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GLOBAL EXISTENCE OF WEAK SOLUTIONS FOR STRONGLY DAMPED WAVE EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS AND BALANCED POTENTIALS

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Abstract

We demonstrate the global existence of weak solutions to a class of semilinear strongly damped wave equations possessing nonlinear hyperbolic dynamic boundary conditions. The associated linear operator is $(-\Delta_W)^\theta \partial_t u$, where $\theta \in [\frac{1}{2}, 1)$ and Δ_W is the Wentzell–Laplacian. A balance condition is assumed to hold between the nonlinearity defined on the interior of the domain and the nonlinearity on the boundary. This allows for arbitrary (supercritical) polynomial growth of each potential, as well as mixed dissipative/antidissipative behaviour.

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1. Introduction

Our aim is to show the global existence of weak solutions to the fractional strongly damped wave equation with nonlinear hyperbolic dynamic boundary conditions. We establish the global existence of weak solutions under a balance condition imposed on the nonlinear terms. This condition is motivated by [20, Lemma 3.1] and allows both nonlinearities to have supercritical polynomial growth. Special attention is given to obtaining the compact resolvent for the associated linear operator which contains (fractional) Wentzell–Laplacians.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\Gamma := \partial\Omega$. Throughout, we assume that $\theta \in [\frac{1}{2}, 1)$, $\omega \in (0, 1]$ and $\alpha \in (0, 1]$. We consider the following equations in the unknown $u = u(t, x)$.

$$\partial_t^2 u - \omega\Delta^\theta \partial_t u + \partial_t u - \Delta u + u + f(u) = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.1}$$

$$\partial_t^2 u + \omega\partial_{\mathbf{n}}^\theta \partial_t u + \partial_{\mathbf{n}} u - \alpha\omega\Delta_\Gamma \partial_t u + \partial_t u - \Delta_\Gamma u + u + g(u) = 0 \quad \text{on } (0, \infty) \times \Gamma. \tag{1.2}$$

Additionally, we impose the initial conditions

$$u(0,x) = u_0(x) \quad \text{and} \quad \partial_t u(0,x) = u_1(x) \quad \text{at } \{0\}\times\Omega, \tag{1.3}$$

and

$$u_{|\Gamma}(0,x) = \gamma_0(x) \quad \text{and} \quad \partial_t u_{|\Gamma}(0,x) = \gamma_1(x) \quad \text{at } \{0\}\times\Gamma. \tag{1.4}$$

Here, Δ_Γ denotes the Laplace–Beltrami operator (see, for example, [6]).

We assume that $f \in C(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ satisfy the sign conditions

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -M_1, \quad g'(s) \geq -M_2 \quad \text{for all } s \in \mathbb{R}, \tag{1.5}$$

for some $M_1, M_2 > 0$, and the growth assumptions, for all $s \in \mathbb{R}$,

$$|f(s)| \leq \ell_1(1+|s|^{r_1-1}), \quad |g(s)| \leq \ell_2(1+|s|^{r_2-1}), \tag{1.6}$$

for some positive constants ℓ_1 and ℓ_2, and where $r_1, r_2 \geq 2$. In addition, we assume that there exists $\varepsilon \in (0,\omega)$ so that the following balance condition holds: that is,

$$\liminf_{|s|\to\infty} \frac{1}{|s|^{r_1}}\Big(f(s)s + \frac{|\Gamma|}{|\Omega|}g(s)s - \frac{C_\Omega^2|\Gamma|^2}{4\varepsilon|\Omega|^2}|g'(s)s+g(s)|^2\Big) > 0, \tag{1.7}$$

for $r_1 \geq \max\{r_2, 2(r_2-1)\}$, where $C_\Omega > 0$ is the best Sobolev constant in the Sobolev–Poincaré inequality

$$\|u - \langle u\rangle_\Gamma\|_{L^2(\Omega)} \leq C_\Omega\|\nabla u\|_{L^2(\Omega)}, \quad \langle u\rangle_\Gamma := \frac{1}{|\Gamma|}\int_\Gamma \mathrm{tr}_D(u)\,d\sigma, \quad \text{for all } u \in H^1(\Omega). \tag{1.8}$$

Let us provide further context for the balance condition (1.7) in our setting (see also [20] and [12] for other settings). Suppose that, for $|y|\to\infty$, the internal and boundary functions satisfy

$$\lim_{|y|\to\infty} \frac{f(y)}{|y|^{r_1-1}} = (r_1-1)c_f, \quad \lim_{|y|\to\infty}\frac{g'(y)}{|y|^{r_2-2}} = (r_2-1)c_g,$$

for some constants $c_f, c_g \in \mathbb{R}\setminus\{0\}$. In particular,

$$f(y)y \sim c_f|y|^{r_1}, \quad g(y)y \sim c_g|y|^{r_2} \quad \text{as } |y|\to\infty.$$

For the case of bulk dissipation (that is, $c_f > 0$) and antidissipative behaviour at the boundary Γ (that is, $c_g < 0$), assumption (1.7) is automatically satisfied provided that $r_1 > \max\{r_2, 2(r_2-1)\}$. Furthermore, if $2 < r_2 < 2(r_2-1) = r_1$ and

$$c_f > \frac{1}{4\varepsilon}\Big(\frac{C_\Omega|\Gamma|c_g r_2}{|\Omega|}\Big)^2,$$

for some $\varepsilon \in (0,\omega)$, then (1.7) is again satisfied. In the case when f and g are sublinear (that is, $r_1 = r_2 = 2$ in (1.6)), the condition (1.7) is also automatically satisfied provided that

$$\Big(c_f + \frac{|\Gamma|}{|\Omega|}c_g\Big) > \frac{1}{\varepsilon}\Big(\frac{C_\Omega|\Gamma|c_g}{|\Omega|}\Big)^2 \quad \text{for some } \varepsilon \in (0,\omega).$$

Notation and conventions. We use the following notation and conventions. Norms are denoted by $\|\cdot\|_B$, where B is the underlying Banach space. The notation $(\cdot,\cdot)_H$ denotes the inner product on the Hilbert space H. The dual product on $H^* \times H$ is denoted by $\langle\cdot,\cdot\rangle_{H^*\times H}$. The notation $\langle\cdot,\cdot\rangle$ is also used to denote the product on the phase space and various other vectorial function spaces. The vector-valued function $\binom{u}{v}$ is denoted by $(u, v)^{\mathrm{tr}}$. Throughout, $C > 0$ will denote a *generic* constant which may depend on various structural parameters such as $|\Omega|$, $|\Gamma|$, M_1, M_2 and so on, and may change from line to line. Also, $Q : \mathbb{R}_+ \to \mathbb{R}_+$ will be a generic monotonically increasing function whose specific dependence on other parameters will be made explicit on each occurrence. All of these quantities are *independent* of the perturbation parameters θ, α and ω.

