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$\{\sigma, \tau\}$-Rota–Baxter Operators, Infinitesimal Hom-bialgebras and the Associative (Bi)Hom-Yang–Baxter Equation

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Abstract. We introduce the concept of a $\{\sigma, \tau\}$-Rota–Baxter operator, as a twisted version of a Rota–Baxter operator of weight zero. We show how to obtain a certain $\{\sigma, \tau\}$-Rota–Baxter operator from a solution of the associative (Bi)Hom-Yang–Baxter equation, and, in a compatible way, a Hom-pre-Lie algebra from an infinitesimal Hom-bialgebra.

1 Introduction

Hom-type algebras appeared in the Physics literature related to quantum deformations of algebras of vector fields; these types of algebras satisfy a modified version of the Jacobi identity involving a homomorphism, and were called Hom-Lie algebras by Hartwig, Larsson and Silvestrov in [10, 12]. Hom-analogues of various classical algebraic structures have subsequently been introduced in the literature, such as Hom-(co)associative (co)algebras, Hom-dendriform algebras, Hom-pre-Lie algebras etc. Recently, structures of a more general type, called *BiHom-type algebras* and for which a classical algebraic identity is twisted by two commuting homomorphisms (called *structure maps*), were introduced in [8].

Infinitesimal bialgebras were introduced by Joni and Rota in [11] (under the name infinitesimal coalgebra). The current term is due to Aguiar, who developed a theory for them in a series of papers ([2–4]). It turns out that infinitesimal bialgebras have connections with some other concepts such as Rota–Baxter operators, pre-Lie algebras, Lie bialgebras etc. Aguiar discovered a large class of examples of infinitesimal bialgebras; namely, he showed that the path algebra of an arbitrary quiver carries a natural structure of infinitesimal bialgebra. In an analytical context, infinitesimal bialgebras have been used in [21] by Voiculescu in free probability theory.

The Hom-analogues of infinitesimal bialgebras, called *infinitesimal Hom-bialgebras*, were introduced and studied by Yau in [22]. He extended to the Hom-context some of Aguiar's results; however, there exist several basic results of Aguiar that do

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not have a Hom-analogue in Yau's paper. It is our aim here to complete the study by proving those Hom-analogues.

The associative Yang–Baxter equation was introduced by Aguiar in [2]. Let (A, μ) be an associative algebra and $r = \sum_i x_i \otimes y_i \in A \otimes A$; then r is called a *solution* of the associative Yang–Baxter equation if

$$\sum_{i,j} x_i \otimes y_i x_j \otimes y_j = \sum_{i,j} x_i x_j \otimes y_j \otimes y_i + \sum_{i,j} x_i \otimes x_j \otimes y_j y_i.$$

In this situation, Aguiar noticed in [1] that the map $R: A \to A$, $R(a) = \sum_i x_i a y_i$, is a Rota–Baxter operator of weight zero. We recall (see, for instance, [9]) that if B is an algebra and $R: B \to B$ is a linear map, then R is called a *Rota–Baxter operator of weight zero* if

$$R(a)R(b) = R(R(a)b + aR(b)), \quad \forall\, a, b \in B.$$

Rota–Baxter operators appeared first in the work of Baxter in probability and the study of fluctuation theory, and were intensively studied by Rota in connection with combinatorics. Rota–Baxter operators occurred also in other areas of mathematics and physics, notably in the seminal work of Connes and Kreimer [6] concerning a Hopf algebraic approach to renormalization in quantum field theory.

The Hom-analogue of the associative Yang–Baxter equation was introduced by Yau in [22] but without exploring the relation between this new equation and Rota–Baxter operators. Our first aim is to obtain Hom and BiHom-analogues of Aguiar's observation mentioned above, expressing a relationship between Hom and BiHom-analogues of the associative Yang–Baxter equation and certain generalized Rota–Baxter operators. The BiHom-analogue of the associative Yang–Baxter equation is defined as follows. Let (A, μ, α, β) be a BiHom-associative algebra and let $r = \sum_i x_i \otimes y_i \in A \otimes A$ such that $(\alpha \otimes \alpha)(r) = r = (\beta \otimes \beta)(r)$; we say that r is a solution of the associative BiHom-Yang–Baxter equation if

$$\sum_{i,j} \alpha(x_i) \otimes y_i x_j \otimes \beta(y_j) = \sum_{i,j} x_i x_j \otimes \beta(y_j) \otimes \beta(y_i) + \sum_{i,j} \alpha(x_i) \otimes \alpha(x_j) \otimes y_j y_i.$$

With such an element r we want to associate a certain linear map $R: A \to A$ that will turn out to be a twisted version of a Rota–Baxter operator of weight zero. More precisely, the map R is defined by

$$R: A \longrightarrow A, \quad R(a) = \sum_i \alpha\beta^3(x_i)\big(a\alpha^3(y_i)\big) = \sum_i \big(\beta^3(x_i)a\big)\alpha^3\beta(y_i), \quad \forall\, a \in A,$$

which, in the Hom case (*i.e.,* for $\alpha = \beta$), reduces to

$$R(a) = \sum_i \alpha(x_i)(a y_i) = \sum_i (x_i a)\alpha(y_i),$$

for all $a \in A$, and the equation it satisfies is (see Theorem 4.4)

$$R\big(\alpha\beta(a)\big)R\big(\alpha\beta(b)\big) = R\big(\alpha\beta(a)R(b) + R(a)\alpha\beta(b)\big), \quad \forall\, a, b \in A.$$

We call a linear map satisfying this equation an *αβ-Rota–Baxter operator* (of weight zero). This is a particular case of the following concept we introduce and study in this paper. Let B be an algebra, let $\sigma, \tau: B \to B$ be algebra maps, and let $R: B \to B$ be a linear map. We call R a *{σ, τ}-Rota–Baxter operator* if

$$R\big(\sigma(a)\big)R\big(\tau(b)\big) = R\big(\sigma(a)R(b) + R(a)\tau(b)\big), \quad \forall\, a, b \in B.$$

This concept is a sort of modification of the concept of (σ, τ)-Rota–Baxter operator introduced in [20] (inspired by an example in [7]). In Section 3 we prove that certain classes of $\{\sigma, \tau\}$-Rota–Baxter operators have similar properties to those of a usual Rota–Baxter operator of weight zero (see Theorem 3.12 and its corollaries, and Proposition 3.17).

Our second aim is to extend to infinitesimal Hom-bialgebras the following result from [4] providing a left pre-Lie algebra from a given infinitesimal bialgebra.

Theorem 1.1 (Aguiar) *Let (A, μ, Δ) be an infinitesimal bialgebra, with notation $\mu(a \otimes b) = ab$ and $\Delta(a) = a_1 \otimes a_2$, for all $a, b \in A$. If we define a new operation on A by $a \bullet b = b_1 a b_2$, then $(A, \bullet)$ is a left pre-Lie algebra.*

Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra, with notation $\mu(a \otimes b) = ab$ and $\Delta(a) = a_1 \otimes a_2$, for all $a, b \in A$. We want to define a new multiplication $\bullet$ on A, turning it into a left Hom-pre-Lie algebra. It is not clear what the formula for this multiplication should be (note, for instance, that the obvious choice $a \bullet b = \alpha(b_1)(ab_2) = (b_1 a)\alpha(b_2)$ does not work), so we need to guess. We proceed as follows. Recall first the following old result.

Theorem 1.2 (Gel'fand–Dorfman) *Let (A, μ) be an associative and commutative algebra, with notation $\mu(a \otimes b) = ab$, and let $D: A \to A$ be a derivation. Define a new multiplication on A by $a \star b = aD(b)$. Then $(A, \star)$ is a left pre-Lie algebra (it is actually even a Novikov algebra).*

We make the following observation: if the infinitesimal bialgebra in Aguiar's Theorem is commutative, then his theorem is a particular case of the theorem of Gel'fand and Dorfman. Indeed, by using commutativity, the multiplication $\bullet$ becomes $a \bullet b = b_1 a b_2 = a b_1 b_2 = aD(b)$, where we denoted by D the linear map $D: A \to A$, $D(b) = b_1 b_2$, *i.e.*, $D = \mu \circ \Delta$, and it is well known (see [2]) that D is a derivation.

