)\, dr.$$

Whenever (U_tf, g) is a uniformly bounded measurable function on $(-\infty, \infty)$, (F_tf, g) is a square integrable function for which we can repeat the argument in the proof of the previous theorem to obtain

THEOREM 2. *For a uniformly bounded group of operators U_t $(-\infty < t < \infty)$ on $L^p(I)$ and any f in $L^p(I)$ and any g in $L^q(I)$, $1/p + 1/q = 1$, we have $(F_sF_tf, g) = (F_{s\wedge t}f, g)$ for almost all (s, t) in $(-\infty, \infty) \times (-\infty, \infty)$.*

By Lamperti's theorem [10, p. 333] an invertible isometry W on $L^p(I)$ $(p \neq 2, \infty)$ is necessarily induced by an invertible endomorphism T on I such that $Wf(x) = f(T^{-1}x)_r^{1/p}(x)$ where r is the Radon-Nikodym derivative of mT^{-1}

with respect to the measure m on I. We now introduce the family of isometries $\{W_{p'}: 1 \le p' < \infty\}$ where $W_{p'}f(x) = f(T^{-1}x)r^{1/p'}(x)$ for all f in $L^{p'}(I)$. If f lies in $L^p(I)$ and g in $L^q(I)$, $1/p+1/q=1$, then for all integers n and n' we have

$$\begin{aligned}(W_p^{n+n'}f(x))(W_q^n g(x)) &= f(T^{-n-n'}x)(dm(T^{-n-n'}x)/dm(x))^{1/p} \\ &\quad \times g(T^{-n}x)(dm(T^{-n}x)/dm(x))^{1/q} \\ &= f(T^{-n-n'}x)\left(\frac{dm(T^{-n'}(T^{-n}x))}{dm(T^{-n}x)}\right)^{1/p} g(T^{-n}x)\frac{dm(T^{-n}x)}{dm(x)} \\ &= W_1^n(f(T^{-n'}x)(dm(T^{-n'}x)/dm(x))^{1/p}g(x)) \\ &= W_1^n(g(x)W_p^{n'}f(x)).\end{aligned}$$

Similarly, $(W_p^n f)(W_q^{n+n'}g) = W_1^n(fW_q^{n'}g)$. Thus we can show the following result.

THEOREM 3. *To any given invertible isometry W on a particular $L^{p'}(I)$, $p' \ne 2$, ∞, there corresponds a family of operators $\{W_p: 1 \le p < \infty\}$ induced by a common endomorphism such that W_p is an invertible isometry on $L^p(I)$. Furthermore, if G_t^p given by equation* (1) *corresponds to W_p and if we put $E^p(J) = G_t^p - G_{t'}^p$ for all intervals $J = \{e^{is}: t' \le s < t\}$ on the unit circle in the complex plane, then for almost all intervals J and J',*

$$(E^p(J)f)(E^q(J')g) = E^1(JJ')((E^p(J)f)(E^q(J')g))$$

for all f in $L^p(I)$ and all g in $L^q(I)$, $1/p+1/q=1$, $p \ne 1$, ∞, and where $JJ' = \{zz'; z$ in J, z' in $J'\}$.

By the expression "almost all intervals $J = \{e^{iu}: t' \le u < t\}$ and $J' = \{e^{iu}: s' \le u < s\}$" we obviously mean almost all (t', t, s', s) in $[0, 2\pi)^4$. The proof of the second part of the theorem is analogous to that of the same result for operators with the multiplicative property on $L^2(I)$ [2, p. 807–808]. The same property was obtained by C. Foiaş [7, p. 641] whenever f, g, and their product all lie in $L^2(I)$.