Outline of the article. In Section 2 we establish the variational formulation of Problem **P** and define weak solutions. A proof of the existence of global weak solutions is developed in Section 3. Because of the nature of the balance condition, a continuous dependence-type estimate is not available. We give some remarks on this difficulty and plans for further research. Appendix A contains some explicit characterisations for the fractional Wentzell–Laplacian used throughout the article, as well as a compact embedding result that we need to draw upon.

2. Formulation of the model problem

In this section, we first recall the Wentzell–Laplacian defined on vectorial Hilbert spaces (see [1, Section 2] and [10, Section 2 and Appendix]). Then we give the basic functional set-up in order to formulate the model problem. We also provide various results pertaining to the problem.

Let $\lambda_\Omega > 0$ denote the best constant satisfying the Sobolev inequality in Ω: that is:

$$\lambda_\Omega \int_\Omega u^2 dx \le \int_\Omega (|\nabla u|^2 + u^2)\, dx.$$

The Laplace–Beltrami operator $-\Delta_\Gamma$ on the surface Γ is positive definite and self-adjoint on $L^2(\Gamma)$ with domain $D(\Delta_\Gamma)$. The Sobolev spaces $H^s(\Gamma)$, for $s \in \mathbb{R}$, may be defined as $H^s(\Gamma) = D((\Delta_\Gamma)^{s/2})$ endowed with the norm whose square is given by

$$\|u\|^2_{H^s(\Gamma)} := \|u\|^2_{L^2(\Gamma)} + \|(-\Delta_\Gamma)^{s/2} u\|^2_{L^2(\Gamma)} \quad \text{for all } u \in H^s(\Gamma).$$

Let $\lambda_\Gamma > 0$ denote the best constant satisfying the Sobolev inequality on Γ: that is:

$$\lambda_\Gamma \int_\Gamma u^2 d\sigma \le \int_\Gamma (|\nabla_\Gamma u|^2 + u^2)\, d\sigma.$$

Next, recall that Ω is a bounded domain of $\mathbb{R}^3$ with boundary Γ, which we assume is of class C^2. To this end, consider the space $\mathbb{X}^2 = L^2(\overline{\Omega}, d\mu)$, where $d\mu = dx_{|\Omega} \oplus d\sigma$ is such that dx denotes the Lebesgue measure on Ω and $d\sigma$ denotes the natural surface measure on Γ. Then $\mathbb{X}^2 = L^2(\Omega, dx) \oplus L^2(\Gamma, d\sigma)$ may be identified by the natural norm

$$\|u\|^2_{\mathbb{X}^2} = \int_\Omega |u(x)|^2 dx + \int_\Gamma |u(x)|^2 d\sigma.$$

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If we identify every $u \in C(\overline{\Omega})$ with $U = (u_{|\Omega}, u_{|\Gamma})^{\mathrm{tr}} \in C(\Omega) \times C(\Gamma)$, we may also define $\mathbb{X}^2$ to be the completion of $C(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{\mathbb{X}^2}$. Thus, in general, any function $u \in \mathbb{X}^2$ will be of the form $u = \binom{u_1}{u_2}$ with $u_1 \in L^2(\Omega, dx)$ and $u_2 \in L^2(\Gamma, d\sigma)$. It is important to note that there need not be any connection between u_1 and u_2. From now on, the inner product in the Hilbert space $\mathbb{X}^2$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathbb{X}^2}$. The Dirichlet trace map $\mathrm{tr}_D : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$, defined by $\mathrm{tr}_D(u) = u_{|\Gamma}$, extends to a linear continuous operator $\mathrm{tr}_D : H^r(\Omega) \to H^{r-1/2}(\Gamma)$, for all $r > 1/2$, which is onto for $1/2 < r < 3/2$. This map also possesses a bounded right inverse $\mathrm{tr}_D^{-1} : H^{r-1/2}(\Gamma) \to H^r(\Omega)$ such that $\mathrm{tr}_D(\mathrm{tr}_D^{-1}\psi) = \psi$ for any $\psi \in H^{r-1/2}(\Gamma)$. We can thus introduce the subspaces of $H^r(\Omega) \times H^{r-1/2}(\Gamma)$ and $H^r(\Omega) \times H^r(\Gamma)$, respectively, by

$$
\begin{aligned}
\mathbb{V}_0^r &:= \{U = (u, \gamma) \in H^r(\Omega) \times H^{r-1/2}(\Gamma) : \mathrm{tr}_D(u) = \gamma\}, \\
\mathbb{V}^r &:= \{U = (u, \gamma) \in \mathbb{V}_0^r : \mathrm{tr}_D(u) = \gamma \in H^r(\Gamma)\},
\end{aligned}
$$

for $r > 1/2$, and note that $\mathbb{V}_0^r$, $\mathbb{V}^r$ are not product spaces. However, there are dense and compact embeddings $\mathbb{V}_0^{r_1} \subset \mathbb{V}_0^{r_2}$ for any $r_1 > r_2 > 1/2$ (by definition, this also true for the sequence of spaces $\mathbb{V}^{r_1} \subset \mathbb{V}^{r_2}$). The norms on the spaces $\mathbb{V}_0^r$, $\mathbb{V}^r$ are defined by

$$
\|U\|_{\mathbb{V}_0^r}^2 := \|u\|_{H^r(\Omega)}^2 + \|\gamma\|_{H^{r-1/2}(\Gamma)}^2, \quad \|U\|_{\mathbb{V}^r}^2 := \|u\|_{H^r(\Omega)}^2 + \|\gamma\|_{H^r(\Gamma)}^2.
$$

We consider the basic (linear) operator associated with the model problem (1.1)–(1.4), the so-called Wentzell–Laplacian. Let

$$
\Delta_W \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} \Delta u_1 - u_1 \\ -\partial_{\mathbf{n}} u_1 + \Delta_\Gamma u_2 - u_2 \end{pmatrix},
$$

with

$$
D(\Delta_W) := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{V}^1 : -\Delta u_1 \in L^2(\Omega),\ \partial_{\mathbf{n}} u_1 - \Delta_\Gamma u_2 \in L^2(\Gamma) \right\}.
$$

By [10, see Appendix and, in particular, Theorem 5.3], the operator $(\Delta_W, D(\Delta_W))$ is a self-adjoint and strictly positive operator on $\mathbb{X}^2$, and the resolvent operator $(I + \Delta_W)^{-1} \in \mathcal{L}(\mathbb{X}^2)$ is compact. Since Γ is of class C^2, then $D(\Delta_W) = \mathbb{V}^2$. Indeed, the map $L : U \mapsto \Delta_W U$, as a mapping from $\mathbb{V}^2$ into $\mathbb{X}^2 = L^2(\Omega) \times L^2(\Gamma)$, is an isomorphism and there is a positive constant C_*, independent of $U = (u, \gamma)^{\mathrm{tr}}$, such that

$$
C_*^{-1} \|U\|_{\mathbb{V}^2} \le \|L(U)\|_{\mathbb{X}^2} \le C_* \|U\|_{\mathbb{V}^2} \quad \text{for all } U \in \mathbb{V}^2
$$

(see Lemma 2.1 and also [7]).

