LOCALIZATION THEORIES FOR SIMPLICIAL PRESHEAVES

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ABSTRACT. Most extant localization theories for spaces, spectra and diagrams of such can be derived from a simple list of axioms which are verified in broad generality. Several new theories are introduced, including localizations for simplicial presheaves and presheaves of spectra at homology theories represented by presheaves of spectra, and a theory of localization along a geometric topos morphism. The f-localization concept has an analog for simplicial presheaves, and specializes to the $\mathbb{A}^1$-local theory of Morel-Voevodsky. This theory answers a question of Soulé concerning integral homology localizations for diagrams of spaces.

This work was motivated in part by the following question of Soulé: given a simplicial presheaf X on a site $\mathcal{C}$, how does one produce a map of simplicial presheaves $X \longrightarrow L_{H\mathbb{Z}}X$ in such a way that each of the maps in sections $X(U) \longrightarrow L_{H\mathbb{Z}}X(U)$, $U \in \mathcal{C}$, is an integral homology localization map in the sense of Bousfield? Secondly, if Y is a simplicial presheaf which is integrally homology local in a suitable sense, is it the case that the map $X \longrightarrow L_{H\mathbb{Z}}X$ induces an isomorphism

$$[L_{H\mathbb{Z}}X, Y] \cong [X, Y]$$

relating sets of morphisms in the homotopy category of simplicial presheaves on $\mathcal{C}$? These questions are related to the definition of the K-theory of simplicial sheaves that appears in [8].

The first of these questions is easily answered by observing that associated fibrations in the closed model category describing Bousfield's homology localizations are created with small object constructions and are therefore natural; in particular there is a functorial method of picking out a fibrant model $Y \longrightarrow L_{H\mathbb{Z}}Y$ for arbitrary simplicial sets Y, which restricts in particular to a natural simplicial presheaf map $X(U) \longrightarrow L_{H\mathbb{Z}}X(U)$, $U \in \mathcal{C}$.

The second question involves homotopy coherence, and is therefore much more subtle: the analogous space-level problem can be solved by Bousfield's original techniques, but this does not imply the functorial global solution that Soulé requires. The problem is solved by using methods introduced in this paper, and in particular by applying Theorem 3.9 below. In the case corresponding to the identity functor on the site $\mathcal{C}$, the chaotic topology on $\mathcal{C}$ and the constant presheaf of spectra associated to the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$, Theorem 3.9 implies that there is a closed simplicial model structure in the sense of Quillen on the category $\mathbf{S}\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves on $\mathcal{C}$ such

1048

that the cofibrations are the pointwise monomorphisms and the weak equivalences are the pointwise integral homology isomorphisms. The map $i: X \longrightarrow L_{H\mathbb{Z}}X$ is then just a choice of fibrant model (*i.e.*, trivial cofibration, taking values in fibrant object) for this closed model structure, and the induced maps in sections $i: X(U) \longrightarrow L_{H\mathbb{Z}}X(U)$ are fibrant models for the corresponding theory on simplicial sets (*i.e.*, integral homology localizations in Bousfield's original sense), because the U-sections functor has a left adjoint which preserves cofibrations and takes integral homology isomorphisms to pointwise integral homology isomorphisms. Furthermore, if we say that a simplicial presheaf Y is integral homology local if it's fibrant with respect to this new closed model structure on $\mathbf{S}\operatorname{Pre}(\mathcal{C})$, then Y is globally fibrant in the traditional sense, and the closed simplicial model structure gives an isomorphism

$$\pi(L_{H\mathbb{Z}}X, Y) \cong \pi(X, Y)$$

in naive homotopy classes of maps which is induced by the $H\mathbb{Z}$-trivial cofibration i (here $\pi(X, Y) = \pi_0\mathbf{hom}(X, Y)$, for example).

This application is based on a very special case of the results that appear here, which hold in striking generality. The overall point is that a very wide class of results, which includes objects as apparently diverse as Theorem 3.9, the closed model structure for simplicial presheaves [15] (see also Remark 2.9), a general f-localization theory for simplicial presheaves (Theorem 4.6), the closed model structures of various stable categories (Theorem 3.7) and a homology localization technique for presheaves of spectra (Theorem 3.10), all arise from a simple collection of axioms for classes of cofibrations and weak equivalences (see axioms **E1**–**E7** almost immediately below, and then **sE1**–**sE7** for spectra in Section 3). The proofs, in all cases, involve relatively simple cardinality counts which are modelled simultaneously on Bousfield's original work on homology localization and the derivation of the closed model structures for simplicial presheaves. Some new theories have been discovered along the way, including a notion of localization along a geometric topos morphism (Theorem 2.7, Theorem 3.10) and a resulting method of localizing a space or a spectrum at a generalized homology theory arising from a presheaf of spectra on an arbitrary site.

There is a further application for these techniques, in that the Morel-Voevodsky $\mathbb{A}^1$-localization theory [17], [18] is an instance of the f-localization results of Section 4 (see Remark 4.11, but note that we do not discuss properness). The collection of known applications is, however, still quite small. It's rather difficult, in particular, to know what localization along an arbitrary geometric morphism of toposes should mean in the context of traditional homotopy theory. The fibrant objects in all of these theories continue to be really quite mysterious.

1. **Fundamental results.** Suppose that C is a small Grothendieck site, and that α is an infinite cardinal. A simplicial presheaf X on C is said to be α-*bounded* if α is an upper bound on the cardinality of all sets of sections of X in the sense that $|X_n(U)| \le \alpha$ for all $n \ge 0$ and all objects U of C.

Suppose that $\mathbf{E}$ is a class of morphisms of $\mathbf{S}\,\mathrm{Pre}(C)$, and say that a monomorphism of simplicial presheaves is a cofibration. Say that a simplicial presheaf cofibration which is also a member of $\mathbf{E}$ is an $\mathbf{E}$-trivial cofibration.

In many examples, we shall see that the class $\mathbf{E}$ and the class of cofibrations together satisfy the following axioms:

E1: The class of morphisms $\mathbf{E}$ is closed under retracts.

E2: Given a composable pair of morphisms

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z,$$

if any two of f, g and gf are in the class $\mathbf{E}$, then so is the third.

E3: Every pointwise weak equivalence is in $\mathbf{E}$.

E4: The class of $\mathbf{E}$-trivial cofibrations is closed under pushout.

E5: Suppose that γ is a limit ordinal, and there is a functor $X\colon \gamma \to \mathbf{S}\,\mathrm{Pre}(C)$ such that for each morphism $i \le j$ of γ, the induced map $X(i) \to X(j)$ is an $\mathbf{E}$-trivial cofibration. Then the canonical maps

$$X(i) \xrightarrow{\ \tau_i\ } \varinjlim_{j \in \gamma} X(j)$$

are $\mathbf{E}$-trivial cofibrations.

E6: Suppose that the morphisms $f_i\colon X_i \to Y_i$ are $\mathbf{E}$-trivial cofibrations for $i \in I$. Then the morphism

$$\bigsqcup_{i \in I} f_i \colon \bigsqcup_{i \in I} X_i \to \bigsqcup_{i \in I} Y_i$$

is an $\mathbf{E}$-trivial cofibration.

E7: There is an infinite cardinal α which is an upper bound for the cardinality of the set of morphisms of C, such that for every simplicial presheaf diagram

$$
\begin{array}{c}
X \\
\downarrow{\scriptstyle i} \\
A \hookrightarrow Y
\end{array}
$$

with i an $\mathbf{E}$-trivial cofibration and A α-bounded, there is a subobject $B \subset Y$ such that $A \subset B$, the object B is α-bounded, and the inclusion $B \cap X \hookrightarrow B$ is an $\mathbf{E}$-trivial cofibration.

We shall refer to condition $\mathbf{E7}$ as the *bounded cofibration condition*. It is the only axiom of the list that is not a standard part of a closed model structure, and is almost always the most difficult to verify.

Say that a morphism $p: X \longrightarrow Y$ is an **E**-fibration if it has the right lifting property with respect to all **E**-trivial cofibrations. An **E**-weak equivalence is a member of the class **E**.

If K is a simplicial set and X is a simplicial sheaf, then the tensor object $X \times K$ is defined in sections for $U \in C$ by

$$(X \times K)(U) = X(U) \times K.$$

For the simplicial presheaves X and Y, the function complex $\mathbf{hom}(X, Y)$ is the simplicial set whose set of n-simplices is defined by

$$\mathbf{hom}(X, Y)_n = \hom(X \times \Delta^n, Y)$$

where the morphism set on the right is in the category of simplicial presheaves on C. The ordinary exponential law for simplicial sets bootstraps immediately to a simplicial category structure on the simplicial presheaf category $\mathbf{S}\operatorname{Pre}(C)$. The exponential object X^K associated to a simplicial presheaf X and a simplicial set K is the simplicial presheaf which is defined in sections by the function spaces

$$X^K(U) = \mathbf{hom}\big(K, X(U)\big)$$

for $U \in C$.

THEOREM 1.1.
(1) *Under the conditions* **E1**–**E7** *listed above, there is a closed model structure on* $\mathbf{S}\operatorname{Pre}(C)$ *such that the cofibrations are the monomorphisms, the weak equivalences are the* **E***-weak equivalences, and fibrations (i.e.,* **E***-fibrations) are defined by a right lifting property.*
(2) *Suppose further that, given an inclusion* $i: K \longrightarrow L$ *of finite simplicial sets and a cofibration* $j: X \hookrightarrow Y$, *then the induced monomorphism*

$$X \times L \cup_{X \times K} Y \times K \hookrightarrow Y \times L$$

is an **E***-trivial cofibration if either* i *is a weak equivalence of simplicial sets or* j *is an* **E***-weak equivalence of simplicial presheaves. Then* $\mathbf{S}\operatorname{Pre}(C)$ *has the structure of a closed simplicial model category.*

The proof of this result is a distillation of ideas which are common to Bousfield's work on homology localizations [2], and the homotopy theory of simplicial presheaves [12].

PROOF. We only need to prove the first statement. Suppose that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms of a site C. Say that a cofibration $A \hookrightarrow B$ of $\mathbf{S}\operatorname{Pre}(C)$ is α-bounded if the object B is α-bounded. We begin by showing that a morphism of $\mathbf{S}\operatorname{Pre}(C)$ is an **E**-fibration if and only it has the right lifting property with respect to all α-bounded cofibrations which are **E**-weak equivalences.

Suppose that K is a simplicial set and U is an object of $\mathcal{C}$, then the simplicial presheaf $L_U K$ is defined for $V \in \mathcal{C}$ by

$$L_U K(V) = \bigsqcup_{\phi: V \to U} K.$$

Observe that morphisms of simplicial presheaves $L_U K \to X$ are in one to one correspondence with simplicial set maps $K \to X(U)$. If the simplicial set K is α-bounded in the sense that $|K_n| \leq \alpha$ for $n \geq 0$, then the simplicial presheaf $L_U K$ is α-bounded.

Suppose given a diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
i \downarrow & \nearrow & \downarrow p \\
B & \longrightarrow & Y
\end{array}
$$

where i is a cofibration and an **E**-weak equivalence, and p has the right lifting property with respect to all α-bounded **E**-trivial cofibrations. We shall show that the indicated dotted arrow exists, making the diagram commute. Assume that the map i is not an isomorphism, for otherwise the problem is solved trivially.