We want to exploit this observation in order to guess the formula for the multiplication in the Hom case. There, we already have an analogue of the Gel'fand–Dorfman Theorem, due to Yau (see [23]), saying that if (A, μ, α) is a commutative Hom-associative algebra, $D: A \to A$ is a derivation (in the usual sense) commuting with α, and we define a new multiplication on A by $a * b = aD(b)$, then $(A, *, \alpha)$ is a left Hom-pre-Lie algebra (it is actually even Hom-Novikov). So, we begin with a commutative infinitesimal Hom-bialgebra (A, μ, Δ, α) and we define the map $D: A \to A$ also by the formula $D = \mu \circ \Delta$. The problem is that, because of the condition from the definition of an infinitesimal Hom-bialgebra satisfied by Δ (which involves the map α), D is not a derivation (so we cannot use Yau's result mentioned above). Instead, it turns out that D is a so-called α^2-derivation, that is, it satisfies $D(ab) = \alpha^2(a)D(b) + D(a)\alpha^2(b)$. So what we need first is a generalization of Yau's version of the Gel'fand–Dorfman Theorem, one that would apply not only to derivations but also to α^2-derivations. A generalization dealing with α^k-derivations, for k an arbitrary natural number, is achieved in Proposition 5.1. The outcome is a left Hom-pre-Lie algebra (actually, a Hom-Novikov algebra) whose structure map is α^{k+1}. Coming back to the case $k = 2$, by applying this result we obtain that, for the commutative infinitesimal Hom-bialgebra we started with, we are able to obtain a left Hom-pre-Lie algebra structure on it, with structure map α^3 and multiplication $x \bullet y = \alpha^2(x)D(y) = \alpha^2(x)(y_1 y_2)$, which, by using commutativity and Hom-associativity, can be written as $x \bullet y = \alpha(y_1)(\alpha(x)y_2)$.

We can consider this formula even if the infinitesimal Hom-bialgebra is not commutative, and it turns out that this is the formula we were trying to guess (see Proposition 5.4).

Let (A, μ, Δ_r) be a quasitriangular infinitesimal bialgebra; *i.e.*, the comultiplication is given by the principal derivation corresponding to a solution $r = \sum_i x_i \otimes y_i$ of the associative Yang–Baxter equation. There are two left pre-Lie algebras associated with A: the first one is obtained by Theorem 1.1; the second is obtained from the fact that the Rota–Baxter operator $R: A \to A$, $R(a) = \sum_i x_i a y_i$ provides a dendriform algebra, which in turn provides a left pre-Lie algebra. Aguiar proved in [4] that these two left pre-Lie algebras coincide. Our last result shows that the Hom-analogue of this fact is also true.

In a subsequent paper we will introduce the BiHom-analogue of infinitesimal bialgebras and prove the BiHom-analogue of Theorem 1.1. It turns out that things are more complicated than in the Hom case, and, moreover, the result in the Hom case is not a particular case of the corresponding result in the BiHom case. This comes essentially from the following phenomenon. A BiHom-associative algebra (A, μ, α, β) for which $\alpha = \beta$ is the same thing as the Hom-associative algebra (A, μ, α). But a left BiHom-pre-Lie algebra (A, μ, α, β) for which $\alpha = \beta$ is not the same thing as the left Hom-pre-Lie algebra (A, μ, α), unless α is bijective.

2 Preliminaries

We work over a base field $\Bbbk$. All algebras, linear spaces, etc., will be over $\Bbbk$; unadorned $\otimes$ means $\otimes_{\Bbbk}$. By an *algebra* we mean a pair (A, μ), where A is a linear space and $\mu: A \otimes A \to A$ is a linear map, usually denoted by $\mu(a \otimes a') = aa'$, for $a, a' \in A$. Unless otherwise specified, the (co)algebras that will appear in what follows are not supposed to be (co)associative or (co)unital, and for a comultiplication $\Delta: C \to C \otimes C$ on a linear space C, we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. For the composition of two maps f and g, we will write either $g \circ f$ or simply gf. For the identity map on a linear space V we will use the notation id_V.

Definition 2.1 ([8]) A BiHom-associative algebra is a 4-tuple (A, μ, α, β), where A is a linear space, $\alpha, \beta: A \to A$ and $\mu: A \otimes A \to A$ are linear maps, such that

$$\alpha \circ \beta = \beta \circ \alpha, \quad \alpha(xy) = \alpha(x)\alpha(y), \quad \beta(xy) = \beta(x)\beta(y),$$

and the so-called BiHom-associativity condition

$$\alpha(x)(yz) = (xy)\beta(z) \tag{2.1}$$

hold, for all $x, y, z \in A$. The maps α and β (in this order) are called the *structure maps* of A.

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A Hom-associative algebra, as defined in [17], is a BiHom-associative algebra (A,μ,α,β) for which $\alpha=\beta$. The defining relation,

$$\alpha(x)(yz)=(xy)\alpha(z),\quad \forall\, x,y,z\in A, \tag{2.2}$$

is called the *Hom-associativity condition*, and the map α is called the *structure map*.

If (A,μ) is an associative algebra and $\alpha,\beta\colon A\to A$ are two commuting algebra maps, then $A_{(\alpha,\beta)}:=(A,\mu\circ(\alpha\otimes\beta),\alpha,\beta)$ is a BiHom-associative algebra, called the *Yau twist* of A via the maps α and β.

Definition 2.2 ([18]) A Hom-coassociative coalgebra is a triple (C,Δ,α), in which C is a linear space, $\alpha\colon C\to C$, and $\Delta\colon C\to C\otimes C$ are linear maps, such that $(\alpha\otimes\alpha)\circ\Delta=\Delta\circ\alpha$ and

$$(\Delta\otimes\alpha)\circ\Delta=(\alpha\otimes\Delta)\circ\Delta. \tag{2.3}$$

The map α is called the *structure map*, and (2.3) is called the *Hom-coassociativity condition*.

For a Hom-coassociative coalgebra (C,Δ,α), we will use the extra notation

$$(\mathrm{id}\otimes\Delta)(\Delta(c))=c_1\otimes c_{(2,1)}\otimes c_{(2,2)}\quad\text{and}\quad(\Delta\otimes\mathrm{id})(\Delta(c))=c_{(1,1)}\otimes c_{(1,2)}\otimes c_2,$$

for all $c\in C$.

Definition 2.3 A left pre-Lie algebra is a pair (A,μ), where A is a a linear space and $\mu\colon A\otimes A\to A$ is a linear map satisfying the condition

$$x(yz)-(xy)z=y(xz)-(yx)z,\quad \forall\, x,y,z\in A.$$

A morphism of left pre-Lie algebras from (A,μ) to (A',μ') is a linear map $\alpha\colon A\to A'$ satisfying $\alpha(xy)=\alpha(x)\alpha(y)$, for all $x,y\in A$.

Definition 2.4 ([17, 23]) A left Hom-pre-Lie algebra is a triple (A,μ,α), where A is a linear space, and $\mu\colon A\otimes A\to A$ and $\alpha\colon A\to A$ are linear maps satisfying $\alpha(xy)=\alpha(x)\alpha(y)$ and

$$\alpha(x)(yz)-(xy)\alpha(z)=\alpha(y)(xz)-(yx)\alpha(z),$$

for all $x,y,z\in A$. We call α the structure map of A. If, moreover, the condition

$$(xy)\alpha(z)=(xz)\alpha(y),\quad \forall\, x,y,z\in A,$$

is satisfied, then (A,μ,α) is called a *Hom-Novikov algebra*.

If (A,μ) is a left pre-Lie algebra and $\alpha\colon A\to A$ is a morphism of left pre-Lie algebras, then $A_\alpha:=(A,\alpha\circ\mu,\alpha)$ is a left Hom-pre-Lie algebra, called the *Yau twist* of A via the map α.

Definition 2.5 ([2]) An infinitesimal bialgebra is a triple (A,μ,Δ), in which (A,μ) is an associative algebra, (A,Δ) is a coassociative coalgebra, and $\Delta\colon A\to A\otimes A$ is a derivation; that is, $\Delta(ab)=ab_1\otimes b_2+a_1\otimes a_2b$ for all $a,b\in A$.

A morphism of infinitesimal bialgebras from (A,μ,Δ) to (A',μ',Δ') is a linear map $\alpha\colon A\to A'$ that is a morphism of algebras and a morphism of coalgebras.

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Definition 2.6 ([22]) An infinitesimal Hom-bialgebra is a 4-tuple (A,μ,Δ,α) in which (A,μ,α) is a Hom-associative algebra, (A,Δ,α) is a Hom-coassociative coalgebra and

$$\Delta(ab)=\alpha(a)b_1\otimes\alpha(b_2)+\alpha(a_1)\otimes a_2\alpha(b),\quad\forall\ a,b\in A. \tag{2.4}$$

Definition 2.7 ([22]) Let (A,μ,α) be a Hom-associative algebra and $r=\sum_i x_i\otimes y_i\in A\otimes A$ such that $(\alpha\otimes\alpha)(r)=r$. Define the following elements in $A\otimes A\otimes A$:

$$r_{12}r_{23}=\sum_{i,j}\alpha(x_i)\otimes y_ix_j\otimes\alpha(y_j),\qquad r_{13}r_{12}=\sum_{i,j}x_ix_j\otimes\alpha(y_j)\otimes\alpha(y_i),$$
$$r_{23}r_{13}=\sum_{i,j}\alpha(x_i)\otimes\alpha(x_j)\otimes y_jy_i,\qquad A(r)=r_{13}r_{12}-r_{12}r_{23}+r_{23}r_{13}.$$

We say that r is a solution of the associative Hom-Yang–Baxter equation if $A(r)=0$, that is

$$\sum_{i,j}\alpha(x_i)\otimes y_ix_j\otimes\alpha(y_j)=\sum_{i,j}x_ix_j\otimes\alpha(y_j)\otimes\alpha(y_i)+\sum_{i,j}\alpha(x_i)\otimes\alpha(x_j)\otimes y_jy_i. \tag{2.5}$$

We introduce the following variation of the concept introduced by Yau in [22].