The following basic elliptic estimate is taken from [11, Lemma 2.2].

Lemma 2.1. *Consider the linear boundary value problem,*

$$
\begin{cases} -\Delta u = p_1 & \text{in } \Omega, \\ -\Delta_\Gamma u + \partial_{\mathbf{n}} u + u = p_2 & \text{on } \Gamma. \end{cases}
$$

If $(p_1, p_2) \in H^s(\Omega) \times H^s(\Gamma)$ for $s \ge 0$ and $s + \frac{1}{2} \notin \mathbb{N}$ then, for some constant $C > 0$,

$$
\|u\|_{H^{s+2}(\Omega)} + \|u\|_{H^{s+2}(\Gamma)} \le C(\|p_1\|_{H^s(\Omega)} + \|p_2\|_{H^s(\Gamma)}).
$$

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We also recall the following basic inequality which gives interior control over some boundary terms (see [9, Lemma A.2]).

Lemma 2.2. *Let $s>1$ and $u\in H^1(\Omega)$. Then, for every $\varepsilon>0$, there exists a positive constant $C_\varepsilon\sim\varepsilon^{-1}$ such that,*

$$\|u\|^s_{L^s(\Gamma)}\le\varepsilon\|\nabla u\|^2_{L^2(\Omega)}+C_\varepsilon(\|u\|^\gamma_{L^\gamma(\Omega)}+1),$$

where $\gamma=\max\{s,2(s-1)\}$.

For more details, we refer the reader to [5, 7] and [13].

Finally, since the operator Δ_W with domain $D(\Delta_W)$ is positive and self-adjoint on $\mathbb{X}^2$, we may define fractional powers of Δ_W (see Appendix A). Indeed, with $\theta\in[\frac{1}{2},1)$, $\alpha\in(0,1]$ and $\omega\in(0,1]$, we define

$$\Delta_W^\theta\begin{pmatrix}u_1\\u_2\end{pmatrix}:=\begin{pmatrix}\Delta^\theta u_1-u_1\\-\partial^\theta_{\mathbf{n}}u_1+\Delta_\Gamma u_2-u_2\end{pmatrix}$$

and

$$\Delta_W^{\theta,\alpha,\omega}\begin{pmatrix}u_1\\u_2\end{pmatrix}:=\begin{pmatrix}\omega\Delta^\theta u_1-u_1\\-\omega\partial^\theta_{\mathbf{n}}u_1+\alpha\omega\Delta_\Gamma u_2-u_2\end{pmatrix}$$

with domain

$$D(\Delta_W^{\theta,\alpha,\omega}):=\Big\{U=\begin{pmatrix}u_1\\u_2\end{pmatrix}\in\mathbb{V}^1:-\omega\Delta^\theta u_1\in L^2(\Omega),\ \omega\partial^\theta_{\mathbf{n}}u_1-\alpha\omega\Delta_\Gamma u_2\in L^2(\Gamma)\Big\}.$$

Hence, $\Delta_W^{\theta,1,1}=\Delta_W^\theta$. The fractional flux $\partial^\theta_{\mathbf{n}}$ is defined as follows. Consider $\partial_{\mathbf{n}}u=\nabla u\cdot\mathbf{n}$, and recall that $\partial_{\mathbf{n}}u\in L^2(\Gamma)$ whenever $u\in H^{3/2}(\Omega)$. So we can define $\partial^\theta_{\mathbf{n}}u=\nabla_W^{\theta/2}u\cdot\mathbf{n}$ when $u\in H^{\frac{1}{2}+\theta}(\Omega)$, which guarantees the fractional flux $\partial^\theta_{\mathbf{n}}u\in L^2(\Gamma)$. (These fractional flux operators are explicitly written in Appendix A.) To define the linear operator associated with the model problem (1.1)–(1.4), let $U=(u_1,u_2)\in\mathbb{V}^1$, $V=(v_1,v_2)\in\mathbb{X}^2$ and $\mathcal{X}=(U,V)$. Motivated by [4], we define the unbounded linear operator $\mathcal{A}_{\theta,\alpha,\omega}$ by

$$\mathcal{A}_{\theta,\alpha,\omega}\mathcal{X}:=\begin{pmatrix}0&I_{2\times2}\\\Delta_W&\Delta_W^{\theta,\alpha,\omega}\end{pmatrix}\begin{pmatrix}U\\V\end{pmatrix}=\begin{pmatrix}V\\\Delta_WU+\Delta_W^{\theta,\alpha,\omega}V\end{pmatrix}=\begin{pmatrix}V\\\Delta_W^{\theta,1,1}(\Delta_W^{1-\theta,1,1}U+\Delta_W^{0,\alpha,\omega}V)\end{pmatrix}$$

with domain

$$D(\mathcal{A}_{\theta,\alpha,\omega}):=\Big\{\mathcal{X}=\begin{pmatrix}U\\V\end{pmatrix}\in\mathbb{V}^1\times\mathbb{X}^2:\Delta_W^{1-\theta,1,1}U+\Delta_W^{0,\alpha,\beta}V\in D(\Delta_W^{\theta,1,1})\Big\}.$$

By [16, Theorem 3.1(a)], the resolvent $(I_{4\times4}+\mathcal{A}_{\theta,\alpha,\omega})^{-1}\in\mathcal{L}(\mathbb{V}^1\times\mathbb{X}^2)$ is compact. Hence, we can support the local existence of weak solutions (defined below) with a Galerkin method. Next, we define a nonlinear mapping on $\mathbb{V}^1\times\mathbb{X}^2$ by

$$F(U):=\begin{pmatrix}0\\-f(u)\end{pmatrix},\quad G(U):=\begin{pmatrix}0\\-g(\gamma)\end{pmatrix}$$

and

$$\mathcal{F}(\mathcal{X}) := \begin{pmatrix} F(U) \\ G(U) \end{pmatrix} = \begin{pmatrix} 0 \\ -f(u) \\ 0 \\ -g(\gamma) \end{pmatrix} \quad \text{for } U \in \mathbb{V}^1.$$