The object B is a filtered colimit of its α-bounded subobjects, since all generating simplicial presheaves $L_U \Delta^n$ are α-bounded. The map i is not an isomorphism, so there is an α-bounded subobject D of B such that D is not a subobject of A. But then the bounded cofibration condition **E7** says that there is an α-bounded E such that $D \subset E \subset B$, and such that the inclusion $E \cap A \hookrightarrow E$ is an **E**-weak equivalence. Form the diagram

$$
\begin{array}{ccccc}
E \cap A & \longrightarrow & A & \longrightarrow & X \\
\downarrow & & \downarrow i_* & \nearrow & \uparrow \\
E & \longrightarrow & E \cup A & & \downarrow p \\
& & \downarrow & & \uparrow \\
& & B & \longrightarrow & Y
\end{array}
$$

where the indicated partial lift exists since p is assumed to have the right lifting property with respect to all α-bounded cofibrations which are **E**-weak equivalences. Observe further that the map i_* is an **E**-weak equivalence since the class of **E**-trivial cofibrations is closed under pushout, by **E4**. It follows that the category of all such partial lifts is non-empty. This category has maximal elements, by a Zorn's lemma argument and **E5**, and any such maximal element must be a solution to the lifting problem.

Every simplicial presheaf map $f: X \to Y$ has a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{}_i \searrow & & \nearrow {}_p \\
& Z &
\end{array}
$$

where p is an **E**-fibration and i is a **E**-trivial cofibration. In effect, take a cardinal $\beta > 2^\alpha$, and define a functor $F: \beta \longrightarrow \mathbf{S}\operatorname{Pre}(\mathcal{C}) \downarrow Y$ by first setting $F(0) = f: X \longrightarrow Y$. We let

$$X(\zeta) = \varinjlim_{\gamma < \zeta} X(\gamma)$$

for limit ordinals ζ. Further, the map $X(\gamma) \longrightarrow X(\gamma + 1)$ is defined by taking the set of all diagrams

$$D: \quad \begin{array}{ccc} U_D & \longrightarrow & X(\gamma) \\ {\scriptstyle i_D}\downarrow & & \downarrow{\scriptstyle F(\gamma)} \\ V_D & \longrightarrow & Y \end{array}$$

such that i_D is an α-bounded **E**-trivial cofibration, and then forming the pushout

$$\begin{array}{ccc} \coprod\limits_D U_D & \longrightarrow & X(\gamma) \\ {\scriptstyle \coprod_D i_D}\downarrow & & \downarrow{\scriptstyle i_*} \\ \coprod\limits_D V_D & \longrightarrow & X(\gamma + 1). \end{array}$$

Then i_* is an **E**-trivial cofibration, by axioms **E4** and **E6**, as is the map i_β in the resulting diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle i_\beta}\searrow \quad \nearrow {\scriptstyle F(\beta)} & \\ & X(\beta), & \end{array}$$

by axiom **E5**, where $X(\beta) = \varinjlim_{\gamma < \beta} X(\gamma)$, and $F(\beta)$ is induced by all maps $F(\gamma)$. In any diagram

$$\begin{array}{ccc} U & \xrightarrow{\ g\ } & X(\beta) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle F(\beta)} \\ V & \longrightarrow & Y \end{array}$$

where i is an α-bounded **E**-trivial cofibration, the simplicial presheaf U is α-bounded, so that g must factor through some subcomplex $X(\gamma) \subset X(\beta)$ with $\gamma < \beta$ (for otherwise U has too many subobjects). It follows that the dotted arrow exists, making the diagram commute.

Now, if a map $f: X \longrightarrow Y$ has the right lifting property with respect to all morphisms of the form $A \subset L_U \Delta^n$, then f has the right lifting property with respect to all cofibrations, by an argument similar to that which characterizes **E**-fibrations above. A corresponding transfinite small object argument then shows that f has a factorization

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ & {\scriptstyle j}\searrow \quad \nearrow {\scriptstyle q} & \\ & W & \end{array}$$

where j is a cofibration, and q has the right lifting property with respect to all cofibrations. In particular, q is an **E**-fibration; it is also a pointwise trivial fibration and hence a pointwise weak equivalence, since it has the right lifting property with respect to all maps $L_U \partial \Delta^n \subset L_U \Delta^n$. Axiom **E3** says that every pointwise weak equivalence is an **E**-weak equivalence, so q is an **E**-trivial fibration.

We have therefore proved the factorization axioms. To verify the lifting axiom **CM4**, observe that if $p \colon X \longrightarrow Y$ is a **E**-trivial fibration, then p has a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\ p\ } & Y \\
& {\scriptstyle j}\searrow \quad \nearrow {\scriptstyle q} & \\
& W &
\end{array}
$$

where j is a cofibration and q has the right lifting property with respect to all cofibrations. But then j is a **E**-trivial cofibration by **E2** and **E3**, since q is a pointwise weak equivalence, so that there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ 1_X\ } & X \\
{\scriptstyle j}\downarrow & \nearrow & \downarrow {\scriptstyle p} \\
W & \xrightarrow[\ q\ ]{} & Y,
\end{array}
$$

and so p is a retract of q. In particular, p has the right lifting property with respect to all cofibrations. The weak equivalence axiom **CM2** is axiom **E2**, and the retract axiom **CM3** is a direct consequence of axiom **E1** and the definitions. ∎

We say that a closed model structure on the category of simplicial presheaves on a site C which arises from a class **E** of cofibrations which satisfies conditions **E1**–**E7** is a *localization theory*. This usage is consistent with and specializes to the standard examples of localization theories in ordinary homotopy theory. There are corresponding localization theories for simplicial sheaf categories, according to the following result:

THEOREM 1.2. *Suppose that the conditions* **E1**–**E7** *are satisfied for some cardinal* α *which is an upper bound on the cardinality of the set of morphisms of the site* C. *Suppose further that the canonical map* $\eta \colon X \longrightarrow L^2 X$ *from a simplicial presheaf to its associated simplicial sheaf is always an* **E**-*weak equivalence. Then we have the following:*

(1) *The category* **S** $\mathrm{Shv}(C)$ *inherits a closed model structure from the corresponding simplicial presheaf category, for which the the cofibrations are the monomorphisms, and the weak equivalences are the maps which are* **E**-*weak equivalences of simplicial presheaves.*

(2) *If the simplicial presheaf category on* C *has a closed simplicial model structure with respect to these definitions, then the simplicial sheaf category* **S** $\mathrm{Shv}(C)$ *is also a closed simplicial model category.*

(3) *The associated sheaf functor and the inclusion* **S** $\mathrm{Shv}(C) \subset$ **S** $\mathrm{Pre}(C)$ *together induce an equivalence*

$$
\mathrm{Ho}_{\mathbf{E}}\big(\mathbf{S}\,\mathrm{Shv}(C)\big) \simeq \mathrm{Ho}_{\mathbf{E}}\big(\mathbf{S}\,\mathrm{Pre}(C)\big)
$$

of the respective homotopy categories.

PROOF. We shall only prove statements (1) and (2). The third statement is an automatic consequence of (1) and the assumption that the canonical map $\eta\colon X \to L^2X$ is an **E**-weak equivalence.

To verify statement (1), observe that the closed model axioms **CM1**, **CM2** and **CM3** are immediate. The assumption on η guarantees that the associated sheaf functor preserves **E**-weak equivalences, and it follows that a map $p\colon X \to Y$ is an **E**-fibration of simplicial sheaves if and only if p is an **E**-fibration of simplicial presheaves. The lifting axiom **CM4** for simplicial sheaves on C is an easy consequence.

It therefore remains only to prove the factorization axiom **CM5**. We will show that any map $f\colon X \to Y$ of simplicial sheaves has a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
 & {\scriptstyle i}\searrow \quad \nearrow {\scriptstyle p} & \\
 & Z &
\end{array}
$$

where Z is a simplicial sheaf, p is an **E**-fibration and i is a **E**-trivial cofibration. The other part of **CM5** has a similar proof.

As before, take a cardinal $\beta > 2^{\alpha}$, and form the factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\;f\;} & Y \\
 & {\scriptstyle i_{\beta}}\searrow \quad \nearrow {\scriptstyle F(\beta)} & \\
 & X(\beta), &
\end{array}
$$

in the category of simplicial sheaves on C, by analogy with the transfinite small object argument appearing in the proof of Theorem 1.1. In particular, $X(\beta)$ is the colimit in the simplicial sheaf category of a functor $X\colon \beta \to \mathbf{S}\,\mathrm{Shv}(C)$ having

$$
X(\zeta) = \varinjlim_{\gamma < \zeta} X(\gamma)
$$

at all limit ordinals $\zeta \leq \beta$. The map $X(\gamma) \to X(\gamma + 1)$ is defined by taking the set of all diagrams

$$
\begin{array}{ccc}
U_D & \longrightarrow & X(\gamma) \\
{\scriptstyle i_D}\downarrow & & \downarrow \\
V_D & \longrightarrow & Y
\end{array}
$$

of simplicial presheaf morphisms with i_D an α-bounded **E**-trivial cofibration, and then forming the pushout

$$
\begin{array}{ccc}
L^2\big(\bigsqcup_D U_D\big) & \longrightarrow & X(\gamma) \\
\downarrow & & \downarrow \\
L^2\big(\bigsqcup_D V_D\big) & \longrightarrow & X(\gamma + 1)
\end{array}
$$

of simplicial sheaves. Then the induced canonical map $i_\beta \colon X \longrightarrow X(\beta)$ is a trivial **E**-cofibration of simplicial sheaves.

To prove that the map $F(\beta) \colon X(\beta) \longrightarrow Y$ is an **E**-fibration it suffices to show that any simplicial presheaf map $g \colon B \longrightarrow X(\beta)$ has a factorization

$$
\begin{array}{ccc}
 & & X(\gamma) \\
 & \nearrow & \downarrow \\
B & \xrightarrow[g]{} & X(\beta)
\end{array}
$$

for some $\gamma < \beta$, provided that B is α-bounded.

Suppose that this is not so. Write $B(\gamma) = B \times_{X(\beta)} X(\gamma)$. Then the idea is to show that, for each $\gamma < \beta$, there is an ordinal ζ with $\gamma < \zeta < \beta$ and such that $B(\zeta) - B(\gamma)$ is non-empty. This would give a contradiction, for then the α-bounded simplicial presheaf B would have too many subobjects.

Finally, if $B(\gamma) \neq B$, then there is a section $x \in B(U)$ such that $g(x)$ is not in $X(\gamma)(U)$. The element $g(x)$ lifts to elements y_V in the presheaf colimit $\varinjlim_{\zeta < \beta} X(\zeta)$, after refinement along members $\phi \colon V \longrightarrow U$ of a covering sieve for U, since the associated sheaf map

$$
\eta \colon \varinjlim_{\zeta < \beta} X(\zeta) \longrightarrow X(\beta)
$$

is a local epimorphism. At least one of the sections y_V is not in $X(\gamma)(V)$, for otherwise $g(x)$ is in the image of the simplicial sheaf monomorphism $X(\gamma) \longrightarrow X(\beta)$. We can therefore assume that B has a section x such that $g(x)$ lifts to some $z \in \varinjlim_{\zeta < \beta} X(\zeta)$ but is not in $X(\gamma)$. But then $z \in X(\zeta)$ for some ζ with $\gamma < \zeta < \beta$, so we're done.

For statement (2), suppose that X is a simplicial sheaf and that K is a simplicial set. The tensor object $X \otimes K$ is defined to be the simplicial sheaf $L^2(X \times K)$ which is obtained by applying the associated sheaf functor to the corresponding tensor object in the simplicial presheaf category. The function complex $\mathbf{hom}(X, Y)$ is the simplicial set having n-simplices

$$
\mathbf{hom}(X, Y)_n = \hom(X \otimes \Delta^n, Y),
$$

where the morphisms on the right are in the simplicial sheaf category. The simplicial category structure on $\mathbf{S}\,\mathrm{Shv}(\mathcal{C})$ is then induced from the simplicial category structure on the simplicial presheaf category, through the associated sheaf functor.