Definition 2.8 An infinitesimal Hom-bialgebra (A,μ,Δ,α) is called *quasitriangular* if there exists an element $r\in A\otimes A$, $r=\sum_i x_i\otimes y_i$, such that $(\alpha\otimes\alpha)(r)=r$ and r is a solution of the associative Hom-Yang–Baxter equation, with the property that

$$\Delta(b)=\sum_i\alpha(x_i)\otimes y_ib-\sum_i bx_i\otimes\alpha(y_i),\quad\forall\ b\in A.$$

In this situation, we denote Δ by Δ_r.

Yau's definition requires $\Delta(b)=\sum_i bx_i\otimes\alpha(y_i)-\sum_i\alpha(x_i)\otimes y_ib$, for all $b\in A$. This is consistent with Aguiar's convention in [2]; our choice is consistent with the convention in [4].

Definition 2.9 ([13]) A BiHom-dendriform algebra is a 5-tuple $(A,\prec,\succ,\alpha,\beta)$ consisting of a linear space A, linear maps $\prec,\succ: A\otimes A\to A$ and commuting linear maps $\alpha,\beta: A\to A$ such that α and β are multiplicative with respect to $\prec$ and $\succ$ and satisfying the conditions

$$(x\prec y)\prec\beta(z)=\alpha(x)\prec(y\prec z+y\succ z),$$
$$(x\succ y)\prec\beta(z)=\alpha(x)\succ(y\prec z),$$
$$\alpha(x)\succ(y\succ z)=(x\prec y+x\succ y)\succ\beta(z),$$

for all $x,y,z\in A$. We call α and β (in this order) the structure maps of A.

A dendriform algebra, as introduced by Loday in [15], is just a BiHom-dendriform algebra $(A,\prec,\succ,\alpha,\beta)$ for which $\alpha=\beta=\mathrm{id}_A$. A Hom-dendriform algebra, as introduced in [16], is a BiHom-dendriform algebra $(A,\prec,\succ,\alpha,\beta)$ for which $\alpha=\beta$.

Let $(A,\prec,\succ)$ be a dendriform algebra and $\alpha,\beta: A\to A$ two commuting linear maps that are multiplicative with respect to $\prec$ and $\succ$. Define two new operations on A by $x\prec_{(\alpha,\beta)}y=\alpha(x)\prec\beta(y)$ and $x\succ_{(\alpha,\beta)}y=\alpha(x)\succ\beta(y)$, for all $x,y\in A$. Then

$A_{(\alpha,\beta)} := (A, \prec_{(\alpha,\beta)}, \succ_{(\alpha,\beta)}, \alpha, \beta)$ is a BiHom-dendriform algebra, called the *Yau twist* of A via the maps α and β.

Proposition 2.10 ([13,16,19]) *Let $(A, \prec, \succ, \alpha, \beta)$ be a BiHom-dendriform algebra and define a new multiplication on A by $x * y = x \prec y + x \succ y$. Then $(A, *, \alpha, \beta)$ is a BiHom-associative algebra. Moreover, if $\alpha = \beta$ and we define a new operation on A by $x \circ y = x \succ y - y \prec x$, then $(A, \circ, \alpha)$ is a left Hom-pre-Lie algebra.*

3 $\{\sigma, \tau\}$-Rota–Baxter operators

In this section we introduce and study some classes of modified Rota–Baxter operators that are twisted by algebra maps. We recall first the following well-known concept.

Definition 3.1 Let A be an algebra, let $\sigma, \tau: A \to A$ be algebra maps, and let $D: A \to A$ be a linear map. We call D a (τ, σ)-derivation if $D(ab) = D(a)\tau(b) + \sigma(a)D(b)$, for all $a, b \in A$.

The following concept is a variation of the one introduced in [20] for associative algebras.

Definition 3.2 Let A be an algebra, let $\sigma, \tau: A \to A$ be algebra maps, and let $R: A \to A$ be a linear map. We call R a (σ, τ)-Rota–Baxter operator (of weight zero) if

$$R(a)R(b) = R(\sigma(R(a))b + a\tau(R(b))), \quad \forall\ a, b \in A.$$

Remark 3.3 For associative algebras, an (id, τ)-Rota–Baxter operator is the same thing as a τ-twisted Rota–Baxter operator, a concept introduced in [5].

Remark 3.4 Let R be a (σ, τ)-Rota–Baxter operator on an associative algebra A. One can easily check that the triple $(A, \sigma \circ R, \tau \circ R)$ is a Rota–Baxter system, as defined by Brzeziński in [5] (the case $\sigma = \mathrm{id}_A$ may be found in [5]). Consequently, by [5], if we define two operations on A by $a \prec b = a\tau(R(b))$ and $a \succ b = \sigma(R(a))b$, then $(A, \prec, \succ)$ is a dendriform algebra.

It is well known that, if A is an algebra and $D: A \to A$ is a bijective linear map, then D is a derivation (in the usual sense) if and only D^{-1} is a Rota–Baxter operator of weight zero. This fact may be easily generalized, as follows:

Proposition 3.5 *Let A be an algebra, $\sigma, \tau: A \to A$ algebra maps and $D: A \to A$ a bijective linear map with inverse $R =: D^{-1}$. Then D is a (τ, σ)-derivation if and only if R is a (σ, τ)-Rota–Baxter operator.*

We are interested in the following modification of the concept of (σ, τ)-Rota–Baxter operator.

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Definition 3.6 Let A be an algebra, $\sigma, \tau\colon A \to A$ algebra maps and $R\colon A \to A$ a linear map. We call R a $\{\sigma, \tau\}$-Rota–Baxter operator (of weight zero) if

$$R(\sigma(a))R(\tau(b)) = R(\sigma(a)R(b) + R(a)\tau(b)), \quad \forall\ a, b \in A. \tag{3.1}$$

Remark 3.7 Let A be an algebra, $\sigma, \tau\colon A \to A$ bijective algebra maps and $R\colon A \to A$ a linear map commuting with σ and τ. Then one can easily see that R is a (σ, τ)-Rota–Baxter operator if and only if R is a $\{\sigma^{-1}, \tau^{-1}\}$-Rota–Baxter operator.

Remark 3.8 Let (A, μ) be an algebra, let $\sigma\colon A \to A$ be an algebra map, and let $R\colon A \to A$ be a Rota–Baxter operator of weight zero commuting with σ. Then one can easily see that $R\circ\sigma$ is a $\{\sigma, \sigma\}$-Rota–Baxter operator both for (A, μ) and for $(A, \sigma\circ\mu)$.

We will be particularly interested in the following two classes of $\{\sigma, \tau\}$-Rota–Baxter operators.

Definition 3.9 Let A be an algebra, let $\alpha\colon A \to A$ be an algebra map, and let $R\colon A \to A$ be a linear map commuting with α and n a natural number. We call R an α^n*-Rota–Baxter operator* if it is an $\{\alpha^n, \alpha^n\}$-Rota–Baxter operator, *i.e.*,

$$R\big(\alpha^n(a)\big)R\big(\alpha^n(b)\big) = R\big(\alpha^n(a)R(b) + R(a)\alpha^n(b)\big), \quad \forall\ a, b \in A.$$

Obviously, an α^0-Rota–Baxter operator is just a usual Rota–Baxter operator of weight zero commuting with α.

Remark 3.10 From previous remarks it follows that, if A is an algebra, $\alpha\colon A \to A$ a bijective algebra map, $D\colon A \to A$ a bijective linear map commuting with α and n a natural number, then $R := D^{-1}$ is an α^n-Rota–Baxter operator if and only if D is an $(\alpha^{-n}, \alpha^{-n})$-derivation.

Definition 3.11 Let (A, μ, α, β) be a BiHom-associative algebra and let $R\colon A \to A$ be a linear map commuting with α and β. We call R an $\alpha\beta$-Rota–Baxter operator if it is an $\{\alpha\beta, \alpha\beta\}$-Rota–Baxter operator, that is

$$R(\alpha\beta(a))R(\alpha\beta(b)) = R\big(\alpha\beta(a)R(b) + R(a)\alpha\beta(b)\big), \quad \forall\ a, b \in A.$$

Theorem 3.12 *Let (A, μ, α, β) be a BiHom-associative algebra and let $\sigma, \tau, \eta, R\colon A \to A$ be linear maps such that σ, τ, η are algebra maps, R is a $\{\sigma, \tau\}$-Rota–Baxter operator and any two of the maps $\alpha, \beta, \sigma, \tau, \eta, R$ commute. Define new operations on A by*

$$x \prec y = \sigma(x)R\eta(y) \quad \text{and} \quad x \succ y = R(x)\tau\eta(y),$$

for all $x, y \in A$. Then $(A, \prec, \succ, \alpha\sigma, \beta\tau\eta)$ is a BiHom-dendriform algebra.