We are now in a position to define the operators $\varphi(U)$ for certain functions φ on $[0, 2\pi)$ when U is an invertible operator on $L^p(I)$ whose iterates are uniformly bounded. Whenever φ is an integrable function on $[0, 2\pi)$ we shall write φ' for the series whose terms are the derivatives of the terms in the Fourier series of φ. Write $S(U, f, g)$ for the class of functions φ for which the product $\varphi'(t)(G_tf, g)$ is integrable on $[0, 2\pi)$ and for all such functions define

$$(3) \qquad (\varphi(U)f, g) = \varphi(2\pi)(f, g) - \int_0^{2\pi} \varphi'(t)(G_tf, g)\, dt.$$

In $S(U, f, g)$ will be included those functions with L^2 derivative and certainly all analytic functions in a neighborhood of the spectrum of U, in which case (3) yields the same result as the usual methods of operational calculus [5, p. 1941, 2007–2010]. If we enlarge $S(U, f, g)$ to also include all square integrable functions φ on $[0, 2\pi)$ with nth partial Fourier series φ_n for which $\int_0^{2\pi} \varphi_n'(t)$ $(G_tf, g)\,dt$ converges as $n \to \infty$ and take this limit for the value of the integral in (3), then $S(U, f, g)$ will contain all functions on $[0, 2\pi)$ with absolutely convergent Fourier series since the kth Fourier coefficient of (G_tf, g) is of the order $1/|k|$ $(k = \pm 1, \pm 2, \ldots)$, so obtaining a generalization of the operational calculus of I. Colojoară and C. Foiaş [3, p. 154].

THEOREM 4. *Let φ and ψ be elements of $S(U, f, g)$ and α, β any complex numbers. Then*

(i) *$\alpha\varphi + \beta\psi$ lies in $S(U, f, g)$ and $(\alpha\varphi + \beta\psi)(U) = \alpha\varphi(U) + \beta\psi(U)$.*
(ii) *If the product $\varphi\psi$ lies is $S(U, f, g)$ then $(\varphi\psi)(U) = \varphi(U)\psi(U)$.*
(iii) *If φ is a trigonometric polynomial $\sum_m^n a_k \exp(ikt)$ then $\varphi(U) = \sum_m^n a_k U^k$*

We need but prove (ii) as the other two are easy to show.

$$\begin{aligned}
\varphi(U)\psi(U) &= \varphi(2\pi)\psi(U) - \int \varphi'(s)\psi(U)G_s\,ds \\
&= \varphi(2\pi)\left(\psi(2\pi) - \int \psi'(t)G_t\,dt\right) \\
&\quad - \int \varphi'(s)\left(\psi(2\pi) - \int \psi'(t)G_t\,dt\right)G_s\,ds \\
&= \varphi(2\pi)\psi(2\pi) - \varphi(2\pi)\int \psi'(t)G_t\,dt \\
&\quad - \psi(2\pi)\int \varphi'(s)G_s\,ds + \iint \varphi'(s)\psi'(t)G_sG_t\,ds\,dt
\end{aligned}$$

The last term in the line above is equal to $\iiiint \varphi'(s)\psi'(t)(G_s - G_{s'})(G_t - G_{t'})$ which becomes, by theorem 1,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^t \varphi'(s)\psi'(t)G_s\,ds\,dt + \int_0^{2\pi}\int_t^{2\pi} \varphi'(s)\psi'(t)G_t\,ds\,dt \\
&\qquad = \int_0^{2\pi} \varphi'(s)(\psi(2\pi) - \psi(s))G_s\,ds + \int_0^{2\pi} \psi'(t)(\varphi(2\pi) - \varphi(t))G_t\,dt
\end{aligned}$$

and thus by equation (3) we get (ii).

Almost Everywhere Convergence: Let U be a contraction on the Hilbert space $L^2(I)$ with generalized resolution of the identity E_t [9, p. 448–455]. Suppose that U can be extended to contractions on $L^1(I)$ and $L^\infty(I)$ (as in the case of automorphisms [8, p. 13]). For such operators we prove the following.

THEOREM 5. *If U be a contraction on $L^2(I)$ which can be extended to contractions on $L^1(I)$ and $L^\infty(I)$ and if $\{p_k(e^{it})\colon k = 1, 2, \ldots\}$ be a pointwise* *convergent sequence of uniformly bounded trigonometric polynomials on* $[0, 2\pi)$ *then for all* f *in* $L^1(I)$, $p_k(U)f(x)$ *converges pointwise almost everywhere on* I.