Due to the two embeddings, $H^1(\Omega) \hookrightarrow L^{s_1}(\Omega)$, $s_1 \in [1,6]$, and $H^1(\Gamma) \hookrightarrow L^{s_2}(\Omega)$, $s_2 \in [1,\infty)$, one can show that, for $r_1 \in [1,3]$ in (1.6), $\mathcal{F} : \mathbb{V}^1 \times \mathbb{X}^2 \to \mathbb{V}^1 \times \mathbb{X}^2$ is locally Lipschitz (see [14, Lemma 2.6]). With $r_1 \geq 1$ arbitrary, this motivates us to set

$$\widetilde{\mathbb{V}}^{s,r_1} = \{U = (u,\gamma)^{\mathrm{tr}} \in [H^s(\Omega) \cap L^{r_1}(\Omega)] \times H^s(\Gamma) : tr_D(u) = \gamma\}$$

with the canonical norm whose square is given by

$$\|U\|^2_{\widetilde{\mathbb{V}}^{s,r_1}} := \|u\|^2_{H^s(\Omega)} + \|u\|^{r_1}_{L^{r_1}(\Omega)} + \|\gamma\|^2_{H^s(\Gamma)},$$

and to set $\mathcal{H}_0 := \widetilde{\mathbb{V}}^{1,r_1} \times \mathbb{X}^2$. The space $\mathcal{H}_0$ is a Hilbert space with the norm whose square is given, for $\mathcal{X} = (U,V) \in \mathcal{H}_0$, by

$$\begin{aligned}\|\mathcal{X}\|^2_{\mathcal{H}_0} &:= \|U\|^2_{\widetilde{\mathbb{V}}^{1,r_1}} + \|V\|^2_{\mathbb{X}^2} = \|u\|^2_{H^1(\Omega)} + \|u\|^{r_1}_{L^{r_1}(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\gamma\|^2_{H^1(\Gamma)} + \|\delta\|^2_{L^2(\Gamma)} \\ &= \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|u\|^{r_1}_{L^{r_1}(\Omega)} + \|v\|^2_{L^2(\Omega)} + \|\nabla_\Gamma \gamma\|^2_{L^2(\Gamma)} + \|\gamma\|^2_{L^2(\Gamma)} + \|\delta\|^2_{L^2(\Gamma)}.\end{aligned}$$

The space $\mathcal{H}_0$ is our weak energy phase space. Given $\mathcal{X}_0 = (U_0, U_1) \in \mathcal{H}_0 = \widetilde{\mathbb{V}}^{1,r_1} \times \mathbb{X}^2$, the abstract formulation of Problem **P** takes the form

$$\begin{cases} \dfrac{d}{dt}\mathcal{X}(t) = \mathcal{A}_{\theta,\alpha,\omega}\mathcal{X}(t) + \mathcal{F}(\mathcal{X}(t)) & \text{for } t > 0, \\ \mathcal{X}(0) = \mathcal{X}_0. \end{cases}$$

We can now introduce the variational formulation of Problem **P**.

DEFINITION 2.3. Let $\theta \in [\frac{1}{2}, 1)$, $\alpha \in (0,1]$, $\omega \in (0,1]$, $T > 0$ and $\mathcal{X}_0 = (U_0, U_1) \in \mathcal{H}_0$. A function $\mathcal{X}(t) = (U(t), \partial_t U(t)) = (u(t), u_{|\Gamma}(t), \partial_t u(t), \partial_t u_{|\Gamma}(t))$ satisfying

$$\begin{aligned} U &\in L^\infty(0,T;\mathbb{V}^1), \\ \partial_t U &\in L^\infty(0,T;\mathbb{X}^2), \\ \sqrt{\omega}\partial_t u &\in L^2(0,T;H^\theta(\Omega)), \\ \partial_t^2 U &\in L^\infty(0,T;(\mathbb{V}^1)^*), \end{aligned}$$

for almost all $t \in (0,T]$ is called a *weak solution* to Problem **P** with initial data $\mathcal{X}_0$ if

$$\frac{d}{dt}\langle \mathcal{X}(t), \Xi\rangle_{\mathcal{V}^{-1}\times\mathcal{V}^1} = \langle \mathcal{A}_{\theta,\alpha,\omega}\mathcal{X}(t), \Xi\rangle_{\mathcal{H}_0} + \langle \mathcal{F}(\mathcal{X}(t)), \Xi\rangle_{\mathcal{H}_0} \tag{2.1}$$

holds almost everywhere on $[0,T]$ and for all $\Xi = (\Xi_1, \Xi_2) \in \mathbb{V}^1 \times \mathbb{V}^1$. The initial conditions (1.3)–(1.4) hold in the L^2-sense, that is,

$$\langle \mathcal{X}(0), \Xi\rangle_{\mathcal{H}_0} = \langle \mathcal{X}_0, \Xi\rangle_{\mathcal{H}_0} \quad \text{for every } \Xi \in \mathbb{V}^1 \times \mathbb{V}^1.$$

We say that $\mathcal{X}(t) = (U(t), \partial_t U(t))$ is a *global weak solution* of Problem **P** if it is a weak solution on $[0,T]$ for any $T > 0$.

REMARK 2.4. Observe that we are solving a more general problem because γ_0 and γ_1, from U_0 and U_1, respectively, may be taken to be initial data *independent* of u and $\partial_t u$. However, if $\partial_t u(t) \in H^s(\Omega)$ for all $t > 0$ and for some $s > 1/2$, then $\gamma_t(t) = \partial_t u_{|\Gamma}(t)$.

3. Global existence

Theorem 3.1. *Let $\mathcal{X}_0 = (U_0, U_1) \in \mathcal{H}_0$ satisfy $\|\mathcal{X}_0\|_{\mathcal{H}_0} \le R$ for some $R > 0$. Then there exists a global weak solution to Problem* **P** *satisfying the additional regularity*

$$\sqrt{\alpha\omega}\partial_t u \in L^2(0, T; H^1(\Gamma)). \tag{3.1}$$

Proof. We proceed in four steps.