Proof One can see that $\alpha\sigma$ and $\beta\tau\eta$ are multiplicative with respect to $\prec$ and $\succ$. We compute as follows:

$$\begin{aligned}(x \prec y) \prec \beta\tau\eta(z) &= \big(\sigma(x)R\eta(y)\big) \prec \beta\tau\eta(z) = \sigma\big(\sigma(x)R\eta(y)\big)R\beta\tau\eta^2(z)\\ &= \big(\sigma^2(x)\sigma R\eta(y)\big)\beta R\tau\eta^2(z)\end{aligned}$$

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$$
\begin{aligned}
&\overset{(2.1)}{=} \alpha\sigma^2(x)(R\sigma\eta(y)R\tau\eta^2(z))\\
&\overset{(3.1)}{=} \alpha\sigma^2(x)R(\sigma\eta(y)R\eta^2(z)+R\eta(y)\tau\eta^2(z))\\
&= \alpha\sigma^2(x)R\eta\big(\sigma(y)R\eta(z)+R(y)\tau\eta(z)\big)\\
&= \alpha\sigma(x)\prec\big(\sigma(y)R\eta(z)+R(y)\tau\eta(z)\big)\\
&= \alpha\sigma(x)\prec(y\prec z+y\succ z).
\end{aligned}
$$

Then we compute as follows:

$$
\begin{aligned}
(x\succ y)\prec\beta\tau\eta(z) &= \big(R(x)\tau\eta(y)\big)\prec\beta\tau\eta(z) = \sigma\big(R(x)\tau\eta(y)\big)R\beta\tau\eta^2(z)\\
&= \big(\sigma R(x)\sigma\tau\eta(y)\big)\beta R\tau\eta^2(z)\\
&\overset{(2.1)}{=} \alpha\sigma R(x)\big(\sigma\tau\eta(y)R\tau\eta^2(z)\big) = R\alpha\sigma(x)\tau\eta\big(\sigma(y)R\eta(z)\big)\\
&= \alpha\sigma(x)\succ\big(\sigma(y)R\eta(z)\big) = \alpha\sigma(x)\succ(y\prec z).
\end{aligned}
$$

Also, by using (3.1) again, one proves that $\alpha\sigma(x)\succ(y\succ z)=(x\prec y+x\succ y)\succ\beta\tau\eta(z)$, finishing the proof. ∎

We have some particular cases of this theorem.

Corollary 3.13 *Let A be an associative algebra, let $\sigma,\tau\colon A\to A$ be two commuting algebra maps, and let $R\colon A\to A$ be a $\{\sigma,\tau\}$-Rota–Baxter operator commuting with σ and τ. Define new operations on A by $x\prec y=\sigma(x)R(y)$ and $x\succ y=R(x)\tau(y)$, for $x,y\in A$. Then $(A,\prec,\succ,\sigma,\tau)$ is a BiHom-dendriform algebra. Moreover, if we consider $(A,*,\sigma,\tau)$ the BiHom-associative algebra associated with it as in Proposition 2.10, then R is a morphism of BiHom-associative algebras from $(A,*,\sigma,\tau)$ to $A_{(\sigma,\tau)}$, the Yau twist of the associative algebra A via the maps σ and τ.*

Proof In Theorem 3.12, take $\alpha=\beta=\eta=\mathrm{id}_A$. The second statement is obvious. ∎

Remark 3.14 Assume the hypotheses of Corollary 3.13 hold and, moreover, that σ and τ are bijective; denote $\alpha=\sigma^{-1}$, $\beta=\tau^{-1}$. By Remark 3.7, R is an (α,β)-Rota–Baxter operator, so, by Remark 3.4, A becomes a dendriform algebra with operations $a\prec b=a\tau^{-1}(R(b))$ and $a\succ b=\sigma^{-1}(R(a))b$. One can check that the Yau twist of this dendriform algebra via the maps σ and τ is exactly the BiHom-dendriform algebra obtained in Corollary 3.13.

Corollary 3.15 *Let (A,μ,α) be a Hom-associative algebra, n a natural number and $R\colon A\to A$ an α^n-Rota–Baxter operator. Define new operations on A by $x\prec y=\alpha^n(x)R(y)$ and $x\succ y=R(x)\alpha^n(y)$, for all $x,y\in A$. Then $(A,\prec,\succ,\alpha^{n+1})$ is a Hom-dendriform algebra. Consequently, by Proposition 2.10, if we define new operations on A by*

$$
\begin{aligned}
x*y&=x\prec y+x\succ y=\alpha^n(x)R(y)+R(x)\alpha^n(y),\\
x\circ y&=x\succ y-y\prec x=R(x)\alpha^n(y)-\alpha^n(y)R(x),
\end{aligned}
$$

then $(A,*,\alpha^{n+1})$ is a Hom-associative algebra and $(A,\circ,\alpha^{n+1})$ is a left Hom-pre-Lie algebra.

Proof In Theorem 3.12, take $\alpha = \beta$, $\sigma = \tau = \alpha^n$, $\eta = \mathrm{id}_A$. ∎

Corollary 3.16 *Let (A, μ, α, β) be a BiHom-associative algebra and $R: A \to A$ an $\alpha\beta$-Rota–Baxter operator. Let $\eta: A \to A$ be an algebra map commuting with α, β and R. Define new operations on A by $x \prec y = \alpha\beta(x)R\eta(y)$ and $x \succ y = R(x)\alpha\beta\eta(y)$, for all $x, y \in A$. Then $(A, \prec, \succ, \alpha^2\beta, \alpha\beta^2\eta)$ is a BiHom-dendriform algebra.*

Proof In Theorem 3.12, take $\sigma = \tau = \alpha\beta$. ∎

We recall from [10] that a Hom-Lie algebra is a triple $(L, [\cdot, \cdot], \alpha)$ in which L is a linear space, $\alpha: L \to L$ is a linear map, and $[\cdot, \cdot]: L \times L \to L$ is a bilinear map, such that, for all $x, y, z \in L$:

$$\alpha\big([x, y]\big) = \big[\alpha(x), \alpha(y)\big],$$

$$[x, y] = -[y, x], \qquad \text{(skew-symmetry)}$$

$$\big[\alpha(x), [y, z]\big] + \big[\alpha(y), [z, x]\big] + \big[\alpha(z), [x, y]\big] = 0. \qquad \text{(Hom-Jacobi condition)}$$

Proposition 3.17 *Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and let $R: L \to L$ be an α^n-Rota–Baxter operator,* i.e., *R commutes with α and*

$$\text{(3.2)} \quad \big[R(\alpha^n(a)), R(\alpha^n(b))\big] = R\big([\alpha^n(a), R(b)] + [R(a), \alpha^n(b)]\big), \quad \forall\, a, b \in L.$$

Then $(L, \cdot, \alpha^{n+1})$ is a left Hom-pre-Lie algebra, where $a \cdot b = [R(a), \alpha^n(b)]$ for all $a, b \in L$.

Proof Obviously, we have $\alpha^{n+1}(a \cdot b) = \alpha^{n+1}(a) \cdot \alpha^{n+1}(b)$, for all $a, b \in A$. Note that the Hom-Jacobi identity together with the skew-symmetry of the bracket $[\cdot, \cdot]$ imply

$$\text{(3.3)} \qquad \big[\alpha(a), [b, c]\big] = \big[[a, b], \alpha(c)\big] + \big[\alpha(b), [a, c]\big], \quad \forall\, a, b, c \in A.$$

Now for $x, y, z \in A$ we compute as follows:

$$\begin{aligned}
&\alpha^{n+1}(x) \cdot (y \cdot z) - (x \cdot y) \cdot \alpha^{n+1}(z) \\
&\quad = \alpha^{n+1}(x) \cdot [R(y), \alpha^n(z)] - [R(x), \alpha^n(y)] \cdot \alpha^{n+1}(z) \\
&\quad = \big[R(\alpha^{n+1}(x)), [\alpha^n(R(y)), \alpha^{2n}(z)]\big] - \big[R([R(x), \alpha^n(y)]), \alpha^{2n+1}(z)\big] \\
&\quad \overset{(3.2)}{=} \big[R(\alpha^{n+1}(x)), [\alpha^n(R(y)), \alpha^{2n}(z)]\big] - \big[[R(\alpha^n(x)), R(\alpha^n(y))], \alpha^{2n+1}(z)\big] \\
&\qquad + \big[R([\alpha^n(x), R(y)]), \alpha^{2n+1}(z)\big] \\
&\quad = \big[R(\alpha^{n+1}(x)), [R(\alpha^n(y)), \alpha^{2n}(z)]\big] - \big[[\alpha^n(R(x)), R(\alpha^n(y))], \alpha^{2n+1}(z)\big] \\
&\qquad + \big[R([\alpha^n(x), R(y)]), \alpha^{2n+1}(z)\big] \\
&\quad \overset{(3.3)}{=} \big[R(\alpha^{n+1}(x)), [R(\alpha^n(y)), \alpha^{2n}(z)]\big] - \big[\alpha^{n+1}(R(x)), [R(\alpha^n(y)), \alpha^{2n}(z)]\big] \\
&\qquad + \big[\alpha^{n+1}(R(y)), [\alpha^n(R(x)), \alpha^{2n}(z)]\big] + \big[R([\alpha^n(x), R(y)]), \alpha^{2n+1}(z)\big]
\end{aligned}$$

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$$\overset{\text{skew-symmetry}}{=} \big[\alpha^{n+1}(R(y)), [\alpha^n(R(x)), \alpha^{2n}(z)]\big] - \big[R([R(y), \alpha^n(x)]), \alpha^{2n+1}(z)\big]$$
$$= \alpha^{n+1}(y)\cdot(x\cdot z) - (y\cdot x)\cdot\alpha^{n+1}(z).$$

This finishes the proof. ∎

4 The Associative BiHom–Yang–Baxter Equation

In this section we introduce the associative BiHom-Yang–Baxter equation, generalizing the associative Yang–Baxter equation introduced by Aguiar as well as the associative Hom-Yang–Baxter equation introduced by Yau. Moreover, we discuss its connection with the generalized Rota–Baxter operators introduced in Section 3.