Assume that $\mathbf{S}\,\mathrm{Pre}(\mathcal{C})$ has the structure of a closed simplicial model category with respect to the ambient definitions. Suppose that $j \colon X \hookrightarrow Y$ is a cofibration of simplicial sheaves and an that $i \colon K \hookrightarrow L$ is an inclusion of simplicial sets. Then the induced monomorphism

$$
(1.3) \qquad\qquad X \otimes L \cup_{X \otimes K} Y \otimes K \hookrightarrow Y \otimes L
$$

of simplicial sheaves is obtained from the corresponding monomorphism

$$(1.4) \qquad\qquad X \times L \cup_{X \times K} Y \times K \hookrightarrow Y \times L$$

by applying the associated sheaf functor. If either i is a weak equivalence or j is an **E**-weak equivalence, the simplicial presheaf map (1.4) is an **E**-weak equivalence by assumption. The associated sheaf functor preserves **E**-weak equivalences, so that the map (1.3) is an **E**-weak equivalence. ∎

2**. Geometric morphisms.** We shall follow Mac Lane and Moerdijk [16] in saying that a *morphism of Grothendieck sites* is a functor $F: C \to D$ such that
  (a) C and D are sites having all finite limits, and F preserves them
  (b) If $R \subset \mathrm{hom}(\ , U)$ is a covering sieve for $U \in C$, then the image $F(R)$ of the collection of morphisms of R under F generates a covering sieve for $F(U)$.
In the presence of a site morphism, if X is a sheaf on D, then the composite functor $X \circ F$ is a sheaf on C (which is denoted by F_*X and called the *direct image* of X along F); this condition is often paraphrased by saying that F is *continuous*.

REMARK 2.1. Any geometric morphism $f: F \to E$ is induced by a site morphism $F: C \to D$, for a suitable choices of sites C and D underlying the toposes E and F respectively. This is a consequence of the inner workings of Giraud's Theorem. Implicitly, the claim that E is a Grothendieck topos means, in part, that E has a small set of generators $G(E)$. The topos E has all finite limits, so that the original set of generators can be expanded to a new set of generators that we shall call $\mathrm{Ob}(C)$ which is closed under finite limits. The full subcategory $C \subset E$ on this set of generators comes equipped with a canonical choice of Grothendieck topology (the covering families are the epimorphic families), in such a way that the sheaf category $\mathrm{Shv}(C)$ is equivalent to the topos E. The corresponding set $\mathrm{Ob}(D)$ of generators for the topos F is obtained by closing up the set of objects $G(F) \cup f^*\big(\mathrm{Ob}(C)\big)$ under finite limits. The full subcategory $D \subset F$ has a canonical topology such that $\mathrm{Shv}(D)$ is equivalent to the topos F, and the inverse image functor $f^*: E \to F$ restricts to a site morphism $f': C \to D$ which induces the geometric morphism f in the sense that f_* can be identified up to isomorphism with composition with f', and f^* is equivalent to f'^*.

Not to worry: any continuous map of topological spaces or any scheme homomorphism induces a site morphism between the corresponding Grothendieck sites straight up, for all of the favourite topologies, since all such sites have finite limits which are preserved by the corresponding functors.

Recall [12], [15] that the category $\mathbf{S}\,\mathrm{Pre}(C)$ of simplicial presheaves on a small Grothendieck site is a proper closed simplicial model category in which the cofibrations are the monomorphisms. The weak equivalences of simplicial presheaves on C are the local weak equivalences, which can be described as maps that induce isomorphisms in all local sheaves of homotopy groups: in the case where the topos $\mathrm{Shv}(C)$ has enough points, this means that a local weak equivalence is a map which induces a weak equivalence of ordinary simplicial sets in all stalks. The fibrations of $\mathbf{S}\,\mathrm{Pre}(C)$ are the global fibrations,

which are those maps which have the right lifting property with respect to all maps which are simultaneously local weak equivalences and cofibrations. The associated sheaf maps $\eta\colon X \to L^2 X$ are local weak equivalences, and the proper closed simplicial model structure on $\mathbf{S}\operatorname{Pre}(C)$ restricts to a proper closed simplicial model structure on the category $\mathbf{S}\operatorname{Shv}(C)$ for which the cofibrations are the monomorphisms, the weak equivalences (respectively fibrations) are those maps of simplicial sheaves which are local weak equivalences (respectively global fibrations) of simplicial presheaves. The associated sheaf functor and the inclusion functor $\mathbf{S}\operatorname{Shv}(C) \hookrightarrow \mathbf{S}\operatorname{Pre}(C)$ both preserve weak equivalences, and induce an adjoint equivalence

$$\operatorname{Ho}\bigl(\mathbf{S}\operatorname{Shv}(C)\bigr) \simeq \operatorname{Ho}\bigl(\mathbf{S}\operatorname{Pre}(C)\bigr)$$

of associated homotopy categories.

The reader who has visited the first section of this paper might now be experiencing a bit of déjà vu, for good reason: the method of proof of the existence of the closed model structure on $\mathbf{S}\operatorname{Pre}(C)$ that appears in [15] can be interpreted as showing that the collection of local weak equivalences, suitably defined, satisfies axioms **E1**–**E7**, so that Theorems 1.1 and 1.2 apply. In other words, the standard homotopy theory of simplicial presheaves and sheaves on an arbitrary small Grothendieck site C is a type of localization theory.

More explicitly, recall that a Boolean localization of the topos $\operatorname{Shv}(C)$ is a geometric morphism $\wp\colon \operatorname{Shv}(B) \to \operatorname{Shv}(C)$ such that B is a complete Boolean algebra and such that the associated inverse image functor $\wp^*\colon \operatorname{Shv}(C) \to \operatorname{Shv}(B)$ is faithful. Any topos $\operatorname{Shv}(C)$ has a Boolean localization—this is a result of Barr and Diaconescu [16]. The game, as it's played in [15], is first to show that the axiom of choice for the Boolean topos $\operatorname{Shv}(B)$ implies that every locally fibrant simplicial presheaf on B is actually a presheaf of Kan complexes, and that every local weak equivalence $f\colon X \to Y$ of simplicial sheaves on B is a weak equivalence of simplicial sets in each section. We've always known, one way or another (but see Lemma 2.6 below), that inverse image functors preserve local weak equivalences, so one is led to decree that a local weak equivalence on simplicial presheaves on C is a map $g\colon Z \to W$ such that the induced map $g_*\colon \wp^* L^2 \operatorname{Ex}^\infty Z \to \wp^* L^2 \operatorname{Ex}^\infty W$ is a local (hence pointwise) weak equivalence of sheaves of Kan complexes on B. Note the use of Kan's Ex^∞-functor [9]; this is a functorial and combinatorial method of replacing a simplicial set by a Kan complex, and hence can be applied to simplicial presheaves.

The faithfulness of the inverse image functor $\wp^*$ is then used to show that this definition of local weak equivalence coincides with the standard notion involving isomorphisms of local sheaves of homotopy groups. On closer examination, one sees that the faithfulness of the functor $\wp^*$ only appears at the very end, and is in fact independent of the existence of a closed model structure for the simplicial presheaf category $\mathbf{S}\operatorname{Pre}(C)$.

This is the point of this section: we show here that any geometric morphism $f\colon E \to \operatorname{Shv}(C)$ determines a closed model structure on $\mathbf{S}\operatorname{Pre}(C)$ for which the cofibrations are the monomorphisms and the weak equivalences are those maps $g\colon X \to Y$ for which the

induced maps $f^*L^2g \colon f^*L^2X \to f^*L^2Y$ are local weak equivalences of simplicial objects of E. This result is Theorem 2.7. This closed model structure coincides with the standard one on $\mathbf{S}\,\mathrm{Pre}(C)$ if the inverse image functor $f^* \colon \mathrm{Shv}(C) \to E$ is faithful (see Remark 2.9), but is otherwise new.

There is a technical assumption, which will appear from time to time in what follows, that $f \colon C \to D$ is a continuous functor between small sites which induces a geometric morphism $\mathrm{Shv}(D) \to \mathrm{Shv}(C)$. The functor f is the sort of thing that arises most easily when one wants to construct a site level functor underlying a geometric topos morphism $E \to \mathrm{Shv}(C)$ according to the trick involving Giraud's Theorem which is described in Remark 2.1, and in the case where one has reason to avoid adding finite limits to the site C inside the topos $\mathrm{Shv}(C)$. It is typical, for example, that one wouldn't want to fatten up the site C in any way when the focus of concern is the category of simplicial presheaves on C. Note that there is a fairly extensive discussion in [16] concerning the question of when a functor $C \to D$ between Grothendieck sites induces a geometric morphism; in their terminology a continuous functor $f \colon C \to D$ induces a geometric morphism $\mathrm{Shv}(D) \to \mathrm{Shv}(C)$ if and only if f is also "flat", or equivalently "filtering".

We begin by establishing a technical condition (Lemma 2.3) which leads to instances of the bounded cofibration axiom **E7** that hold quite generally for base change along *any* functor between Grothendieck sites.

Suppose that $f \colon X \to Y$ is a morphism of simplicial presheaves on a small Grothendieck site C, and let $i \colon K \subset L$ be an inclusion of finite simplicial sets. We say that f has the *local right lifting property* with respect to i if given any commutative diagram of simplicial set maps

(2.2)
$$
\begin{array}{ccc}
K & \longrightarrow & X(U) \\
\downarrow & & \downarrow f \\
L & \longrightarrow & Y(U)
\end{array}
$$

and for any object U of C there is a covering sieve $R \subset \hom(\ ,U)$ such that for each $\phi \colon V \to U$ in R there is a commutative diagram of simplicial set maps

$$
\begin{array}{ccccc}
K & \longrightarrow & X(U) & \xrightarrow{\ \phi^*\ } & X(U) \\
\downarrow & & & \nearrow & \downarrow f \\
L & \longrightarrow & Y(U) & \xrightarrow{\ \phi^*\ } & Y(V)
\end{array}
$$

In other words, all lifting problems having the form of diagram (2.2) have solutions after refinement along covering sieves. This condition is equivalent, variously, to the requirement that the induced sheaf map

$$
L^2X^L \xrightarrow{\ (i^*,f_*)\ } L^2X^K \times_{L^2Y^K} L^2Y^L
$$

is a sheaf epimorphism, and to the requirement that the presheaf map

$$X^L \xrightarrow{(i^*, f_*)} X^K \times_{Y^K} Y^L$$

is a local epimorphism in the sense that all sections of the base lift to X^L after refinement along some covering sieve. Here, for example, X^L is the presheaf on C which is defined via simplicial set maps in sections by the formula

$$X^L(U) = \hom_{\mathbf{S}}\big(L, X(U)\big).$$

Suppose that the functor $F \colon C \to D$ is *any functor* between two Grothendieck sites. The direct image functor $F_* \colon \mathrm{Pre}(D) \to \mathrm{Pre}(C)$ is nevertheless defined by composition with F—it's just that F_* might not preserve sheaves. Furthermore, the functor F_* always has a left adjoint $F^p \colon \mathrm{Pre}(C) \to \mathrm{Pre}(D)$, which can be defined explicitly for a presheaf Y on D, in sections, by

$$F^p Y(d) = \varinjlim_{d \to F(c) \text{ in } D} Y(c).$$

If the cardinal α is an upper bound for the size of the set of morphisms of D and the presheaf Y on C is α-bounded, then the presheaf $F^p Y$ is α-bounded.