Step 1. An a priori *estimate.* In (2.1), take $\Xi = (\partial_t U, \partial_t U)$ to find the differential identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\{\|\partial_t U\|^2_{\mathbb{X}^2} + \|U\|^2_{\mathbb{V}^1} + 2(F(u), 1)_{L^2(\Omega)} + 2(G(u), 1)_{L^2(\Gamma)}\} \\ &+ \omega\|\nabla^\theta \partial_t u\|^2_{L^2(\Omega)} + \|\partial_t u\|^2_{L^2(\Omega)} + \alpha\omega\|\nabla\partial_t u\|^2_{L^2(\Gamma)} + \|\partial_t u\|^2_{L^2(\Gamma)} = 0. \end{aligned}\tag{3.2}$$

Using (1.6) and setting $\tilde{F}' = f$ and $\tilde{G}' = g$, a simple integration by parts on (1.5) gives, for all $u \in H^1(\Omega)$ and $\gamma \in H^1(\Gamma)$,

$$(\tilde{F}(u), 1)_{L^2(\Omega)} \ge (f(u), u)_{L^2(\Omega)} + \frac{M_1}{2}\|u\|^2_{L^2(\Omega)} \tag{3.3}$$

and

$$(\tilde{G}(\gamma), 1)_{L^2(\Gamma)} \ge (g(\gamma), \gamma)_{L^2(\Gamma)} + \frac{M_2}{2}\|\gamma\|^2_{L^2(\Gamma)}. \tag{3.4}$$

To bound the products on the right-hand sides of (3.3) and (3.4) from below, we utilise (1.7). Following [9, (2.22)], [12, (3.34)] and [20, (3.11)], we estimate the products as

$$\begin{aligned}(f(u), u)_{L^2(\Omega)} &+ (g(u), u)_{L^2(\Gamma)} \\ &= \int_\Omega \Big(f(u)u + \frac{|\Gamma|}{|\Omega|}g(u)u\Big)\,dx - \frac{|\Gamma|}{|\Omega|}\int_\Omega\Big(g(u)u - \frac{1}{|\Gamma|}\int_\Gamma g(u)u\,d\sigma\Big)\,dx. \end{aligned}\tag{3.5}$$

Using the Poincaré inequality (1.8) and Young's inequality, for all $\varepsilon > 0$,

$$\begin{aligned}\frac{|\Gamma|}{|\Omega|}\int_\Omega\Big(g(u)u - \frac{1}{|\Gamma|}\int_\Gamma g(u)u\,d\sigma\Big)\,dx &\le C_\Omega\frac{|\Gamma|}{|\Omega|}\int_\Omega |\nabla(g(u)u)|\,dx \\ &= C_\Omega\frac{|\Gamma|}{|\Omega|}\int_\Omega |\nabla u(g'(u)u + g(u))|\,dx \\ &\le \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + \frac{C_\Omega^2|\Gamma|^2}{4\varepsilon|\Omega|^2}\int_\Omega |g'(u)u + g(u)|^2\,dx. \end{aligned}\tag{3.6}$$

Then combining (3.5) and (3.6) and applying assumption (1.7) yields

$$(f(u), u)_{L^2(\Omega)} + (g(u), u)_{L^2(\Gamma)} \ge \|u\|^{r_1}_{L^{r_1}(\Omega)} - \varepsilon\|\nabla u\|^2_{L^2(\Omega)} - C_\delta$$

for some positive constants δ and C_δ that are independent of t and ε. Hence, together with (3.3) and (3.4), this becomes

$$(F(u), 1)_{L^2(\Omega)} + (G(u), 1)_{L^2(\Gamma)} \ge \|u\|^{r_1}_{L^{r_1}(\Omega)} + \frac{M_1}{2}\|u\|^2_{L^2(\Omega)} + \frac{M_2}{2}\|u\|^2_{L^2(\Gamma)} - \varepsilon\|\nabla u\|^2_{L^2(\Omega)} - C_\delta. \tag{3.7}$$

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Moreover, (3.7) provides a lower bound to the functional

$$E(t) := \|\partial_t U(t)\|^2_{\mathbb{X}^2} + \|U(t)\|^2_{\mathbb{V}^1} + 2(F(u(t)), 1)_{L^2(\Omega)} + 2(G(u(t)), 1)_{L^2(\Gamma)}.$$

Integrating the identity (3.2) over $(0, t)$,

$$E(t) + 2\int_0^t (\omega\|\nabla^\theta \partial_t u(\tau)\|^2_{L^2(\Omega)} + \alpha\omega\|\nabla\partial_t u(\tau)\|^2_{L^2(\Gamma)} + \|\partial_t U(\tau)\|^2_{\mathbb{X}^2})\, d\tau = E(0). \tag{3.8}$$

We can find an upper bound on $E(0)$ with (1.6). Evidently,

$$\begin{aligned} &2(F(u(0)), 1)_{L^2(\Omega)} + 2(G(u(0)), 1)_{L^2(\Gamma)} \\ &\quad \le \ell_1(\|u(0)\|_{L^1(\Omega)} + \|u(0)\|^{r_1}_{L^{r_1}(\Omega)}) + \ell_2(\|u(0)\|_{L^1(\Gamma)} + \|u(0)\|^{r_2}_{L^{r_2}(\Gamma)}). \end{aligned} \tag{3.9}$$

Hence, (3.9) and the embedding $\mathbb{V}^1 \hookrightarrow \mathbb{X}^2$ show that

$$\begin{aligned} E(0) &\le \|\partial_t u(0)\|^2_{L^2(\Omega)} + \|\nabla u(0)\|^2_{L^2(\Omega)} + \|u(0)\|^2_{L^2(\Omega)} + \|\partial_t u(0)\|^2_{L^2(\Gamma)} + \|\nabla u(0)\|^2_{L^2(\Gamma)} \\ &\quad + \|u(0)\|^2_{L^2(\Gamma)} + \ell_1(\|u(0)\|_{L^1(\Omega)} + \|u(0)\|^{r_1}_{L^{r_1}(\Omega)}) + \ell_2(\|u(0)\|_{L^1(\Gamma)} + \|u(0)\|^{r_2}_{L^{r_2}(\Gamma)}) \\ &\le \|\partial_t U(0)\|^2_{\mathbb{X}^2} + \|U(0)\|^2_{\mathbb{V}^1} + C(\|U(0)\|_{\mathbb{V}^1} + \|u(0)\|^{r_1}_{L^{r_1}(\Omega)} + \|u(0)\|^{r_2}_{L^{r_2}(\Gamma)}). \end{aligned} \tag{3.10}$$