Definition 4.1 Let (A,μ,α,β) be a BiHom-associative algebra and $r=\sum_i x_i\otimes y_i \in A\otimes A$ such that $(\alpha\otimes\alpha)(r) = r = (\beta\otimes\beta)(r)$. We define the following elements in $A\otimes A\otimes A$:

$$r_{12}r_{23} = \sum_{i,j}\alpha(x_i)\otimes y_ix_j\otimes\beta(y_j), \qquad r_{13}r_{12} = \sum_{i,j} x_ix_j\otimes\beta(y_j)\otimes\beta(y_i),$$
$$r_{23}r_{13} = \sum_{i,j}\alpha(x_i)\otimes\alpha(x_j)\otimes y_jy_i, \qquad A(r) = r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13}.$$

We say that r is a solution of the associative BiHom-Yang–Baxter equation if $A(r)=0$, *i.e.*,

$$\sum_{i,j}\alpha(x_i)\otimes y_ix_j\otimes\beta(y_j) = \sum_{i,j}x_ix_j\otimes\beta(y_j)\otimes\beta(y_i) + \sum_{i,j}\alpha(x_i)\otimes\alpha(x_j)\otimes y_jy_i. \tag{4.1}$$

Remark 4.2 Obviously, for $\alpha=\beta$ the associative BiHom-Yang–Baxter equation reduces to the associative Hom-Yang–Baxter equation (2.5).

Remark 4.3 Assume that the BiHom-associative algebra A in the previous definition has a unit, that is (see [8]) an element $1_A\in A$ satisfying the conditions $\alpha(1_A) = \beta(1_A) = 1_A$, $a1_A = \alpha(a)$ and $1_Aa = \beta(a)$, for all $a\in A$. Then, by using the unit 1_A, one can define the elements $r_{12}, r_{13}, r_{23}\in A\otimes A\otimes A$ by $r_{12} = \sum_i x_i\otimes y_i\otimes 1_A$, $r_{13} = \sum_i x_i\otimes 1_A\otimes y_i$ and $r_{23}=\sum_i 1_A\otimes x_i\otimes y_i$. Then the element $r_{12}r_{23}$ defined above is just the product between r_{12} and r_{23} in $A\otimes A\otimes A$, but the element $r_{13}r_{12}$ is not the product between r_{13} and r_{12} (which is $\sum_{i,j}x_ix_j\otimes\beta(y_j)\otimes\alpha(y_i)$), and similarly the element $r_{23}r_{13}$ is not the product between r_{23} and r_{13}.

Theorem 4.4 *Let (A,μ,α,β) be a BiHom-associative algebra and let $r=\sum_i x_i\otimes y_i\in A\otimes A$ be such that $(\alpha\otimes\alpha)(r) = r = (\beta\otimes\beta)(r)$ and r is a solution of the associative BiHom-Yang–Baxter equation. Define the linear map*

$$R\colon A\longrightarrow A,\quad R(a) = \sum_i \alpha\beta^3(x_i)(a\alpha^3(y_i)) = \sum_i\big(\beta^3(x_i)a\big)\alpha^3\beta(y_i),\quad \forall a\in A. \tag{4.2}$$

Then R is an $\alpha\beta$-Rota–Baxter operator.

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Proof The fact that R commutes with α and β follows immediately from $(\alpha\otimes\alpha)(r) = r = (\beta\otimes\beta)(r)$. Now for $a, b \in A$, we compute as follows:

$$
\begin{aligned}
&R\big(\alpha\beta(a)\big)R\big(\alpha\beta(b)\big)\\
&= \Big\{\sum_i \big(\beta^3(x_i)\alpha\beta(a)\big)\alpha^3\beta(y_i)\Big\}\Big\{\sum_j \alpha\beta^3(x_j)\big(\alpha\beta(b)\alpha^3(y_j)\big)\Big\}\\
&\overset{(\alpha\otimes\alpha)(r)=r}{=} \Big\{\sum_i \big(\alpha\beta^3(x_i)\alpha\beta(a)\big)\alpha^4\beta(y_i)\Big\}\Big\{\sum_j \alpha\beta^3(x_j)\big(\alpha\beta(b)\alpha^3(y_j)\big)\Big\}\\
&\overset{(2.1)}{=} \sum_{i,j}\Big\{\{(\beta^3(x_i)\beta(a))\alpha^3\beta(y_i)\}\alpha\beta^3(x_j)\Big\}\Big\{\alpha\beta^2(b)\alpha^3\beta(y_j)\Big\}\\
&\overset{(2.1)}{=} \sum_{i,j}\Big\{(\alpha\beta^3(x_i)\alpha\beta(a))(\alpha^3\beta(y_i)\alpha\beta^2(x_j))\Big\}\Big\{\alpha\beta^2(b)\alpha^3\beta(y_j)\Big\}\\
&\overset{(\beta\otimes\beta)(r)=r}{=} \sum_{i,j}\Big\{(\alpha\beta^4(x_i)\alpha\beta(a))(\alpha^3\beta^2(y_i)\alpha\beta^2(x_j))\Big\}\Big\{\alpha\beta^2(b)\alpha^3\beta(y_j)\Big\}\\
&\overset{(\alpha^2\otimes\alpha^2)(r)=r}{=} \sum_{i,j}\Big\{(\alpha\beta^4(x_i)\alpha\beta(a))\alpha^3\beta^2(y_ix_j)\Big\}\Big\{\alpha\beta^2(b)\alpha^5\beta(y_j)\Big\}\\
&\overset{(4.1)}{=} \sum_{i,j}\Big\{(\beta^4(x_ix_j)\alpha\beta(a))\alpha^3\beta^3(y_j)\Big\}\Big\{\alpha\beta^2(b)\alpha^5\beta(y_i)\Big\}\\
&\quad + \sum_{i,j}\Big\{(\alpha\beta^4(x_i)\alpha\beta(a))\alpha^4\beta^2(x_j)\Big\}\Big\{\alpha\beta^2(b)\alpha^5(y_jy_i)\Big\}\\
&\overset{(2.1)}{=} \sum_{i,j}\Big\{\alpha\beta^4(x_ix_j)(\alpha\beta(a)\alpha^3\beta^2(y_j))\Big\}\Big\{\alpha\beta^2(b)\alpha^5\beta(y_i)\Big\}\\
&\quad + \sum_{i,j}\Big\{(\alpha\beta^4(x_i)\alpha\beta(a))\alpha^4\beta^2(x_j)\Big\}\Big\{\alpha\beta^2(b)\alpha^5(y_jy_i)\Big\}\\
&= \sum_{i,j}\Big\{(\beta^3(x_i)\beta^2(x_j))(\alpha\beta(a)\alpha^2(y_j))\Big\}\Big\{\alpha\beta^2(b)\alpha^4(y_i)\Big\}\\
&\quad + \sum_{i,j}\Big\{(\beta^4(x_i)\alpha\beta(a))\beta^2(x_j)\Big\}\Big\{\alpha\beta^2(b)(\alpha(y_j)\alpha^4(y_i))\Big\},
\end{aligned}
$$

where for the last equality we used the identities $\sum_i \alpha\beta^4(x_i)\otimes\alpha^5\beta(y_i) = \sum_i \beta^3(x_i)\otimes\alpha^4(y_i)$ and $\sum_j \alpha\beta^4(x_j)\otimes\alpha^3\beta^2(y_j) = \sum_j \beta^2(x_j)\otimes\alpha^2(y_j)$, for the first term, and $\sum_i \alpha\beta^4(x_i)\otimes\alpha^5(y_i) = \sum_i \beta^4(x_i)\otimes\alpha^4(y_i)$ and $\sum_j \alpha^4\beta^2(x_j)\otimes\alpha^5(y_j) = \sum_j \beta^2(x_j)\otimes\alpha(y_j)$, for the second term, identities that are consequences of the relation $(\alpha\otimes\alpha)(r) = r = (\beta\otimes\beta)(r)$. On the other hand, we have:

$$
\begin{aligned}
&R\big(R(a)\alpha\beta(b) + \alpha\beta(a)R(b)\big)\\
&= R\Big(\sum_j\{\alpha\beta^3(x_j)(a\alpha^3(y_j))\}\alpha\beta(b)\Big) + R\Big(\alpha\beta(a)\Big\{\sum_j \alpha\beta^3(x_j)(b\alpha^3(y_j))\Big\}\Big)\\
&= \sum_{i,j}\alpha\beta^3(x_i)\Big\{\{(\alpha\beta^3(x_j)(a\alpha^3(y_j)))\alpha\beta(b)\}\alpha^3(y_i)\Big\}\\
&\quad + \sum_{i,j}\alpha\beta^3(x_i)\Big\{\{\alpha\beta(a)(\alpha\beta^3(x_j)(b\alpha^3(y_j)))\}\alpha^3(y_i)\Big\}
\end{aligned}
$$