LEMMA 2.3. *Suppose that $F \colon C \to D$ is any functor between small Grothendieck sites. Let α be an infinite cardinal which is an upper bound on the cardinalities of the sets of morphisms for the categories C and D. Suppose that $f \colon X \to Y$ is a map of simplicial presheaves on C and that $Z \subset Y$ is an α-bounded subcomplex of Y. Assume further that*

(1) *$F^p f \colon F^p X \to F^p Y$ has the local right lifting property with respect to a family $K_i \subset L_i$ of simplicial set monomorphisms, where L_i is finite, $i \in I$, and $|I| < \alpha$,*

(2) *there is a functor defined on the α-bounded subcomplexes $W \subset Y$ with $Z \subset W$, taking values in commutative diagrams of simplicial presheaf maps of the form*

$$\begin{array}{ccc} X_W & \longrightarrow & X \\ {\scriptstyle f_W}\downarrow & & \downarrow{\scriptstyle f} \\ W & \hookrightarrow & Y \end{array}$$

such that X_W is α-bounded, and such that $f = \varinjlim_{Z \subset W} f_W$ and further that the functor $V \mapsto f_V$ preserves filtered colimits in the category of α-bounded subobjects $V \subset X$ with $Z \subset V$.

Then there is some α-bounded subcomplex $W \subset Y$ with $Z \subset W$ such that the map $F^p f_W \colon F^p X_W \to F^p W$ has the local right lifting property with respect to all $K_i \subset L_i$.

PROOF. The functor F^p has a right adjoint, so that $V \mapsto F^p f_V$ preserves filtered colimits in V.

The simplicial presheaves $F^p X_Z$ and $F^p Z$ are α-bounded, so there is an α-bounded collection of diagrams of simplicial set maps

(2.4)
$$
\begin{array}{ccccc}
K_i & \longrightarrow & F^p X_Z(U) & \longrightarrow & F^p X(U) \\
\downarrow & & \downarrow{\scriptstyle F^p f_Z} & & \downarrow{\scriptstyle F^p f} \\
L_i & \longrightarrow & F^p Z(U) & \longrightarrow & F^p Y(U)
\end{array}
$$

Given such a diagram, there is a covering sieve $R \subset \hom(U, \ )$ such that for all $\phi\colon V \to U$ there is a lifting

$$
\begin{array}{ccccccc}
K_i & \longrightarrow & F^p X_Z(U) & \longrightarrow & F^p X(U) & \longrightarrow & F^p X(V) \\
\downarrow & & & {\scriptstyle x_\phi} & & & \downarrow{\scriptstyle F^p f} \\
L_i & \longrightarrow & F^p Z(U) & \longrightarrow & F^p Y(U) & \longrightarrow & F^p Y(V)
\end{array}
$$

The simplicial set L_i is finite, and $F^p X = \varinjlim_{Z \subset W} F^p X_W$. It follows that there is an α-bounded W' with $Z \subset W'$ such that all x_ϕ live in $F^p X_{W'}$. Taking the union of all such subcomplexes W' over all diagrams 2.4 gives an α-bounded subcomplex Z_1 of Y such that $Z \subset Z_1$ and such that all lifting problems 2.4 are solved in $F^p X_{Z_1}$. Repeat the construction of obtain a sequence of α-bounded subobjects

$$ Z = Z_0 \subset Z_1 \subset Z_2 \subset \cdots $$

such that all local lifting problems

$$
\begin{array}{ccc}
K_i & \longrightarrow & F^p X_{Z_j}(U) \\
\downarrow & & \downarrow{\scriptstyle F^p f_Z} \\
L_i & \longrightarrow & F^p Z_j(U)
\end{array}
$$

are solved over Z_{j+1}.

Let W be the α-bounded subcomplex defined by

$$ W = \bigcup_{j \geq 0} Z_j. $$

Then $F^p X_W = \varinjlim_{j \geq 0} F^p X_{Z_j}$, and any map $K_i \to F^p X_W(U)$ factors through some map $K_i \to F^p X_{Z_j}(U)$ since K_i is finite. It follows that that the induced map $F^p f_W\colon F^p X_W \to F^p W$ has the local right lifting property with respect to all $K_i \subset L_i$. ∎

REMARK 2.5. There is a natural source of functors $V \mapsto f_V$ satisfying condition (2) of Lemma 2.3. Suppose that $f\colon X \to Y$ is a map of simplicial presheaves on a site C. Then f has a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\ j_* \ } & \\
{\scriptstyle f}\downarrow & & \overline{X} \\
Y & \underset{p_*}{\nearrow} &
\end{array}
$$

such that p_* is a pointwise Kan fibration and j_* is a pointwise weak equivalence, and this factorization is natural and preserves filtered colimits in f. To see this, take the standard factorization

$$\text{Ex}^\infty X \xrightarrow{\quad j \quad} \text{Ex}^\infty X \times_{\text{Ex}^\infty Y} \mathbf{hom}(\Delta^1, \text{Ex}^\infty Y)$$

and pull it back to Y using the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \nu\ } & \text{Ex}^\infty X \\
f \downarrow & & \downarrow \text{Ex}^\infty f \\
Y & \xrightarrow{\ \nu\ } & \text{Ex}^\infty X
\end{array}
$$

so that

$$\overline{X} = Y \times_{\text{Ex}^\infty Y} \text{Ex}^\infty X \times_{\text{Ex}^\infty Y} \mathbf{hom}(\Delta^1, \text{Ex}^\infty Y).$$

The maps ν are pointwise weak equivalences, and weak equivalences are preserved by pullback along fibrations in the simplicial set category, so the induced map j_* is a pointwise weak equivalence. This construction is natural in morphisms f, and preserves filtered colimits in the category of simplicial presheaf morphisms. Note finally that if X and Y are α-bounded simplicial presheaves, then so is $\overline{X}$.

It follows, for example, that if $g: X \to Y$ is a morphism of simplicial presheaves such that $g^{-1}(W)$ is α-bounded whenever W is an α-bounded subcomplex of Y, then the functor defined on α-bounded subcomplexes W of Y containing a fixed α-bounded subobject Z by sending W to the pointwise fibration $p_*: \overline{g^{-1}(W)} \to W$ converges to the map $p_*: \overline{X} \to Y$, and thus satisfies condition (2) of Lemma 2.3, for $f = p_*$. ∎

LEMMA 2.6. *Suppose that $f: C \to D$ is a continuous functor between small Grothendieck sites which induces a geometric morphism* $\text{Shv}(D) \to \text{Shv}(C)$. *Then*

(1) *The functors $f^p: \mathbf{S}\,\text{Pre}(C) \to \mathbf{S}\,\text{Pre}(D)$ and $f^*: \mathbf{S}\,\text{Shv}(C) \to \mathbf{S}\,\text{Shv}(D)$ preserve local weak equivalences.*

(2) *If $p: X \to Y$ is a global fibration of simplicial sheaves on D, then the direct image $f_* p: f_* X \to f_* Y$ is a global fibration of $\mathbf{S}\,\text{Shv}(C)$. If the functor f preserves finite limits, then f_* preserves global fibrations of simplicial presheaves.*

PROOF. Only the statement about preservation of local weak equivalences on the simplicial presheaf level requires proof, because the functor f^* preserves cofibrations in general, and f^p preserves cofibrations under the extra exactness condition on f.

Suppose that $g: X \to Y$ is a local weak equivalence of $\mathbf{S}\,\text{Pre}(C)$. We want to show that $f^* L^2 g \cong L^2 f^p g$ is a local weak equivalence of simplicial presheaves over D, for then $f^p g$ is also a local weak equivalence.

The functor $f^* L^2$ preserves the Ex^∞ construction:

$$f^* L^2 \text{Ex}^\infty X \cong L^2 \text{Ex}^\infty f^* L^2 X$$

by the exactness of f^*, and the associated sheaf functor L^2 takes pointwise weak equivalences to local weak equivalences. It follows that f^*L^2g is a local weak equivalence over D if and only if $f^*L^2 \operatorname{Ex}^\infty g$ is a local weak equivalence. It therefore suffices to assume that $g: X \to Y$ is a morphism of presheaves of Kan complexes. Then there is a commutative diagram

$$
\begin{array}{ccc}
 & Z & \\
{\scriptstyle i}\nearrow & & \searrow{\scriptstyle \pi} \\
X & \xrightarrow[\;g\;]{} & Y
\end{array}
$$

in which π is a pointwise Kan fibration and i is right inverse to a pointwise trivial fibration $\pi': Z \to X$. But then π is a local weak equivalence and a pointwise Kan fibration between presheaves of Kan complexes, and therefore has the local right lifting property with respect to all inclusions $\partial \Delta^n \subset \Delta^n$, as does the map π'. The functor f^*L^2 preserves this local right lifting property, by exactness, so that the maps $f^*L^2\pi$ and $f^*L^2\pi'$ are local weak equivalences over D, as is f^*L^2g. ∎

Suppose that $f: E \to \operatorname{Shv}(C)$ is a geometric topos morphism, and consider the composite functor

$$
\mathbf{S}\operatorname{Pre}(C) \xrightarrow{\;L^2\;} \mathbf{S}\operatorname{Shv}(C) \xrightarrow{\;f^*\;} \mathbf{S}E.
$$

Then we know that the category of simplicial objects in E is a proper closed simplicial model category in a canonical way. Say that a map $g: Z \to W$ of simplicial presheaves on C is an f^*L^2-weak equivalence if the induced map $f^*L^2(g): f^*L^2Z \to f^*L^2W$ is a local weak equivalence of the category $\mathbf{S}E$ of simplicial objects in the Grothendieck topos E.

THEOREM 2.7. *Suppose that $f: E \to \operatorname{Shv}(C)$ is a geometric morphism, where C is a small site. Then the composite functor $f^*L^2: \mathbf{S}\operatorname{Pre}(C) \to \mathbf{S}E$ induces a proper closed simplicial model category structure on $\mathbf{S}\operatorname{Pre}(C)$, for which the ($f^*$-local) weak equivalences are the f^*L^2-weak equivalences, the cofibrations are inclusions, and the (f^*-global) fibrations are defined by a right lifting property.*

PROOF. This result is proved by appealing to Theorem 1.1, where the morphism class $\mathbf{E}$ is the collection of f^*L^2-weak equivalences. We can assume that f is induced by a continuous functor $f: C \to D$. We know from Lemma 2.6 that the functor f^*L^2 preserves local weak equivalences, and therefore takes pointwise weak equivalences to local weak equivalences, giving **E3**.

The class of local trivial cofibrations of $\mathbf{S}\operatorname{Shv}(D)$ is closed under pushout in the standard closed model structure for the closed model structure on D, and the functor f^*L^2 is left and right exact, so axiom **E4** is satisfied. Similar exactness assertions and the closed model structure for $\mathbf{S}\operatorname{Shv}(D)$ together lead to axioms **E5** and **E6**, while **E1** and **E2** are trivial. It remains only to verify the bounded cofibration property **E7**.

Let α be an infinite cardinal which is an upper bound for the cardinalities of the sets of

morphisms for the categories C and D, and suppose given a simplicial presheaf diagram

$$
\begin{array}{c}
X \\
\big\downarrow{\scriptstyle i} \\
A \lhook\joinrel\longrightarrow Y
\end{array}
$$

on C, where A is an α-bounded subcomplex of Y and the map i is a cofibration and a $p^* L^2$-weak equivalence. As seen previously, we can apply the construction of Remark 2.5 to the map i and all of the inclusions $V \cap X \hookrightarrow V$ arising from α-bounded subcomplexes V of Y which contain A, to obtain the pointwise Kan fibration $p_* : \overline{X} \longrightarrow Y$ as a filtered colimit of the pointwise fibrations $p_* : \overline{V \cap X} \longrightarrow V$, where the functor $V \longmapsto \overline{V \cap X}$ preserves filtered colimits, and all objects $\overline{V \cap X}$ are α-bounded. The image $L^2 f^p p_* = f^* L^2 p_*$ in $\mathbf{S}\operatorname{Shv}(D)$ of the map $p_* : \overline{X} \longrightarrow Y$ has the local right lifting property with respect to all inclusions $\partial \Delta^n \subset \Delta^n$. The associated sheaf functor reflects this local left lifting property, so the simplicial presheaf map $f^p p_*$ also has the local right lifting property with respect of all inclusions $\partial \Delta^n \subset \Delta^n$.