Thus (3.8) and (3.10) yield, for all $t \ge 0$,

$$\begin{aligned} &\|\partial_t U(t)\|^2_{\mathbb{X}^2} + \|U(t)\|^2_{\mathbb{V}^1} + 2(F(u(t)), 1)_{L^2(\Omega)} + 2(G(u(t)), 1)_{L^2(\Gamma)} \\ &\qquad + 2\int_0^t (\omega\|\nabla^\theta \partial_t u(\tau)\|^2_{L^2(\Omega)} + \alpha\omega\|\nabla\partial_t u(\tau)\|^2_{L^2(\Gamma)} + \|\partial_t U(\tau)\|^2_{\mathbb{X}^2})\, d\tau \\ &\quad \le \|\partial_t U(0)\|^2_{\mathbb{X}^2} + \|U(0)\|^2_{\mathbb{V}^1} + C(\|U(0)\|_{\mathbb{V}^1} + \|u(0)\|^{r_1}_{L^{r_1}(\Omega)} + \|u(0)\|^{r_2}_{L^{r_2}(\Gamma)}) \\ &\quad \le \|\partial_t U(0)\|^2_{\mathbb{X}^2} + \|U(0)\|^2_{\mathbb{V}^1} + C(\|U(0)\|_{\mathbb{V}^1} + \|u(0)\|^{r_1}_{L^{r_1}(\Omega)} + 1), \end{aligned} \tag{3.11}$$

where the last inequality follows from Lemma 2.2. Now we see that, for any $T > 0$,

$$U \in L^\infty(0, T; \mathbb{V}^1), \tag{3.12}$$

$$\partial_t U \in L^\infty(0, T; \mathbb{X}^2), \tag{3.13}$$

$$\sqrt{\omega}\partial_t u \in L^2(0, T; H^\theta(\Omega)), \tag{3.14}$$

$$\sqrt{\alpha\omega}\partial_t u \in L^2(0, T; H^1(\Gamma)), \tag{3.15}$$

$$F(u) \in L^\infty(0, T; L^1(\Omega)), \tag{3.16}$$

$$G(u) \in L^\infty(0, T; L^1(\Gamma)). \tag{3.17}$$

We have found $\mathcal{X} \in L^\infty(0, T; \mathcal{H}_0)$. Moreover, $\Delta_W U \in L^\infty(0, T; (\mathbb{V}^1)^*)$ because $U \in L^\infty(0, T; \mathbb{V}^1)$ and $\Delta_W^{\theta,\alpha,\omega}\partial_t U \in L^2(0, T; (\mathbb{V}^1)^*)$ because $\sqrt{\alpha\omega}\partial_t U \in L^2(0, T; \mathbb{V}^1)$. Therefore, after comparing terms in the first equation of (3.2), we see that

$$\partial_t^2 U \in L^2(0, T; (\mathbb{V}^1)^*). \tag{3.18}$$

Hence, this justifies our choice of test function in (3.2). With (3.15), we also find (3.1), as claimed. This concludes Step 1.

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Step 2. A Galerkin basis. According to Section 2, for each $\theta \in [\frac{1}{2}, 1)$, the operator $\mathcal{A}_{\theta,\alpha,\omega}$ admits a system of eigenfunctions $\Psi_i^{\theta,\alpha,\omega} = (\psi^{\theta,\alpha,\omega}, \phi^{\theta,\alpha,\omega}, \psi_{|\Gamma}^{\theta,\alpha,\omega}, \phi_{|\Gamma}^{\theta,\alpha,\omega})$ such that $\{\Psi_i^{\theta,\alpha,\omega}\}_{i=1}^{\infty} \subset D(\mathcal{A}_{\theta,\alpha,\omega}) \cap (C^2(\overline{\Omega}) \times C^2(\Gamma) \times C^2(\overline{\Omega}) \times C^2(\Gamma))$ and

$$\mathcal{A}_{\theta,\alpha,\omega}\Psi_i^{\theta,\alpha,\omega} = \Lambda_i \Psi_i^{\theta,\alpha,\omega} \quad \text{for } i = 1, 2, \ldots,$$

where the eigenvalues $\Lambda_i = \Lambda_i^{\theta,\alpha,\omega} \in (0, +\infty)$ may be put into increasing order and counted according to their multiplicity to form a diverging sequence. This means that the pair (Λ_i, Ψ_i), $\Psi_i = \Psi_i^{\theta,\alpha,\omega}$ is a classical solution of the elliptic problem

$$\begin{cases} -\Delta\psi_i + \psi_i + \omega(-\Delta)^\theta \phi_i + \phi_i = \Lambda_i \psi_i & \text{in } \Omega \\ -\alpha\omega\Delta_\Gamma \phi_{i|\Gamma} + \phi_{i|\Gamma} - \Delta_\Gamma \psi_{i|\Gamma} + \psi_{i|\Gamma} = \Lambda_i \psi_{i|\Gamma} & \text{on } \Gamma. \end{cases}$$

By the standard spectral theory, these eigenfunctions form an orthogonal basis in $\mathcal{H}_0$ that is orthonormal in $L^2(\Omega) \times L^2(\Omega) \times L^2(\Gamma) \times L^2(\Gamma)$.

Let $T > 0$ be fixed. For $n \in \mathbb{N}$, define the spaces

$$\mathbb{H}_n := \operatorname{span}\{\Psi_1^{\theta,\alpha,\omega}, \ldots, \Psi_n^{\theta,\alpha,\omega}\} \subset \mathcal{H}_0 \quad \text{and} \quad \mathbb{H}_\infty := \bigcup_{n=1}^{\infty} \mathbb{H}_n.$$

Obviously, $\mathbb{H}_\infty$ is a dense subspace of $\mathcal{H}_0$. For each $n \in \mathbb{N}$, let $\mathbb{P}_n : \mathcal{H}_0 \to \mathbb{H}_n$ denote the orthogonal projection of $\mathcal{H}_0$ onto $\mathbb{H}_n$. We seek functions of the form

$$\mathcal{X}^{(n)}(t) = \sum_{i=1}^{n} A_i(t) \Psi_i^{\theta,\alpha,\omega} \tag{3.19}$$

that will satisfy the associated discretised Problem $\mathbf{P}_n$ described below. The functions A_i are assumed to be (at least) $C^2((0, T))$ for $i = 1, \ldots, n$. Precisely,

$$u^{(n)}(t) = \sum_{i=1}^{n} A_i(t)\psi_i^{\theta,\alpha,\omega}, \quad \partial_t u^{(n)}(t) = \sum_{i=1}^{n} A_i'(t)\psi_i^{\theta,\alpha,\omega}, \tag{3.20}$$

and

$$u_{|\Gamma}^{(n)}(t) = \sum_{i=1}^{n} A_i(t)\phi_{i|\Gamma}^{\theta,\alpha,\omega}, \quad \partial_t u_{|\Gamma}^{(n)}(t) = \sum_{i=1}^{n} A_i'(t)\phi_{i|\Gamma}^{\theta,\alpha,\omega}. \tag{3.21}$$

Using semigroup properties of $\mathcal{A}_{\theta,\alpha,\omega}$, the domain $D(\mathcal{A}^{\theta,\alpha,\omega})$ is dense in $\mathcal{H}_0$. So, to approximate the given initial data $\mathcal{X}_0 \in \mathcal{H}_0$, we may take $\mathcal{X}_0^{(n)} \in D(\mathcal{A}^{\theta,\alpha,\omega})$ such that $\mathcal{X}_0^{(n)} \to \mathcal{X}_0$ in $\mathcal{H}_0$.