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$$
\begin{aligned}
&\overset{(\beta\otimes\beta)(r)=r}{=} \sum_{i,j} \alpha\beta^4(x_i)\big\{\{(\alpha\beta^3(x_j)(a\alpha^3(y_j)))\alpha\beta(b)\}\alpha^3\beta(y_i)\big\}\\
&\quad + \sum_{i,j} \alpha\beta^3(x_i)\big\{\{\alpha\beta(a)(\alpha\beta^3(x_j)(b\alpha^3(y_j)))\}\alpha^3(y_i)\big\}\\
&\overset{(2.1)}{=} \sum_{i,j} \alpha\beta^4(x_i)\big\{\{\alpha^2\beta^3(x_j)(\alpha(a)\alpha^4(y_j))\}\{\alpha\beta(b)\alpha^3(y_i)\}\big\}\\
&\quad + \sum_{i,j} \alpha\beta^3(x_i)\big\{\{(\beta(a)\alpha\beta^3(x_j))(\beta(b)\alpha^3\beta(y_j))\}\alpha^3(y_i)\big\}\\
&\overset{(2.1)}{=} \sum_{i,j} \big\{\beta^4(x_i)\{\alpha^2\beta^3(x_j)(\alpha(a)\alpha^4(y_j))\}\big\}\{\alpha\beta^2(b)\alpha^3\beta(y_i)\}\\
&\quad + \sum_{i,j} \alpha\beta^3(x_i)\big\{\{(\beta(a)\alpha\beta^3(x_j))(\beta(b)\alpha^3\beta(y_j))\}\alpha^3(y_i)\big\}\\
&\overset{(\beta\otimes\beta)(r)=r}{=} \sum_{i,j} \big\{\beta^4(x_i)\{\alpha^2\beta^3(x_j)(\alpha(a)\alpha^4(y_j))\}\big\}\{\alpha\beta^2(b)\alpha^3\beta(y_i)\}\\
&\quad + \sum_{i,j} \alpha\beta^4(x_i)\big\{\{(\beta(a)\alpha\beta^3(x_j))(\beta(b)\alpha^3\beta(y_j))\}\alpha^3\beta(y_i)\big\}\\
&\overset{(2.1)}{=} \sum_{i,j} \big\{\beta^4(x_i)\{\alpha^2\beta^3(x_j)(\alpha(a)\alpha^4(y_j))\}\big\}\{\alpha\beta^2(b)\alpha^3\beta(y_i)\}\\
&\quad + \sum_{i,j} \alpha\beta^4(x_i)\big\{(\alpha\beta(a)\alpha^2\beta^3(x_j))\{(\beta(b)\alpha^3\beta(y_j))\alpha^3(y_i)\}\big\}\\
&\overset{(\alpha\otimes\alpha)(r)=r}{=} \sum_{i,j} \{\alpha\beta^4(x_i)\big\{\alpha^2\beta^3(x_j)(\alpha(a)\alpha^4(y_j))\}\big\}\big\{\alpha\beta^2(b)\alpha^4\beta(y_i)\big\}\\
&\quad + \sum_{i,j} \alpha\beta^4(x_i)\big\{(\alpha\beta(a)\alpha^2\beta^3(x_j))\{(\beta(b)\alpha^3\beta(y_j))\alpha^3(y_i)\}\big\}\\
&\overset{(2.1)}{=} \sum_{i,j} \big\{(\beta^4(x_i)\alpha^2\beta^3(x_j))(\alpha\beta(a)\alpha^4\beta(y_j))\big\}\big\{\alpha\beta^2(b)\alpha^4\beta(y_i)\big\}\\
&\quad + \sum_{i,j} \big\{\beta^4(x_i)(\alpha\beta(a)\alpha^2\beta^3(x_j))\big\}\big\{(\beta^2(b)\alpha^3\beta^2(y_j))\alpha^3\beta(y_i)\big\}\\
&\overset{(\alpha\otimes\alpha)(r)=r}{=} \sum_{i,j} \big\{(\beta^4(x_i)\alpha^2\beta^3(x_j))(\alpha\beta(a)\alpha^4\beta(y_j))\big\}\big\{\alpha\beta^2(b)\alpha^4\beta(y_i)\big\}\\
&\quad + \sum_{i,j} \big\{\alpha\beta^4(x_i)(\alpha\beta(a)\alpha^2\beta^3(x_j))\big\}\big\{(\beta^2(b)\alpha^3\beta^2(y_j))\alpha^4\beta(y_i)\big\}\\
&\overset{(2.1)}{=} \sum_{i,j} \big\{(\beta^4(x_i)\alpha^2\beta^3(x_j))(\alpha\beta(a)\alpha^4\beta(y_j))\big\}\big\{\alpha\beta^2(b)\alpha^4\beta(y_i)\big\}\\
&\quad + \sum_{i,j} \big\{(\beta^4(x_i)\alpha\beta(a))\alpha^2\beta^4(x_j)\big\}\big\{\alpha\beta^2(b)(\alpha^3\beta^2(y_j)\alpha^4(y_i))\big\}.
\end{aligned}
$$

By using the identities

$$
\sum_i \beta^4(x_i)\otimes\alpha^4\beta(y_i) = \sum_i \beta^3(x_i)\otimes\alpha^4(y_i)
$$

$$
\sum_j \alpha^2\beta^3(x_j)\otimes\alpha^4\beta(y_j) = \sum_j \beta^2(x_j)\otimes\alpha^2(y_j),
$$

for the first term and $\sum_j \alpha^2\beta^4(x_j)\otimes\alpha^3\beta^2(y_j) = \sum_j \beta^2(x_j)\otimes\alpha(y_j)$ for the second term, identities that are consequences of the relation $(\alpha\otimes\alpha)(r) = r = (\beta\otimes\beta)(r)$, the final expression becomes

$$\sum_{i,j}\Big\{\big(\beta^3(x_i)\beta^2(x_j)\big)\big(\alpha\beta(a)\alpha^2(y_j)\big)\Big\}\big\{\alpha\beta^2(b)\alpha^4(y_i)\big\}$$
$$+\sum_{i,j}\Big\{\big(\beta^4(x_i)\alpha\beta(a)\big)\beta^2(x_j)\Big\}\Big\{\alpha\beta^2(b)\big(\alpha(y_j)\alpha^4(y_i)\big)\Big\},$$

and this coincides with the expression we obtained for $R(\alpha\beta(a))R(\alpha\beta(b))$. ■

Corollary 4.5 *Let (A,μ,α) be a Hom-associative algebra and let $r = \sum_i x_i\otimes y_i \in A\otimes A$ be such that $(\alpha\otimes\alpha)(r) = r$ and r is a solution of the associative Hom-Yang–Baxter equation. Define $R\colon A\to A$, $R(a) = \sum_i \alpha(x_i)(ay_i) = \sum_i (x_i a)\alpha(y_i)$. Then R is an α^2-Rota–Baxter operator.*

Proof Take $\alpha = \beta$ in the previous theorem and note that since $(\alpha\otimes\alpha)(r) = r$, the formula (4.2) becomes $R(a) = \sum_i \alpha(x_i)(ay_i) = \sum_i (x_i a)\alpha(y_i)$. ■

5 Hom-pre-Lie Algebras from Infinitesimal Hom-bialgebras

In this section we derive Hom-pre-Lie algebras from infinitesimal Hom-bialgebras, generalizing Aguiar's result in the classical case.

Proposition 5.1 *Let (A,μ,α) be a commutative Hom-associative algebra, k a natural number and $D\colon A\to A$ an α^k-derivation; that is, D is a linear map commuting with α and*

$$D(ab) = D(a)\alpha^k(b) + \alpha^k(a)D(b), \quad \forall\, a,b\in A. \tag{5.1}$$

Define a new operation on A by

$$x\bullet y = \alpha^k(x)D(y), \quad \forall\, x,y\in A. \tag{5.2}$$

Then $(A,\bullet,\alpha^{k+1})$ is a Hom-Novikov algebra.