Now apply Lemma 2.3 in the case where the site-level functor is $f : C \longrightarrow D$, the simplicial set inclusions are the morphisms $\partial \Delta^n \subset \Delta^n$, $n \geq 0$, the morphism f is the map $p_* : \overline{X} \longrightarrow Y$, and the functor $W \longmapsto f_W$ in that statement is identified with the functor which takes W to the pointwise fibration $p_* : \overline{W \cap X} \longrightarrow W$. This gives an α-bounded subcomplex $B \subset Y$ with $A \subset B$ such that the map $f^p p_* : f^p \overline{B \cap X} \longrightarrow f^p B$ has the local right lifting property with respect to all inclusions $\partial \Delta^n \subset \Delta^n$. Applying the associated sheaf functor therefore shows that the map $f^* L^2 p_* : f^* L^2 \overline{B \cap X} \longrightarrow f^* L^2 B$ has the same local left lifting property, and is therefore a local weak equivalence of the simplicial sheaf category on D. The corresponding inclusion $B \cap X \hookrightarrow B$ is thus part of a commutative diagram

$$
\begin{array}{ccc}
B \cap X & \xrightarrow{\;\; j_* \;\;} & \overline{B \cap X} \\
& \searrow & \big\downarrow{\scriptstyle p_*} \\
& & B
\end{array}
$$

in which the maps j_* and p_* are $f^* L^2$-weak equivalences.

Suppose that the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\;\; g_* \;\;} & X \\
\big\downarrow & & \big\downarrow{\scriptstyle p} \\
W & \xrightarrow[\;\; g \;\;]{} & Y
\end{array}
$$

is a pullback diagram in $\mathbf{S}\operatorname{Pre}(C)$, where g is an $f^* L^2$-weak equivalence and p is an f^*-global fibration. Every local weak equivalence is an $f^* L^2$-weak equivalence, by Lemma 2.6, so that every f^*-global fibration is a global fibration, and hence a local

fibration. It follows that applying the functor f^*L^2 gives a pullback diagram

$$
\begin{array}{ccc}
f^*L^2Z & \xrightarrow{f^*L^2g_*} & f^*L^2X \\
\downarrow & & \downarrow{\scriptstyle f^*L^2p} \\
f^*L^2W & \xrightarrow[f^*L^2g]{} & f^*L^2Y
\end{array}
$$

of simplicial sheaves on D in which f^*L^2p is a local fibration and f^*L^2g is a local weak equivalence. A Boolean localization argument (see the proof of Theorem 24 in [15]) shows that local weak equivalences are stable under pullback along local fibrations, so $f^*L^2g_*$ is a local weak equivalence. ∎

THEOREM 2.8. *Suppose that $f\colon E \to \mathrm{Shv}(C)$ is a geometric morphism, where C is a small site. Then there is a proper closed simplicial model structure on the simplicial sheaf category $\mathbf{S}\,\mathrm{Shv}(C)$ such that the cofibrations are the monomorphisms, and a map $g\colon X \to Y$ is a weak equivalence if and only if the map $f^*g\colon f^*X \to f^*Y$ is a weak equivalence of $\mathbf{S}E$. Furthermore, the associated sheaf functor preserves f^*-local weak equivalences, so there is an induced adjoint equivalence*

$$
\mathrm{Ho}_f\big(\mathbf{S}\,\mathrm{Pre}(C)\big) \simeq \mathrm{Ho}_f\big(\mathbf{S}\,\mathrm{Shv}(C)\big).
$$

PROOF. Observe that the weak equivalences of simplicial sheaves on C are exactly those maps which are f^*L^2-weak equivalences in the simplicial presheaf category, and that the fibrations are f^*-global fibrations. This result is a consequence of Theorem 2.7 and Theorem 1.2. One uses Lemma 2.6 to show that that the associated sheaf map $\eta\colon X \to L^2X$ is an f^*L^2-weak equivalence. ∎

REMARK 2.9. Generally, for any geometric morphism $f\colon E \to \mathrm{Shv}(C)$, the functor $f^*L^2\colon \mathbf{S}\,\mathrm{Pre}(C) \to \mathbf{S}E$ takes local weak equivalences to local weak equivalences, by Lemma 2.6. However, if the inverse image map $f^*\colon \mathrm{Shv}(C) \to E$ is also faithful, then a simplicial presheaf map $g\colon X \to Y$ on C is a local weak equivalence if and only if f^*L^2g is a local weak equivalence of simplicial sheaves, so that the local weak equivalences and the f^*-local weak equivalences of $\mathbf{S}\,\mathrm{Pre}(C)$ coincide. To see this, take a map $g\colon X \to Y$ and form the factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\ j_*\ } & \\
{\scriptstyle g}\downarrow & \searrow & \overline{X} \\
Y & \nearrow_{\ p_*} &
\end{array}
$$

such that p_* is a pointwise Kan fibration and j_* is a pointwise weak equivalence, according to the method outlined in Remark 2.5. Then the composite functor f^*L^2 preserves local weak equivalences and local fibrations, so that g is an f^*L^2-weak equivalence if and only if the map $f^*L^2p_*$ has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$. In the case that f^* is faithful, the composite f^*L^2 reflects this local left lifting property, so therefore g is an f^*L^2-weak equivalence if and only if the map p_* has the local right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, and hence if and only if g is a local weak equivalence. ∎

3. **Homology theories.** Going further requires that we first back up, and see that the conditions of Theorem 1.1 and its proof can, after minor adjustments, be used as a template for results about presheaves of spectra.

Suppose that $\mathbf{sE}$ is a class of morphisms of the category $\mathbf{Spt}\,\mathrm{Pre}(\mathcal{C})$ of presheaves of spectra on a site $\mathcal{C}$. A cofibration $i\colon X \longrightarrow Y$ presheaves of spectra is the standard thing [5], [14], namely a map such that the level 0 map $i^0\colon X^0 \longrightarrow Y^0$ is a cofibration of pointed simplicial presheaves, as are all induced maps

$$S^1 \wedge Y^n \cup_{S^1 \wedge X^n} X^{n+1} \longrightarrow Y^{n+1}.$$

Say that a cofibration of presheaves of spectra which is also a member of $\mathbf{sE}$ is an $\mathbf{sE}$-trivial cofibration.

For any infinite cardinal α, a presheaf of spectra A is said to be α-bounded if each of its constituent simplicial presheaves A^n, $n \geq 0$, is α-bounded. If α is an upper bound for the cardinality of the set of morphisms of the underlying site $\mathcal{C}$, then every presheaf of spectra X is a filtered colimit of its α-bounded subobjects. To see this, take some section $x \in X^n_m(U)$, and form the corresponding pointed simplicial presheaf map $x\colon (L_U\Delta^m)_+ \longrightarrow X^n$. The simplicial presheaf $(L_U\Delta^m)_+$ is α-bounded by the assumption on the size of the cardinal α. Then x canonically determines a map of presheaves of spectra $x_*\colon \Sigma^\infty(L_U\Delta^m)_+[-n] \longrightarrow X$, whose image is an α-bounded subobject of X which contains the section x. Here,

$$\Sigma^\infty(L_U\Delta^m)_+[-n]^k = \begin{cases} * & \text{if } k - n < 0, \text{ and} \\ \Sigma^\infty(L_U\Delta^m)_+^{k-n} & \text{if } k - n \geq 0, \end{cases}$$

according to the usual indexing conventions, and $\Sigma^\infty(L_U\Delta^m)_+$ is the standard suspension spectrum object for the pointed simplicial presheaf $(L_U\Delta^m)_+$.

The analogue of the list of conditions **E1**–**E7** for Theorem 1.1 is the following collection of statements:

$\mathbf{sE1}$: The class of morphisms $\mathbf{sE}$ is closed under retracts.

$\mathbf{sE2}$: Given a composable pair of morphisms

$$X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z,$$

if any two of f, g and gf are in the class $\mathbf{sE}$, then so is the third.

$\mathbf{sE3}$: Every pointwise strict equivalence is in $\mathbf{sE}$.

$\mathbf{sE4}$: The class of $\mathbf{sE}$-trivial cofibrations is closed under pushout.

$\mathbf{sE5}$: Suppose that γ is a limit ordinal, and there is a functor $X\colon \gamma \longrightarrow \mathbf{Spt}\,\mathrm{Pre}(\mathcal{C})$ such that for each morphism $i \leq j$ of γ, the induced map $X(i) \longrightarrow X(j)$ is an $\mathbf{sE}$-trivial cofibration. Then the canonical maps

$$X(i) \xrightarrow{\;\tau_i\;} \varinjlim_{j \in \gamma} X(j)$$

are $\mathbf{sE}$-trivial cofibrations.

sE6: Suppose that the morphisms $f_i: X_i \rightarrow Y_i$ are **sE**-trivial cofibrations for $i \in I$. Then the morphism

$$\bigvee_{i \in I} f_i: \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} Y_i$$

is an E-trivial cofibration.

sE7: There is an infinite cardinal α which is at least as large as the cardinality of the set of morphisms of C, such that for every diagram

$$
\begin{array}{c}
X \\
\big\uparrow i \\
A \hookrightarrow Y
\end{array}
$$

of maps of presheaves of spectra with i a **sE**-trivial cofibration, and A α-bounded, there is an α-bounded subobject $B \subset Y$ such that $A \subset B$, and such that the inclusion $B \cap X \hookrightarrow B$ is an **sE**-trivial cofibration.

A *pointwise strict equivalence* is a map $f: X \rightarrow Y$ of presheaves of spectra such that all induced maps of simplicial sets $f: X^n(U) \rightarrow Y^n(U)$, $U \in C$ are weak equivalences. Also, perhaps it's hard to believe, but the following requires proof so that the statement **sE7** makes sense:

LEMMA 3.1. *Suppose that $i: A \hookrightarrow B$ is a cofibration of spectra, and that $j: V \subset B$ is a subcomplex of B. Then the induced map $i_*: V \cap A \rightarrow V$ is a cofibration of spectra.*

REMARK 3.2. When we say that $j: V \subset B$ is a subobject or subcomplex of B, we mean simply that all maps $j: V^n \rightarrow B^n$ are monomorphisms. We do *not* mean that j is a cofibration. ∎

PROOF OF LEMMA 3.1. We need to show that all induced maps

$$(S^1 \wedge V^n) \cup_{S^1 \wedge (V^n \cap A^n)} (V^{n+1} \cap A^{n+1}) \xrightarrow{(\sigma, i_*)} V^{n+1}$$

are cofibrations in the pointed simplicial set category. Given that the diagram of spectrum maps

$$
\begin{array}{ccc}
V \cap A & \xrightarrow{j_*} & A \\
i_* \big\downarrow & & \big\uparrow i \\
V & \xrightarrow{j} & B
\end{array}
$$

induces a commutative diagram

(3.3)
$$
\begin{array}{ccc}
(S^1 \wedge V^n) \cup_{S^1 \wedge (V^n \cap A^n)} (V^{n+1} \cap A^{n+1}) & \xrightarrow{(\sigma, i_*)} & V^{n+1} \\
j_* \big\downarrow & & \big\downarrow j \\
(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} & \xrightarrow[(\sigma, i)]{} & B^{n+1}
\end{array}
$$

it suffices to show that the vertical map j_* in (3.3) is a cofibration.