For $T > 0$ and for each integer $n \geq 1$, the weak formulation of the approximate Problem $\mathbf{P}_n$ is as follows: find $\mathcal{X}^{(n)}$ given by (3.19) such that, for all $\overline{\mathcal{X}} = (\overline{U}, \overline{V}) \in \mathbb{H}_n$,

$$\langle \partial_t \mathcal{X}^{(n)}, \overline{\mathcal{X}} \rangle_{\mathcal{H}_0} + \langle \mathcal{A}_{\theta,\alpha,\omega} \mathcal{X}^{(n)}, \overline{\mathcal{X}} \rangle_{\mathcal{H}_0} + \langle \mathbb{P}_n \mathcal{F}(\mathcal{X}^{(n)}), \overline{\mathcal{X}} \rangle_{\mathcal{H}_0} = 0 \tag{3.22}$$

for almost all $t \in (0, T)$, subject to the initial conditions

$$\langle \mathcal{X}^{(n)}(0), \overline{\mathcal{X}} \rangle_{\mathcal{H}_0} = \langle \mathcal{X}_0^{(n)}, \overline{\mathcal{X}} \rangle_{\mathcal{H}_0}. \tag{3.23}$$

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To show the existence of at least one solution to (3.22)–(3.23), we now suppose that n is fixed and we take $\overline{\mathcal{X}} = \mathcal{X}^{(k)}$ for some $1 \le k \le n$. Then, substituting the discretised functions (3.20)–(3.21) into (3.22)–(3.23), we find a system of ordinary differential equations in the unknowns $A_k = A_k(t)$ on $\mathcal{X}^{(n)}$. Also, we recall that

$$\langle \mathbb{P}_n \mathcal{F}(\mathcal{X}^{(n)}), \mathcal{X}^{(k)} \rangle_{\mathcal{H}_0} = \langle \mathcal{F}(\mathcal{X}^{(n)}), \mathbb{P}_n \mathcal{X}^{(k)} \rangle_{\mathcal{H}_0} = \langle \mathcal{F}(\mathcal{X}^{(n)}), \mathcal{X}^{(k)} \rangle_{\mathcal{H}_0}.$$

Since $f \in C(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, we may apply Cauchy's theorem for ordinary differential equations to find that there is $T_n \in (0, T)$ such that $A_k \in C^2((0, T_n))$, for $1 \le k \le n$, and that (3.22) holds in the classical sense for all $t \in [0, T_n]$. This shows the existence of at least one local solution to the approximate Problem $\mathbf{P}_n$ and ends Step 2.

Step 3. Boundedness and continuation of approximate maximal solutions. The *a priori* estimate (3.11) holds for any approximate solution $\mathcal{X}^{(n)}$ of Problem $\mathbf{P}_n$ on the interval $[0, T_n)$, where $T_n < T$. Thanks to the boundedness of the projector $\mathbb{P}_n$,

$$\begin{aligned}
&\|\partial_t U^{(n)}(t)\|^2_{\mathbb{X}^2} + \|U^{(n)}(t)\|^2_{\mathbb{V}^1} + 2(F(u^{(n)}(t)), 1)_{L^2(\Omega)} + 2(G(u^{(n)}(t)), 1)_{L^2(\Gamma)} \\
&\qquad + 2\int_0^t (\omega\|\nabla^\theta \partial_t u^{(n)}(\tau)\|^2_{L^2(\Omega)} + \alpha\omega\|\nabla\partial_t u^{(n)}(\tau)\|^2_{L^2(\Gamma)} + \|\partial_t U^{(n)}(\tau)\|^2_{\mathbb{X}^2})\, d\tau \\
&\quad \le \|\partial_t U(0)\|^2_{\mathbb{X}^2} + \|U(0)\|^2_{\mathbb{V}^1} + C(\|U(0)\|_{\mathbb{V}^1} + \|u(0)\|^{r_1}_{L^{r_1}(\Omega)} + \|u(0)\|^{r_2}_{L^{r_2}(\Gamma)}).
\end{aligned} \tag{3.24}$$

Since the right-hand side of (3.24) is independent of n and t, every approximate solution may be extended to the whole interval $[0, T]$, and because $T > 0$ is arbitrary, any approximate solution is a global one. From Step 1, we also obtain the uniform bounds (3.12)–(3.18) for each approximate solution $\mathcal{X}^{(n)}$. Thus,

$$U^{(n)} \in L^\infty(0, T; \mathbb{V}^1), \tag{3.25}$$

$$\partial_t U^{(n)} \in L^\infty(0, T; \mathbb{X}^2), \tag{3.26}$$

$$\sqrt{\omega}\partial_t u^{(n)} \in L^2(0, T; H^\theta(\Omega)), \tag{3.27}$$

$$\sqrt{\alpha\omega}\partial_t u^{(n)} \in L^2(0, T; H^1(\Gamma)), \tag{3.28}$$

$$F(u^{(n)}) \in L^\infty(0, T; L^1(\Omega)), \tag{3.29}$$

$$G(u^{(n)}) \in L^\infty(0, T; L^1(\Gamma)). \tag{3.30}$$

This concludes Step 3.

Step 4. Convergence of approximate solutions. By applying Alaoglu's theorem (see, for example, [19, Theorem 6.64]) to the uniform bounds (3.25)–(3.30), we see that there is a subsequence of $\mathcal{X}^{(n)}$, generally not relabelled, and a function $\mathcal{X} = (u, \partial_t u, u_{|\Gamma}, \partial_t u_{|\Gamma})$ that obey (3.12)–(3.18) such that, as $n \to \infty$,

$$U^{(n)} \rightharpoonup U \quad \text{weakly-}* \text{ in} \quad L^\infty(0, T; \mathbb{V}^1), \tag{3.31}$$

$$\partial_t U^{(n)} \rightharpoonup \partial_t U \quad \text{weakly-}* \text{ in} \quad L^\infty(0, T; \mathbb{X}^2), \tag{3.32}$$

$$\sqrt{\omega}\partial_t u^{(n)} \rightharpoonup u \quad \text{weakly in} \quad L^2(0, T; H^\theta(\Omega)), \tag{3.33}$$

$$\sqrt{\alpha\omega}\partial_t u^{(n)} \rightharpoonup u \quad \text{weakly in} \quad L^2(0, T; H^1(\Gamma)), \tag{3.34}$$

$$\partial_t U^{(n)} \rightharpoonup \partial_t U \quad \text{weakly in} \quad L^2(0, T; (\mathbb{V}^1)^*). \tag{3.35}$$