Proof Since D commutes with α, it is obvious that $\alpha^{k+1}(x\bullet y) = \alpha^{k+1}(x)\bullet\alpha^{k+1}(y)$ for all $x,y\in A$. Now we compute as follows:

$$\begin{aligned}
\alpha^{k+1}(x)\bullet(y\bullet z) - (x\bullet y)\bullet\alpha^{k+1}(z)
&= \alpha^{k+1}(x)\bullet\big(\alpha^k(y)D(z)\big) - \big(\alpha^k(x)D(y)\big)\bullet\alpha^{k+1}(z)\\
&= \alpha^{2k+1}(x)D\big(\alpha^k(y)D(z)\big) - \alpha^k\big(\alpha^k(x)D(y)\big)D\big(\alpha^{k+1}(z)\big)\\
&\overset{(5.1)}{=} \alpha^{2k+1}(x)\big(D(\alpha^k(y))\alpha^k(D(z)) + \alpha^{2k}(y)D^2(z)\big)\\
&\quad - \big(\alpha^{2k}(x)\alpha^k(D(y))\big)\alpha^{k+1}(D(z))
\end{aligned}$$

$$\begin{aligned}
&\overset{(2.2)}{=} \alpha^{2k+1}(x)(\alpha^k(D(y))\alpha^k(D(z))) + \alpha^{2k+1}(x)\big(\alpha^{2k}(y)D^2(z)\big)\\
&\quad - \alpha^{2k+1}(x)\big(\alpha^k(D(y))\alpha^k(D(z))\big)\\
&\overset{(2.2)}{=} \big(\alpha^{2k}(x)\alpha^{2k}(y)\big)\alpha(D^2(z)) = \alpha^{2k}(xy)\alpha(D^2(z)),
\end{aligned}$$

and since $xy = yx$, this expression is obviously symmetric in x and y, so $(A, \bullet, \alpha^{k+1})$ is a left Hom-pre-Lie algebra. Now we compute as follows:

$$\begin{aligned}
(x \bullet y) \bullet \alpha^{k+1}(z) &= (\alpha^k(x)D(y)) \bullet \alpha^{k+1}(z) = \alpha^k(\alpha^k(x)D(y))D(\alpha^{k+1}(z))\\
&= (\alpha^{2k}(x)\alpha^k(D(y)))\alpha^{k+1}(D(z))\\
&\overset{(2.2)}{=} \alpha^{2k+1}(x)\big(\alpha^k(D(y))\alpha^k(D(z))\big) = \alpha^{2k+1}(x)\alpha^k(D(y)D(z))\\
&\overset{\text{(commutativity)}}{=} \alpha^{2k+1}(x)\alpha^k(D(z)D(y)) = (x \bullet z) \bullet \alpha^{k+1}(y).
\end{aligned}$$

So indeed $(A, \bullet, \alpha^{k+1})$ is a Hom-Novikov algebra. ∎

By taking $k = 0$ in the proposition, we obtain the following corollary.

Corollary 5.2 ([23]) *Let (A, μ, α) be a commutative Hom-associative algebra and let $D: A \to A$ be a derivation (in the usual sense) commuting with α. Define a new operation on A by $x \bullet y = xD(y)$, for all $x, y \in A$. Then $(A, \bullet, \alpha)$ is a Hom-Novikov algebra.*

Proposition 5.3 *Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra. Define the linear map $D: A \to A$, $D(a) = a_1a_2$ for all $a \in A$,* i.e., *$D = \mu \circ \Delta$. Then D is an α^2-derivation.*

Proof Obviously D commutes with α and, for all $a, b \in A$, we have

$$\begin{aligned}
D(ab) &\overset{(2.4)}{=} (\alpha(a)b_1)\alpha(b_2) + \alpha(a_1)(a_2\alpha(b))\\
&\overset{(2.2)}{=} \alpha^2(a)(b_1b_2) + (a_1a_2)\alpha^2(b) = \alpha^2(a)D(b) + D(a)\alpha^2(b),
\end{aligned}$$

finishing the proof. ∎

Now let (A, μ, Δ, α) be a commutative infinitesimal Hom-bialgebra. Using Propositions 5.3 and 5.1, we obtain a Hom-Novikov algebra $(A, \bullet, \alpha^3)$, where

$$\begin{aligned}
x \bullet y &\overset{(5.2)}{=} \alpha^2(x)D(y) = \alpha^2(x)(y_1y_2)\\
&\overset{(2.2)}{=} (\alpha(x)y_1)\alpha(y_2)\\
&\overset{\text{commutativity}}{=} (y_1\alpha(x))\alpha(y_2)\\
&\overset{(2.2)}{=} \alpha(y_1)(\alpha(x)y_2).
\end{aligned}$$

Inspired by this, now we have the following proposition.

Proposition 5.4 *Let (A, μ, Δ, α) be an infinitesimal Hom-bialgebra, and define a new multiplication on A by $x \bullet y = \alpha(y_1)(\alpha(x)y_2) = (y_1\alpha(x))\alpha(y_2)$, for all $x, y \in A$. Then $(A, \bullet, \alpha^3)$ is a left Hom-pre-Lie algebra.*

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Proof Since $(\alpha\otimes\alpha)\circ\Delta=\Delta\circ\alpha$, it is easy to see that $\alpha^3(x\bullet y)=\alpha^3(x)\bullet\alpha^3(y)$, for all $x,y\in A$. Now, for all $x,y,z\in A$, we compute as follows:

$$
\begin{aligned}
&\alpha^3(x)\bullet(y\bullet z)-(x\bullet y)\bullet\alpha^3(z)\\
&\quad=\alpha^3(x)\bullet\big(\alpha(z_1)(\alpha(y)z_2)\big)-\big(\alpha(y_1)(\alpha(x)y_2)\big)\bullet\alpha^3(z)\\
&\quad=\alpha\big([\alpha(z_1)(\alpha(y)z_2)]_1\big)\big(\alpha^4(x)[\alpha(z_1)(\alpha(y)z_2)]_2\big)\\
&\qquad-\alpha^4(z_1)\big\{[\alpha^2(y_1)(\alpha^2(x)\alpha(y_2))]\alpha^3(z_2)\big\}\\
&\overset{(2.4)\text{ twice}}{=}\alpha\big(\alpha^2(z_1)(\alpha^2(y)z_{(2,1)})\big)\big[\alpha^4(x)\alpha^2(z_{(2,2)})\big]\\
&\qquad+\alpha\big(\alpha^2(z_1)\alpha^2(y_1)\big)\big[\alpha^4(x)(\alpha^2(y_2)\alpha^2(z_2))\big]\\
&\qquad+\alpha^3(z_{(1,1)})\big[\alpha^4(x)(\alpha(z_{(1,2)})\alpha(\alpha(y)z_2))\big]\\
&\qquad-\alpha^4(z_1)\big\{[\alpha^2(y_1)(\alpha^2(x)\alpha(y_2))]\alpha^3(z_2)\big\}\\
&\quad=\alpha\big([\alpha^2(z_1)(\alpha^2(y)z_{(2,1)})]\alpha(\alpha^2(x)z_{(2,2)})\big)+\alpha^2\big(\alpha(z_1y_1)[\alpha^2(x)(y_2z_2)]\big)\\
&\qquad+\alpha(\alpha^2(z_{(1,1)})[\alpha^3(x)(z_{(1,2)}(\alpha(y)z_2))])\\
&\qquad-\alpha\big(\alpha^3(z_1)\big\{[\alpha(y_1)(\alpha(x)y_2)]\alpha^2(z_2)\big\}\big).
\end{aligned}
$$

We claim that the second and fourth terms in this expression cancel each other. To show this, it is enough to prove that

$$
\alpha^2(z_1y_1)\big[\alpha^3(x)(\alpha(y_2)\alpha(z_2))\big]=\alpha^3(z_1)\big\{[\alpha(y_1)(\alpha(x)y_2)]\alpha^2(z_2)\big\}.
$$

Repeatedly applying the Hom-associativity condition, we compute as follows:

$$
\begin{aligned}
\alpha^3(z_1)\big\{[\alpha(y_1)(\alpha(x)y_2)]\alpha^2(z_2)\big\}&=\alpha^3(z_1)\big\{\alpha^2(y_1)[(\alpha(x)y_2)\alpha(z_2)]\big\}\\
&=\big[\alpha^2(z_1)\alpha^2(y_1)\big]\big[(\alpha^2(x)\alpha(y_2))\alpha^2(z_2)\big]\\
&=\alpha^2(z_1y_1)\big[\alpha^3(x)(\alpha(y_2)\alpha(z_2))\big].
\end{aligned}
$$

This proves the claim.