Note that any pointed map

$$(S^1 \wedge B^n) \cup_{(S^1 \wedge A^n)} A^{n+1} \longrightarrow Y$$

can be identified with a pair of maps $f: A^{n+1} \longrightarrow Y$ and $g: B^n \longrightarrow \Omega Y$, such that the diagram

$$
\begin{array}{ccc}
A^n & \xrightarrow{\sigma_*} & \Omega A^{n+1} \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Omega f} \\
B^n & \xrightarrow{g} & \Omega Y
\end{array}
$$

commutes, where σ_* is the adjoint of the bonding map $\sigma: S^1 \wedge A^n \longrightarrow A^{n+1}$. In this language, suppose given a commutative diagram

(3.4)
$$
\begin{array}{ccc}
(S^1 \wedge V^n) \cup_{S^1 \wedge (V^n \cap A^n)} (V^{n+1} \cap A^{n+1}) & \xrightarrow{(g'_*,f')} & X \\
{\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle p} \\
S^1 \wedge B^n \cup_{S^1 \wedge A^n} A^{n+1} & \xrightarrow{(g_*,f)} & Y
\end{array}
$$

where p is a trivial fibration. Then there is a map h making the diagram

$$
\begin{array}{ccc}
V^{n+1} \cap A^{n+1} & \xrightarrow{f'} & X \\
{\scriptstyle j_*}\downarrow & \nearrow^{h} & \downarrow{\scriptstyle p} \\
A^{n+1} & \xrightarrow{f} & Y
\end{array}
$$

commute, since j_* is a cofibration of simplicial sets and p is a trivial fibration. The maps $\Omega h \circ \sigma_*: A^n \longrightarrow \Omega X$ and $g': V^n \longrightarrow \Omega X$ together induce a morphism $H: V^n \cup A^n \longrightarrow \Omega X$, and there is a commutative diagram

$$
\begin{array}{ccc}
V^n \cup A^n & \xrightarrow{H} & \Omega X \\
\downarrow & \nearrow^{K} & \downarrow{\scriptstyle \Omega p} \\
B^n & \xrightarrow{g} & \Omega Y
\end{array}
$$

since Ωp is a trivial fibration. The map

$$(K_*, h): S^1 \wedge B^n \cup_{S^1 \wedge A^n} A^{n+1} \longrightarrow X$$

solves the lifting problem posed by the diagram (3.4), so that j_* has the left lifting property with respect to all trivial fibrations, and is therefore a cofibration. ∎

Observe that the class of cofibrations satisfies analogues of the axioms **sE4**–**sE6** (*i.e.*, without the "**sE**-trivial" condition), because they are defined pointwise, and cofibrations of ordinary spectra fit into a (strict) closed model structure—one could also argue directly.

Say that a morphism $p: X \longrightarrow Y$ is an **sE**-fibration if it has the right lifting property with respect to all **sE**-trivial cofibrations. An **sE**-weak equivalence is a member of the class **sE**.

If K is a simplicial set and X is a presheaf of spectra, then the tensor object $X \ltimes K$ is defined in sections for $U \in \mathcal{C}$ by

$$(X \ltimes K)(U) = X(U) \ltimes K = \big(X(U) \times K\big)/(* \times K) = X(U) \wedge K_+,$$

where $K_+ = K \sqcup *$ is a copy of K with a disjoint base point attached. For presheaves of spectra X and Y, the function complex $\mathbf{hom}(X, Y)$ is the simplicial set whose set of n-simplices is defined by

$$\mathbf{hom}(X, Y)_n = \hom(X \ltimes \Delta^n, Y)$$

where the morphism set on the right is in the category of presheaves of spectra on $\mathcal{C}$. The ordinary exponential law for spectra induces a simplicial category structure on $\mathbf{Spt}\,\mathrm{Pre}(\mathcal{C})$.

THEOREM 3.5.

(1) *Under the conditions* **sE1**–**sE7** *listed above, there is a closed model structure on* $\mathbf{Spt}\,\mathrm{Pre}(\mathcal{C})$ *such that the cofibrations are the (pointwise) cofibrations as described above, the weak equivalences are the* **sE***-weak equivalences, and fibrations (i.e.,* **sE***-fibrations) are defined by a right lifting property.*

(2) *Suppose further that, given an inclusion* $i\colon K \to L$ *of finite simplicial sets and a cofibration* $j\colon X \hookrightarrow Y$, *then the induced map*

$$X \ltimes L \cup_{X \ltimes K} Y \ltimes K \hookrightarrow Y \ltimes L$$

is an **sE***-trivial cofibration if either i is a weak equivalence of simplicial sets or j is an* **sE***-weak equivalence of presheaves of spectra. Then* $\mathbf{Spt}\,\mathrm{Pre}(\mathcal{C})$ *has the structure of a closed simplicial model category.*

PROOF. Suppose that α is an infinite cardinal which is an upper bound for the cardinality of the set of morphisms of the site $\mathcal{C}$.

The proof is analogous to the argument for Theorem 1.1, except that we need Lemma 3.1 for the proof of the factorization axiom **CM5**. One further shows, via the techniques in the proof of Theorem 1.1, that a map has the right lifting property with respect to all cofibrations if and only if it has the right lifting property with respect to all α-bounded cofibrations. Then a transfinite small object argument is used to show that every map $f\colon X \to Y$ of presheaves of spectra has a factorization

$$\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
 & {\scriptstyle j}\searrow \quad \nearrow {\scriptstyle q} & \\
 & W &
\end{array}$$

where j is a cofibration, and q has the right lifting property with respect to all cofibrations. In particular, the maps $q\colon W^m \to Y^m$ have the right lifting property with respect to all cofibrations $\partial\Delta^n \subset \Delta^n$, since the induced maps of presheaves of spectra

$$\Sigma^\infty(L_U\partial\Delta^n)_+ \longrightarrow \Sigma^\infty(L_U\Delta^n)_+$$

are cofibrations. It follows that the maps $q: W^m \to Y^m$ are pointwise weak equivalences, so that $q: W \to Y$ is a pointwise stable equivalence and therefore an **sE**-weak equivalence as well as an **sE**-fibration, by **sE3**. ∎

REMARK 3.6.   The proof of Theorem 3.5 does not assume the standard closed model structure for presheaves of spectra. This structure appears as a consequence of Theorem 3.5 in Theorem 3.7 below.

Suppose that X and Y are pointed simplicial sets, and that K is an arbitrary simplicial set. The canonical isomorphism

$$\hom(K, X) \times \hom(K, Y) \xrightarrow{\cong} \hom(K, X \times Y)$$

induces a map

$$\hom(K, X) \wedge \hom(K, Y) \xrightarrow{\phi} \hom(K, X \wedge Y)$$

which is natural in K, X and Y. It follows that the maps

$$\hom(\mathrm{sd}^n \Delta^m, X) \wedge \hom(\mathrm{sd}^n \Delta^m, Y) \xrightarrow{\phi} \hom(\mathrm{sd}^n \Delta^m, X \wedge Y)$$

induce a pointed simplicial set map

$$\mathrm{Ex}^\infty X \wedge \mathrm{Ex}^\infty Y \xrightarrow{\phi} \mathrm{Ex}^\infty(X \wedge Y)$$

which is natural in X and Y. In particular, any pointed simplicial set map

$$S^1 \wedge X \xrightarrow{\sigma} Y$$

induces a composite pointed map

$$S^1 \wedge \mathrm{Ex}^\infty X \xrightarrow{\nu \wedge 1} \mathrm{Ex}^\infty S^1 \wedge \mathrm{Ex}^\infty X \xrightarrow{\phi} \mathrm{Ex}^\infty(S^1 \wedge X) \xrightarrow{\mathrm{Ex}^\infty \sigma} \mathrm{Ex}^\infty Y,$$

which will be denoted by $\tilde{\sigma}$. Here, ν is a special case of Kan's natural weak equivalence $\nu: Z \to \mathrm{Ex}^\infty Z$.

It follows that, if X is a spectrum with bonding maps $\sigma: S^1 \wedge X^n \to X^{n+1}$, then there is a spectrum $\mathrm{Ex}^\infty X$ consisting of the pointed spaces $\mathrm{Ex}^\infty X^n$, and with bonding maps given by the induced maps $\tilde{\sigma}: S^1 \wedge \mathrm{Ex}^\infty X^n \to \mathrm{Ex}^\infty X^{n+1}$. The spectrum $\mathrm{Ex}^\infty X$ is strictly fibrant (*i.e.*, consists of Kan complexes) and that the maps $\nu: X^n \to \mathrm{Ex}^\infty X^n$ define a strict weak equivalence of spectra $\nu: X \to \mathrm{Ex}^\infty X$.

Suppose, generally, that Z is a presheaf of spectra such that each of the objects Z^n is a presheaf of Kan complexes. Then Z has sheaves of stable homotopy groups $\pi_i Z$, defined for $i \in \mathbb{Z}$, where $\pi_i Z$ is the colimit in the sheaf category of the system of maps

$$\pi_i Z^{n+i} \to \pi_{i+1} Z^{n+i+1} \to \cdots$$

of sheaves of ordinary homotopy groups. The homotopy group sheaf $\pi_i Z^{n+i}$ is, by definition, the sheaf associated to the presheaf $U \mapsto \pi_i\big(Z^{n+i}(U), *\big)$, where $*$ denotes the

implicit global choice of base point. We say that a map $f: X \longrightarrow Y$ is a *local stable equivalence* if f induces sheaf isomorphisms $\pi_i \operatorname{Ex}^\infty X \cong \pi_i \operatorname{Ex}^\infty Y$ for all $i \in \mathbb{Z}$.

Suppose again that Z is a presheaf of spectra which consists of presheaves of Kan complexes. The presheaf of spectra QZ has space at level n defined by the filtered colimit

$$Z^n \xrightarrow{\sigma_*} \Omega Z^{n+1} \xrightarrow{\Omega \sigma_*} \Omega^2 Z^{n+2} \xrightarrow{\Omega^2 \sigma_*} \cdots,$$

with bonding map $\sigma: QZ^n \longrightarrow \Omega QZ^{n+1}$ induced by the maps $\Omega^k \sigma: \Omega^k Z^{n+k} \longrightarrow \Omega^{k+1} Z^{n+k+1}$. A cofinality argument implies that all bonding maps $\sigma: QZ^n \longrightarrow \Omega QZ^{n+1}$ are isomorphisms of simplicial presheaves, and that the canonical map $Z \longrightarrow QZ$ induces an isomorphism of presheaves of stable homotopy groups, and hence is a local stable equivalence. Finally, the canonical map

$$\pi_i QZ^n \longrightarrow \pi_{n-i} QZ$$

is an isomorphism for all $n \geq 0$ and $i \geq 0$, so that a map $f: X \longrightarrow Y$ of presheaves of spectra is a local stable equivalence if and only if all induced maps $f_*: \pi_i Q \operatorname{Ex}^\infty X^n \longrightarrow \pi_i Q \operatorname{Ex}^\infty Y^n$ of sheaves of (ordinary) homotopy groups are isomorphisms.

In particular, all the morphisms $f_*: Q \operatorname{Ex}^\infty X^n \longrightarrow Q \operatorname{Ex}^\infty Y^n$ are local weak equivalences. This is proved with a Boolean localization argument: if $p: \operatorname{Shv}(\mathcal{B}) \longrightarrow \operatorname{Shv}(\mathcal{C})$ is a Boolean localization, then each of induced map

$$p^* f_*: p^* L^2 Q \operatorname{Ex}^\infty X^n(b) \longrightarrow p^* L^2 Q \operatorname{Ex}^\infty Y^n(b)$$

in sections is a map of Kan complexes which are also loop spaces, and this map preserves the loop space structure. Furthermore, this map induces an isomorphism

$$\pi_i p^* L^2 Q \operatorname{Ex}^\infty X^n(b) \cong \pi_i p^* L^2 Q \operatorname{Ex}^\infty X^n(b),$$

for $i \geq 0$, since Boolean localization commutes with the formation of all sheaves of homotopy groups (see [15]), and so the ambient H-space structures can be used to show that it is a weak equivalence, in all sections.