Proof. First we prove this for functions in $L^\infty(I)$. For any such function f, if $p_k(U)f$ does not converge pointwise almost everywhere, there is a constant δ such that for all x in a subset Y of I of measure $|Y|>0$,

$$\sup_{k',k''\geq m} |p_{k'}(U)f(x)-p_{k''}(U)f(x)|>\delta$$

for all integers m. Thus, given m, there exists an integer $M>m$ and measurable functions $k'(x)$, $k''(x)$ with values between m and M such that for some function h, $|h|=1$, we have

$$((p_{k'(x)}(U)-p_{k''(x)}(U))f(x),\, h(x))>\frac{\delta\,|Y|}{2} \tag{4}$$

whenever M is chosen so that

$$\sup_{m\leq k',k''\leq M} |p_{k'}(U)f(x)-p_{k''}(U)f(x)|>\delta$$

for all x in a set Z in Y of measure greater than $|Y|/2$. Partitioning the set Z into sets $Y_1,\ldots,Y_j$ such that $k'(x)$ and $k''(x)$ are constant on each Y_i $(i=1,\ldots,j)$ we get by writing $\mathscr{X}_i$ for the characteristic function of Y_i, k_i' for the value of $k'(x)$ on Y_i and k_i'' for that of $k''(x)$,

$$((p_{k'(x)}(U)-p_{k''(x)}(U))f(x),\, h(x))=\sum\int (p_{k_i'}(e^{it})-p_{k_i''}(e^{it}))\, d(E_tf,\, \mathscr{X}_ih). \tag{5}$$

where h is taken with support on Z and E_t is the generalized resolution of the identity for U.

For $\varepsilon>0$ and M, m large enough we could choose $k'(x)$ and $k''(x)$ such that $|p_{k_i'}(e^{it})-p_{k_i''}(e^{it})|<\varepsilon/2^i$ for all t and all i. Choosing h with support on Z we still have the inequality (4). But by (5)

$$\begin{aligned}|((p_{k'(x)}(U)-p_{k''(x)}(U))f(x),\, h(x))| &\leq \sum\int |p_{k_i'}(e^{it})-p_{k''}(e^{it})|\,|d(E_tf,\, \mathscr{X}_ih)|\\ &\leq \sum (\varepsilon/2^i)\, \|f\|_2\, \|\mathscr{X}_ih\|_2\\ &\leq \varepsilon\, \|f\|_2\, \|h\|_2\end{aligned}$$

which is less than $\delta\,|Y|/2$ for ε small enough, thus contradicting (4). So we get the theorem for any $L^\infty(I)$ function.

If f lies in $L^1(I)$, then writing $\sum_j a_j^k \exp(ijt)$ for $p_k(e^{it})$ we get the inequality

(4) in the form

$$(6)\qquad (f(x), \sum_j U^{*j}(a_j^{k'(x)} - a_j^{k''(x)})h(x)) > \delta\,|Y|/2$$

which will also hold for some $L^2(I)$ function f close in $L^1(I)$ norm to the original f, where U^* is the adjoint of U. But the left hand side of this inequality can be handled as equation (5) to yield the inequalities

$$\begin{aligned} |((p_{k'(x)}(U) - p_{k''(x)}(U))f(x), h(x))| &= |(f(x), \sum_j U^{*j}(a_j^{k'(x)} - a_j^{k''(x)})h(x))| \\ &\leq \sum_i \left|\left(f, \sum_j (a_j^{k'i} - a_j^{k''i})\, U^{*j}\mathscr{X}_i h\right)\right| \\ &\leq \sum (\varepsilon/2^i)\, \|f\|_2\, \|\mathscr{X}_i h\|_2 \\ &\leq \varepsilon\, \|f\|_2\, \|h\|_2 \end{aligned}$$

which contradicts (6) for ε small enough. Q.E.D.

Theorem 5 is a generalization of a result to be found in [1, pp. 161]. The proof given here is an improvement over that found there.

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