So we can now write (using both Hom-associativity and Hom-coassociativity)

$$
\begin{aligned}
&\alpha^3(x)\bullet(y\bullet z)-(x\bullet y)\bullet\alpha^3(z)\\
&\quad=\alpha\big([\alpha^2(z_1)(\alpha^2(y)z_{(2,1)})]\alpha(\alpha^2(x)z_{(2,2)})\big)\\
&\quad+\alpha(\alpha^2(z_{(1,1)})[\alpha^3(x)(z_{(1,2)}(\alpha(y)z_2))]\big)\\
&\quad=\alpha\big([(\alpha(z_1)\alpha^2(y))\alpha(z_{(2,1)})]\alpha(\alpha^2(x)z_{(2,2)})\big)\\
&\qquad+\alpha([\alpha(z_{(1,1)})\alpha^3(x)]\alpha(z_{(1,2)}(\alpha(y)z_2))\big)\\
&\quad=\alpha\big([\alpha^2(z_1)\alpha^3(y)][\alpha(z_{(2,1)})(\alpha^2(x)z_{(2,2)})]\\
&\qquad+[\alpha(z_{(1,1)})\alpha^3(x)][\alpha(z_{(1,2)})(\alpha^2(y)\alpha(z_2))]\big)\\
&\quad=\alpha\big([\alpha(z_{(1,1)})\alpha^3(y)][\alpha(z_{(1,2)})(\alpha^2(x)\alpha(z_2))]\\
&\qquad+[\alpha(z_{(1,1)})\alpha^3(x)][\alpha(z_{(1,2)})(\alpha^2(y)\alpha(z_2))]\big),
\end{aligned}
$$

and this expression is obviously symmetric in x and y. ∎

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Remark 5.5 The construction introduced in Proposition 5.4 is compatible with the Yau twist, in the following sense. Let (A,μ,Δ) be an infinitesimal bialgebra and let $\alpha\colon A\to A$ be a morphism of infinitesimal bialgebras. Consider the Yau twist $A_\alpha = (A,\mu_\alpha=\alpha\circ\mu,\Delta_\alpha=\Delta\circ\alpha,\alpha)$ (with notation $\mu_\alpha(x\otimes y)=x*y=\alpha(xy)$ and $\Delta_\alpha(x)=x_{[1]}\otimes x_{[2]}=\alpha(x_1)\otimes\alpha(x_2)$), which is an infinitesimal Hom-bialgebra, to which we can apply Proposition 5.4 and obtain a left Hom-pre-Lie algebra with structure map α^3 and multiplication

$$x\bullet y=\alpha(y_{[1]})*(\alpha(x)*y_{[2]})=\alpha^2(y_1)*\alpha^2(xy_2)=\alpha^3(y_1xy_2).$$

This is exactly the Yau twist via the map α^3 of the left pre-Lie algebra obtained from (A,μ,Δ) by Theorem 1.1.

Assume now that (A,μ,Δ_r,α) is a quasitriangular infinitesimal Hom-bialgebra as in Definition 2.8; there are two left Hom-pre-Lie algebras that can be associated with A, and we want to show that they coincide.

The first one is $(A,\bullet,\alpha^3)$ obtained from A by using Proposition 5.4, with multiplication

$$\begin{aligned} a\bullet b=\alpha(b_1)(\alpha(a)b_2) &= \sum_i \alpha^2(x_i)(\alpha(a)(y_ib))-\sum_i\alpha(bx_i)(\alpha(a)\alpha(y_i))\\ &\overset{(2.2)}{=}\sum_i\alpha^2(x_i)\big[(ay_i)\alpha(b)\big]-\sum_i\big[\alpha(b)\alpha(x_i)\big]\alpha(ay_i)\\ &\overset{(2.2)}{=}\sum_i\big[\alpha(x_i)(ay_i)\big]\alpha^2(b)-\sum_i\alpha^2(b)\big[\alpha(x_i)(ay_i)\big].\end{aligned}$$

The second is obtained by applying Corollary 3.15 (for $n=2$) to the α^2-Rota–Baxter operator R defined in Corollary 4.5. So, its structure map is α^3 and the multiplication is

$$a\circ b=R(a)\alpha^2(b)-\alpha^2(b)R(a)=\sum_i[\alpha(x_i)(ay_i)]\alpha^2(b)-\sum_i\alpha^2(b)[\alpha(x_i)(ay_i)],$$

so indeed $\bullet$ and $\circ$ coincide.

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References

[1] M. Aguiar, *Pre-Poisson algebras*. Lett. Math. Phys. **54**(2000), 263–277. http://dx.doi.org/10.1023/A:1010818119040
[2] ———, *Infinitesimal Hopf algebras*. In: New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., 267, American Mathematical Society, Providence, RI, 2000, pp. 1–29. http://dx.doi.org/10.1090/conm/267/04262
[3] ———, *On the associative analog of Lie bialgebras*. J. Algebra **244**(2001), no. 2, 492–532. http://dx.doi.org/10.1006/jabr.2001.8877
[4] ———, *Infinitesimal bialgebras, pre-Lie and dendriform algebras*. In: Hopf algebras, Lecture Notes in Pure and Appl. Math., 237, Dekker, New York, 2004, pp. 1–33.
[5] T. Brzeziński, *Rota–Baxter systems, dendriform algebras and covariant bialgebras*. J. Algebra **460**(2016), 1–25. http://dx.doi.org/10.1016/j.jalgebra.2016.04.018

[6] A. Connes and D. Kreimer, *Hopf algebras, renormalization and noncommutative geometry*. Comm. Math. Phys. **199**(1998), no. 1, 203–242. http://dx.doi.org/10.1007/s002200050499
[7] K. Ebrahimi-Fard and D. Manchon, *Twisted dendriform algebras and the pre-Lie Magnus expansion*. J. Pure Appl. Algebra **215**(2011), 2615–2627. http://dx.doi.org/10.1016/j.jpaa.2011.03.004
[8] G. Graziani, A. Makhlouf, C. Menini, and F. Panaite, *BiHom-associative algebras, BiHom-Lie algebras and BiHom-bialgebras*. SIGMA Symmetry Integrability Geom. Methods Appl. **11**(2015), 086. http://dx.doi.org/10.3842/SIGMA.2015.086
[9] L. Guo, *An introduction to Rota–Baxter algebra*. Surveys of Modern Mathematics, 4, International Press, Somerville, MA; Higher Education Press, Beijing, 2012.
[10] J. T. Hartwig, D. Larsson, and S. D. Silvestrov, *Deformations of Lie algebras using σ-derivations*. J. Algebra **295**(2006), 314–361. http://dx.doi.org/10.1016/j.jalgebra.2005.07.036
[11] S. A. Joni and G.-C. Rota, *Coalgebras and bialgebras in combinatorics*. Stud. Appl. Math. **61**(1979), 93–139. http://dx.doi.org/10.1002/sapm197961293
[12] D. Larsson and S. D. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*. J. Algebra **288**(2005), 321–344. http://dx.doi.org/10.1016/j.jalgebra.2005.02.032
[13] L. Liu, A. Makhlouf, C. Menini, and F. Panaite, *Rota–Baxter operators on BiHom-associative algebras and related structures*. 2017. arxiv:1703.07275
[14] ———, *BiHom-pre-Lie algebras, BiHom-Leibniz algebras and Rota–Baxter operators on BiHom-Lie algebras*. 2017. arxiv:1706.00474
[15] J.-L. Loday, *Dialgebras*. In: Dialgebras and other operads, Lecture Notes in Mathematics, 1763, Springer, Berlin, 2001, pp. 7–66. http://dx.doi.org/10.1007/3-540-45328-8_2
[16] A. Makhlouf, *Hom-dendriform algebras and Rota–Baxter Hom-algebras*. In: Operads and universal algebra, Nankai Ser. Pure Appl. Math. Theor. Phys., 9, World Sci. Publ., Hackensack, NJ, 2012, pp. 147–171. http://dx.doi.org/10.1142/9789814365123_0008
[17] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*. J. Gen. Lie Theory Appl. **2**(2008), 51–64. http://dx.doi.org/10.4303/jglta/S070206
[18] ———, *Hom-algebras and Hom-coalgebras*. J. Algebra Appl. **9**(2010), 553–589. http://dx.doi.org/10.1142/S0219498810004117
[19] A. Makhlouf and D. Yau, *Rota–Baxter Hom-Lie-admissible algebras*. Comm. Algebra **42**(2014), 1231–1257. http://dx.doi.org/10.1080/00927872.2012.737075
[20] F. Panaite and F. Van Oystaeyen, *Twisted algebras and Rota–Baxter type operators*. J. Algebra Appl. **16**(2017), no. 4, 1750079. http://dx.doi.org/10.1142/S0219498817500797
[21] D. Voiculescu, *The coalgebra of the free difference quotient in free probability*. Internat. Math. Res. Notices **2000**, no. 2, 79–106. http://dx.doi.org/10.1155/S1073792800000064
[22] D. Yau, *Infinitesimal Hom-bialgebras and Hom-Lie bialgebras*. 2010. arxiv:1001.5000
[23] ———, *Hom-Novikov algebras*. J. Phys. A. **44**(2011), 085202. http://dx.doi.org/10.1088/1751-8113/44/8/085202

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