Conversely, or at least partially so, if $f: X \longrightarrow Y$ is a map of presheaves of spectra such that each of the maps $f: X^n \longrightarrow Y^n$ is a local weak equivalence, then f induces an isomorphism in sheaves of stable homotopy groups.

The following result was first proved in [13]. We give here a new "one step" proof which is based on Theorem 3.5. The proof of this result specializes to an alternative demonstration of the existence of the Bousfield-Friedlander stable closed model structure for ordinary spectra [5]. Note, however, that we still require the strict closed model structure for ordinary spectra to take care of some of the standard assertions about cofibrations. The existence of the strict structure is just an exercise.

THEOREM 3.7. *Say that a map $f: X \longrightarrow Y$ of presheaves of spectra on a small site $\mathcal{C}$ is a local stable equivalence if it induces an isomorphism in all sheaves of stable homotopy groups, and that a map is a global fibration if it has the right lifting property with respect to all maps which are cofibrations and local stable equivalences. Then with these definitions, the category $\mathbf{Spt} \operatorname{Pre}(\mathcal{C})$ of presheaves of spectra on a Grothendieck site $\mathcal{C}$ satisfies the axioms for a proper closed simplicial model category.*

PROOF. We verify the conditions of Theorem 3.5 to show that the category of presheaves of spectra on C is a closed simplicial model category where, in the notation of that result, **sE** stands for the class of local stable equivalences.

The stable homotopy group functor preserves filtered colimits and takes wedges to direct sums, so the only axiom having any content is **sE7**. Suppose that α is an infinite cardinal which is a strict upper bound for the cardinality of the set of morphisms of the underlying site, and suppose given a diagram

$$
\begin{array}{c}
X \\
\Big\uparrow{\scriptstyle i} \\
A \hookrightarrow Y
\end{array}
$$

where A is an α-bounded subobject of Y and i is a cofibration which is a local stable equivalence. This means in particular that the presheaves of stable homotopy groups $\pi_i^p Y/X$ are locally trivial in the sense that any section $x \in \pi_i^p Y/X(U)$ maps to 0 along some covering sieve $R \subset \mathrm{hom}(\ ,U)$ in the sense that $\phi^*(x) = 0$ for all $\phi\colon V \longrightarrow U$ in R. The presheaf of spectra Y is a filtered colimit of its α-bounded subobjects, so it follows that any $x \in \pi_*^p(A/A \cap X)(U)$ maps to 0 in $\pi_*^p(A_1/A_1 \cap X)$ for some α-bounded A_1 with $A \subset A_1 \subset Y$. Continue inductively, to produce a sequence of α-bounded subobjects

$$
A = A_0 \subset A_1 \subset A_2 \subset \cdots
$$

of Y such that the induced maps

$$
\pi_*^p(A_i/A_i \cap X) \longrightarrow \pi_*^p(A_{i+1}/A_{i+1} \cap X)
$$

of presheaves of stable homotopy groups are all 0-maps. Let $B = \cup_{i \geq 0} A_i$. Then B is an α-bounded subobject of Y containing A such that $\pi_*^p(B/B \cap X) = 0$, so that the cofibration $B \cap X \hookrightarrow B$ is a local stable equivalence.

The class of maps which are both cofibrations and local stable equivalences is closed under pushout, by a standard long exact sequence argument. Suppose that $i\colon K \hookrightarrow L$ is an inclusion of finite simplicial sets and that $j\colon X \longrightarrow Y$ is a cofibration of presheaves of spectra. Then the induced cofibration

$$
(3.8) \qquad\qquad X \ltimes L \cup_{X \ltimes K} Y \ltimes K \hookrightarrow Y \ltimes L
$$

is a local stable equivalence if either i is a weak equivalence or j is a local stable equivalence. If i is a weak equivalence, then the maps $X \ltimes i$ and $Y \ltimes i$ are strict and hence local stable equivalences, as is the pushout map

$$
Y \ltimes K \xrightarrow{\ i_*\ } X \ltimes L \cup_{X \ltimes K} Y \ltimes K.
$$

For the other case, if $i\colon X \longrightarrow Y$ is a local stable equivalence, then the maps

$$
i_*\colon \bigvee_{\sigma \in K_n} X \longrightarrow \bigvee_{\sigma \in K_n} Y
$$

are local stable equivalences, so that the map $i\colon X \ltimes K \longrightarrow Y \ltimes K$ is a local stable equivalence, by a spectral sequence argument (see [14, 4.3]). This holds for every simplicial set K, and in particular for L.

Every global fibration $p\colon X \longrightarrow Y$ is a strict fibration, in the sense that all maps $p\colon X^n \longrightarrow Y^n$ are global fibrations of simplicial presheaves, since the functors $U \longmapsto \Sigma^\infty U_+[-n]$ take trivial cofibrations of simplicial presheaves to cofibrations of presheaves of spectra which are local stable equivalences. Suppose that the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\ g_* \ } & X \\
\downarrow & & \downarrow{\scriptstyle p} \\
W & \xrightarrow[\ g \ ]{} & Y
\end{array}
$$

is a pullback in the category of presheaves of spectra, where p is a global fibration and g is a local stable equivalence. We want to show that the map g_* is a local stable equivalence. Each induced diagram

$$
\begin{array}{ccc}
Q\,\mathrm{Ex}^\infty Z^n & \xrightarrow{\ Q\,\mathrm{Ex}^\infty g_* \ } & Q\,\mathrm{Ex}^\infty X^n \\
\downarrow & & \downarrow{\scriptstyle Q\,\mathrm{Ex}^\infty p} \\
Q\,\mathrm{Ex}^\infty W^n & \xrightarrow[\ Q\,\mathrm{Ex}^\infty g \ ]{} & Q\,\mathrm{Ex}^\infty Y^n
\end{array}
$$

is a pullback diagram of simplicial presheaves in which all the objects are presheaves of Kan complexes, the map $Q\,\mathrm{Ex}^\infty g$ is a local weak equivalence, and the map $Q\,\mathrm{Ex}^\infty p$ is a local fibration. We saw in the proof of Theorem 2.7 that local weak equivalences are stable under pullback along local fibrations, so that $Q\,\mathrm{Ex}^\infty g_*$ is a local weak equivalence, and so the map g_* is a local stable equivalence.

One shows that local stable equivalences are preserved by pushing out along cofibrations by comparing long exact sequences for cofibrations. ∎

THEOREM 3.9. *Suppose that $F\colon C \longrightarrow D$ is a continuous functor between small Grothendieck sites which induces a geometric morphism $\mathrm{Shv}(D) \longrightarrow \mathrm{Shv}(C)$, and that E is a presheaf of spectra on D. Say that a map $g\colon X \longrightarrow Y$ of simplicial presheaves on C is an E_*-weak equivalence if the induced map $g_*\colon E \wedge F^p X_+ \longrightarrow E \wedge F^p Y_+$ is a local stable equivalence of presheaves of spectra on D. Say that a map $p\colon Z \longrightarrow W$ is an E_*-global fibration if p has the right lifting property with respect to all cofibrations which are E_*-weak equivalences. Then the classes of cofibrations, E_*-weak equivalences and E_*-global fibrations give the category $\mathbf{S}\,\mathrm{Pre}(C)$ the structure of a closed simplicial model category.*

PROOF. The functor $X \longmapsto E \wedge F^p X_+$ takes values in the category of presheaves of spectra on the site D, which is a closed simplicial model category by the main result of [13], or Theorem 3.7. The notation $F^p X_+$, as usual, denotes $F^p X$ with a disjoint base point attached. We shall prove this result by invoking Theorem 1.1, where the class $\mathbf{E}$ is the collection of E_*-weak equivalences.

We can assume that the presheaf of spectra E is cofibrant, because otherwise there is a trivial strict fibration $\pi: E' \longrightarrow E$, where E' is cofibrant, and the induced map

$$\pi \wedge 1: E' \wedge F^p X_+ \longrightarrow E \wedge F^p X_+$$

is a strict and hence local stable weak equivalence for all X, so that a map $g: X \longrightarrow Y$ of simplicial presheaves on C is an E_*-weak equivalence if and only if it is an E'_*-weak equivalence.

If $i: A \longrightarrow B$ is a cofibration of $\mathbf{S}\operatorname{Pre}(C)$, then the induced map $F^p(i): F^p A_+ \longrightarrow F^p B_+$ is canonically locally weakly equivalent to a natural choice of cofibration, namely the $L^2 F^p(i): L^2 F^p A_+ \longrightarrow L^2 F^p B_+$. In particular, if the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
i\downarrow & & \downarrow i_* \\
B & \longrightarrow & D
\end{array}
$$

is a pushout diagram of simplicial presheaves on the site C, where i is an E_*-trivial cofibration, then the pushout diagram

$$
\begin{array}{ccc}
L^2 F^p A & \longrightarrow & L^2 F^p C \\
L^2 F^p(i)\downarrow & & \downarrow L^2 F^p(i_*) \\
L^2 F^p B & \longrightarrow & L^2 F^p D
\end{array}
$$

of simplicial sheaves on D contains a cofibration $L^2 F^p(i)$ which induces a local stable equivalence

$$E \wedge L^2 F^p(i)_+: E \wedge L^2 F^p A_+ \longrightarrow E \wedge L^2 F^p B_+.$$

Cofibrations of $\mathbf{S}\operatorname{Pre}(D)$ which induce a local stable equivalence after smashing with E are stable under pushout, and so the same is true for cofibrations of simplicial sheaves on D, since the associated sheaf functor preserves pushouts. It follows that the class of E_*-trivial cofibrations of $\mathbf{S}\operatorname{Pre}(C)$ is closed under pushout.

We used a piece of Lemma 2.6 implicitly in this last argument: if $f: X \longrightarrow Y$ is a pointwise weak equivalence, then the simplicial presheaf map $f_*: F^p X_+ \longrightarrow F^p Y_+$ on D is a local weak equivalence. It follows that the induced map $f_*: E \wedge F^p X_+ \longrightarrow E \wedge F^p Y_+$ is a strict and hence local stable equivalence, so that every pointwise equivalence is an E_*-weak equivalence, giving **E3**.

With the exception of **E7**, the remaining axioms have trivial proofs. It remains to verify the bounded cofibration condition.

Let α be an infinite cardinal which is an upper bound on the cardinalities of the morphisms set of C, the morphism set of D, and all sets of sections of all simplicial presheaves E^n, $n \geq 0$ making up the presheaf of spectra E. Suppose that we are given a

diagram of simplicial presheaf maps

$$
\begin{array}{c}
X \\
\Big\downarrow{\scriptstyle i} \\
A \hookrightarrow Y
\end{array}
$$

on C, where i is a cofibration which is an E_*-equivalence, and A is an α-bounded subobject of Y. The assertion that i is an E_*-equivalence means exactly that all stable homotopy group presheaves $\pi_i^p\big(E \wedge F^p(Y/X)\big)$ are locally trivial in the sense that every section $x \in \pi_i^p\big(E \wedge F^p(Y/X)\big)(U)$ maps to 0 along all maps in some covering sieve $R \subset \mathrm{hom}(\ ,U)$, and for all objects U of the site D. The presheaves of stable homotopy groups $\pi_i^p\Big(E \wedge F^p\big(A/(A \cap X)\big)\Big)$ are α-bounded, by the choice of the size of the cardinal α. The functor $B \mapsto \pi_i^p(E \wedge F^p B)$ preserves filtered colimits in pointed simplicial presheaves B on C, and Y is a filtered colimit of its α-bounded subobjects. It follows that there is an α-bounded subobject A_1 of Y such that $A \subset A_1$, and such that the induced map of presheaves of stable homotopy groups

$$
\pi_i^p\Big(E \wedge F^p\big(A/(A \cap X)\big)\Big) \longrightarrow \pi_i^p\Big(E \wedge F^p\big(A_1/(A_1 \cap X)\big)\Big)
$$

is the zero map in all sections, and for all i. Repeating this construction inductively gives a sequence of inclusions

$$
A = A_0 \subset A_1 \subset A_1 \subset \cdots
$$

of α-bounded subobjects of Y such that all induced maps

$$
\pi_i^p\Big(E \wedge F^p\big(A_j/(A_j \cap X)\big)\Big) \longrightarrow \pi_i^p\Big(E \wedge F^p\big(A_{j+1}/(A_{j+1} \cap X)\big)\Big)
$$

of presheaves of stable homotopy groups are 0. Let $B = \cup_i A_i$: then B is an α-bounded subobject of Y containing A such that $\pi_i\Big(E \wedge F^p\big(B/(B \cap X)\big)\Big) = 0$ for all i, so that the inclusion $B \cap X \hookrightarrow B$ is an E_*-equivalence.