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Using the above convergences (3.31) and (3.32), as well as the fact that the injection $\mathbb{V}^1 \hookrightarrow \mathbb{X}^2$ is compact, we draw upon the conclusion of the Aubin–Lions lemma (see Lemma A.1) to deduce that the embedding

$$\mathcal{W} := \{U \in L^2(0,T;\mathbb{V}^1) : \partial_t U \in L^2(0,T;\mathbb{X}^2)\} \hookrightarrow L^2(0,T;\mathbb{X}^2) \tag{3.36}$$

is compact (see, for example, [22]). Thus,

$$U^{(n)} \to U \quad \text{strongly in } L^2(0,T;\mathbb{X}^2) \tag{3.37}$$

and $U^{(n)}$ converges to U almost everywhere in $\Omega \times (0,T)$. The last strong convergence property is enough to pass to the limit in the nonlinear terms since $f, g \in C^1(\mathbb{R})$ (see, for example, [9, 13]). Indeed, on account of standard arguments (see also [5]),

$$\mathbb{P}_n \mathcal{F}(\mathcal{X}^{(n)}) \rightharpoonup \mathcal{F}(\mathcal{X}) \text{ weakly in } L^2(0,T;\mathcal{H}_0). \tag{3.38}$$

At this point, the convergence properties (3.31)–(3.38) are sufficient to pass to the limit as $n \to \infty$ in equation (3.22). Additionally, we recover (2.1) using standard density arguments. The proof of the theorem is finished. □

Concerning uniqueness. A proof of the following conjecture is needed to show that the weak solutions to Problem **P**, constructed above, depend continuously on initial data, and hence are unique.

Conjecture 3.2. Let $T > 0$, $R > 0$ and $\mathcal{X}_{01} = (U_{01}, U_{11}), \mathcal{X}_{02} = (U_{02}, U_{12}) \in \mathcal{H}_0$ be such that $\|\mathcal{X}_{01}\|_{\mathcal{H}_0} \le R$ and $\|\mathcal{X}_{02}\|_{\mathcal{H}_0} \le R$. Any two weak solutions, $\mathcal{X}^1(t)$ and $\mathcal{X}^2(t)$, to Problem **P** on $[0,T]$ corresponding to the initial data $\mathcal{X}_{01}$ and $\mathcal{X}_{02}$, respectively, satisfy

$$\|\mathcal{X}^1(t) - \mathcal{X}^2(t)\|_{\mathcal{H}_0} \le e^{Q(R)t}\|\mathcal{X}_{01} - \mathcal{X}_{02}\|_{\mathcal{H}_0} \quad \text{for all } t \in [0,T].$$

In order to prove the conjecture, typically, one needs to control products of the form

$$(f(u^1) - f(u^2), \partial_t \bar{u})_{L^2(\Omega)} \quad \text{and} \quad (g(u^1) - g(u^2), \partial_t \bar{u})_{L^2(\Gamma)},$$

where u^1 and u^2 are two weak solutions corresponding to (possibly the same) data $\mathcal{X}_{01} = (U_{01}, U_{11}) = (u_{01}, \gamma_{01}, u_{11}, \gamma_{11})$ and $\mathcal{X}_{02} = (U_{02}, U_{12}) = (u_{02}, \gamma_{02}, u_{12}, \gamma_{12})$. A suitable control on $\|f(u^1) - f(u^2)\|_{L^q(\Omega)}$, for example, is readily available when we assume (1.6) with $r_1 \in [1,3]$ (see [14, Lemma 2.6])), but this is no longer valid when we assume that $r_1 \ge 1$ is arbitrary. In the later case, it would be interesting to investigate whether a *generalised semiflow* in the sense of [2, 3] exists. Under certain conditions, such generalised semiflows admit global attractors which have similar properties to their well-posed counterparts (see [15]).

Appendix A

As introduced in Section 2, the Wentzell–Laplacian Δ_W on $\mathbb{X}^2$ with domain

$$D(\Delta_W) := \{U = (u,\gamma)^{\mathrm{tr}} \in \mathbb{V}^1 : -\Delta u \in L^2(\Omega), \partial_{\mathbf{n}} u = -\gamma + \Delta_\Gamma \gamma \in L^2(\Gamma), \gamma = \mathrm{tr}_D(u)\}$$

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is positive, self-adjoint and has compact resolvent [1]. From [18, Theorem A.37 (Spectral Theorem) and (A.28)], for each $\theta \in [\frac{1}{2}, 1)$,

$$D(\Delta_W^\theta) = \Big\{U = (u, \gamma)^{\mathrm{tr}} \in D(\Delta_W) : \sum_{j=1}^{\infty} \Lambda_j^{2\theta} |(U, W_j)|^2 < \infty\Big\}, \quad \text{where } \Delta_W W_j = \Lambda_j W_j,$$

and the sequence $(\Lambda_j)_{j=1}^{\infty}$ contains real, strictly positive eigenvalues, each having finite multiplicity, which can be ordered into a nondecreasing sequence in which

$$\lim_{j\to\infty} \Lambda_j = +\infty.$$

We mention some results from [10, Theorem 5.2 (c)] concerning the regularity of the eigenfunctions W_j. If Γ is Lipschitz, then every eigenfunction $W_j \in \mathcal{V}^1$ and $W_j \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ for every j. If Γ is of class C^2, then every eigenfunction $W_j \in \mathcal{V}^1 \cap C^2(\overline{\Omega})$ for every j.

We define the fractional powers of the Wentzell–Laplacian with a Fourier series,

$$\Delta_W^\theta U = \sum_{j=1}^{\infty} \Lambda_j^{2\theta} (U, W_j) W_j,$$

and we can rely on [8, (2.6)] to define the fractional flux, where

$$\Delta_W^{\theta/2} U = \nabla_W^\theta U = \sum_{i=1}^{N} \frac{\partial^\theta U}{\partial x_i^\theta} \mathbf{e}_i$$

and

$$\frac{d^\theta U}{dx^\theta} = \frac{1}{\Gamma(1-\theta)} \frac{d}{dx} \int_{-\infty}^{x} (x-y)^{-\theta} U(y)\, dy.$$

The following result is the classical Aubin–Lions lemma (see [17] and, for example, [21, Lemma 5.51] or [23, Theorem 3.1.1]).

LEMMA A.1. *Let X, Y, Z be Banach spaces, where $Z \hookrightarrow Y \hookrightarrow X$ with continuous injections, the second being compact. Then the following embeddings are compact:*

$$W := \{\chi \in L^2(0, T; X),\ \partial_t \chi \in L^2(0, T; Z)\} \hookrightarrow L^2(0, T; Y).$$

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