The natural simplicial set isomorphism

$$
K_+ \ltimes L \cong (K \times L)_+
$$

implies that the functor $X \mapsto E \wedge F^p X_+$ is a simplicial functor which preserves tensors, along with pushouts and cofibrations. It follows that if $i : X \hookrightarrow Y$ is a cofibration of simplicial presheaves on C and $j : K \hookrightarrow L$ is an inclusion of finite simplicial sets, then the induced cofibration

$$
(X \times L) \cup_{(X \times K)} (Y \times K) \hookrightarrow Y \times L
$$

is an E_*-weak equivalence if either i is an E_*-weak equivalence or j is a trivial cofibration of simplicial sets. $\blacksquare$

We say that the closed model structure arising from the presheaf of spectra E and the functor $F: C \to D$ is the E_*-theory. An E_*-fibrant model $X \to L_E X$ is an E_*-weak equivalence such that $L_E X$ is E_*-fibrant. Such things always exist, and any two E_*-fibrant models for X are non-canonically homotopy equivalent, by the Whitehead Theorem. An E_*-fibrant model for X is also said to be an E_*-localization of X.

Recall from [14] that a bispectrum X can be defined to be a spectrum object in the category of spectra, in the sense that X consists of spectra X^n, $n \geq 0$, and maps of spectra $\sigma: X^n \wedge S^1 \to X^{n+1}$. The maps σ are usually called bonding maps. A morphism $i: X \to Y$ of bispectra is a cofibration if the map $i^0: X^0 \to Y^0$ at level 0 is a cofibration of ordinary spectra, along with all induced maps

$$(Y^n \wedge S^1) \cup_{(X^n \wedge S^1)} X^{n+1} \to Y^{n+1}.$$

The spectra Y^n themselves consist of spaces $Y^{n,k}$, $k \geq 0$, and there is a diagonal spectrum $d(Y)$ associated to the bispectrum Y, such that $d(Y)^{2r} = Y^{r,r}$, $d(Y)^{2r+1} = Y^{r,r+1}$, and such that the bonding maps are given by the pointed simplicial set maps

$$S^1 \wedge Y^{r,r} \to Y^{r,r+1},$$

and the composites

$$S^1 \wedge Y^{r,r+1} \xrightarrow[\cong]{\tau} Y^{r,r+1} \wedge S^1 \to Y^{r+1,r+1}$$

arising from the spectrum structure for Y^r and the bispectrum structure for Y, respectively. A map $f: X \to Y$ of bispectra is said to be a stable equivalence if the induced map $f_*: d(X) \to d(Y)$ is a stable equivalence of ordinary spectra.

These definitions generalize quite naturally to presheaves of bispectra. A map $f: X \to Y$ of presheaves of bispectra is said to be a cofibration if each of the maps $f: X(U) \to Y(U)$, $U \in C$, in sections of ordinary bispectra is a cofibration in the sense described above. The map $f: X \to Y$ is said to be a local stable equivalence if the induced map $f_*: d(X) \to d(Y)$ of diagonal presheaves of spectra is a local stable equivalence. Write $\mathbf{Spt}^2 \operatorname{Pre}(C)$ for the category of presheaves of bispectra on the site C. We shall use the result, proved in [14, 2.1], that the category $\mathbf{Spt}^2 \operatorname{Pre}(C)$ has a proper closed simplicial model structure for which the weak equivalences are the local stable equivalences and the cofibrations are as above.

THEOREM 3.10. *Suppose that $F: C \to D$ is a continuous functor between small Grothendieck sites which induces a geometric morphism $\operatorname{Shv}(D) \to \operatorname{Shv}(C)$, and that E is a presheaf of spectra on D. Say that a map $f: X \to Y$ of presheaves of spectra on C is an E_*-weak equivalence if the induced map $f_*: F^p X \wedge E \to F^p Y \wedge E$ is a local stable equivalence of presheaves of bispectra on D. Say that a map $p: Z \to W$ is an E_*-global fibration if p has the right lifting property with respect to all cofibrations which are E_*-weak equivalences. Then the classes of cofibrations, E_*-weak equivalences and E_*-global fibrations give the category $\mathbf{Spt} \operatorname{Pre}(C)$ the structure of a closed simplicial model category.*

PROOF.   We can assume that the spectrum E is cofibrant, by replacing it up to strict weak equivalence by a cofibrant object. Axioms **sE1** and **sE2** are trivial.

If the map $g: Z \to W$ of presheaves of spectra on C is a pointwise strict weak equivalence, then each level map $g: Z^n \to W^n$ is a local weak equivalence of simplicial presheaves, so that each map $F^p(g): F^p Z^n \to F^p W^n$ is a local weak equivalence of simplicial presheaves on D. But then each of the induced maps $F^p Z^n \wedge E^m \to F^p W^n \wedge E^m$ is a local weak equivalence of simplicial presheaves on D, so that $g_*: F^p Z \wedge E \to F^p W \wedge E$ is a local stable equivalence of presheaves of bispectra on D, giving **sE3**.

Suppose given a pushout diagram

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
i \downarrow & & \downarrow i_* \\
B & \longrightarrow & D
\end{array}
$$

of presheaves of spectra on the site C for which the map i is a cofibration and an E_*-weak equivalence. In order to prove **sE4**, it suffices to show that the map i_* is an E_*-weak equivalence. There is an induced pushout diagram

$$
\begin{array}{ccc}
F^p A & \longrightarrow & F^p C \\
F^p(i) \downarrow & & \downarrow F^p(i_*) \\
F^p B & \longrightarrow & F^p D
\end{array}
$$

of presheaves of spectra on D. The map $F^p(i)$ may not be a cofibration, but it induces a map $L^2 F^p(i): L^2 F^p A \to L^2 F^p B$ which is a level cofibration [14, Ch. 2] in the sense that all maps $L^2 F^p(i): L^2 F^p A^n \to L^2 F^p B^n$ are cofibrations of simplicial (pre-) sheaves, after applying the associated sheaf functor, by exactness of the inverse image functor $F^*: \mathrm{Shv}(C) \to \mathrm{Shv}(D)$. Furthermore, the induced map $E \wedge L^2 F^p A \to E \wedge L^2 F^p B$ is a local stable equivalence of presheaves of bispectra on D, since it is strictly locally weakly equivalent to the map $E \wedge F^p A \to E \wedge F^p B$. The collection of level cofibrations of presheaves of spectra on D which induce a local stable equivalence of presheaves of bispectra after smashing with E is closed under pushout, by a standard long exact sequence argument. It follows that the map i_* induces local stable equivalences $E \wedge L^2 F^p C \to E \wedge L^2 F^p D$ and $E \wedge F^p C \to E \wedge F^p D$.

Axioms **sE5** and **sE6** are easily verified. The proof of the bounded cofibration axiom **sE7** proceeds just as in the proof of Theorem 3.9.   ∎

Note the use of sheaves of spectra in the proof of Theorem 3.10. In general, a sheaf of spectra Y is a presheaf of spectra such that all of the objects Y^n, $n \geq 0$, are simplicial sheaves.

If X is a pointed simplicial presheaf on a site C and K is a pointed simplicial set, there is a natural canonical map

$$
\eta_*: K \wedge L^2 X \to L^2(K \wedge X)
$$

which respects the associated sheaf maps $\eta\colon X \longrightarrow L^2 X$ and $\eta\colon K \wedge X \longrightarrow L^2(K \wedge X)$ in the obvious way. It follows that, given a presheaf of spectra Z, there is an associated sheaf of spectra $L^2 Z$ having bonding maps given by the composites

$$S^1 \wedge L^2 Z^n \xrightarrow{\eta_*} L^2(S^1 \wedge Z^n) \xrightarrow{L^2(\sigma)} L^2 Z^{n+1}.$$

This construction is functorial in Z, and there is a natural map of presheaves of spectra $\eta\colon Z \longrightarrow L^2 Z$ which consists of the associated sheaf map $\eta\colon Z^n \longrightarrow L^2 Z^n$ in all levels. This map η is a strict local weak equivalence.

REMARK 3.11. There is a stable proper closed simplicial model structure for the category $\mathbf{Spt}\,\mathrm{Shv}(\mathcal{C})$, for which the weak equivalences are those maps which are local stable equivalences in the simplicial presheaf category. The cofibrations, however, are different in that they need to be defined within the category of simplicial sheaves. Explicitly, a map $i\colon X \longrightarrow Y$ is a cofibration of sheaves of spectra if the following two conditions hold:
  (1) the map $i\colon X^0 \longrightarrow Y^0$ is a cofibration of simplicial sheaves, and
  (2) for each $n \geq 0$ the map

$$L^2\big((S^1 \wedge Y^n) \cup_{(S^1 \wedge X^n)} X^{n+1}\big) \hookrightarrow Y^{n+1}$$

   is a cofibration of simplicial sheaves.
Observe that every map of sheaves of spectra which is a cofibration of presheaves of spectra is also a cofibration of sheaves of spectra. The stable proper closed simplicial model structure for $\mathbf{Spt}\,\mathrm{Shv}(\mathcal{C})$ follows from the existence of the corresponding structure for the category $\mathbf{Spt}\,\mathrm{Pre}(\mathcal{C})$ of presheaves of spectra on $\mathcal{C}$ (Theorem 3.7), in the same way that Theorem 1.2 follows from Theorem 1.1. Furthermore, the natural map $\eta\colon Z \longrightarrow L^2 Z$ is a strict local equivalence, so the stable closed model structures for presheaves of spectra and sheaves of spectra have equivalent associated homotopy categories.

The level cofibration $L^2 F^p(i)\colon L^2 F^p A \longrightarrow L^2 F^p B$ of presheaves of spectra which appears in the proof of Theorem 3.10 is actually a cofibration of sheaves of spectra, and the proof of that result can be rewritten on this basis.

REMARK 3.12. Theorem 3.10 specializes to a closed model structure for ordinary spectra, where E is any spectrum, and the weak equivalences are those maps which induce stable equivalences after smashing with E.

4. **Localization with respect to a map.** Let $\mathbf{S}$ be the category of simplicial sets and $f\colon A \longrightarrow B$ a cofibration. A great deal of thought has gone into the notion of an f-localization. We refer the reader especially to the work of Dror-Farjoun ([6], [7]), Hirschhorn [10], and Bousfield [4]. All of these contain further references and applications. We begin this section by presenting a synopsis of the theory.

A simplicial set X is f-local if it is fibrant and if the fibration

$$f^*\colon \mathbf{hom}(B, X) \longrightarrow \mathbf{hom}(A, X)$$

is a weak equivalence. An f-localization of $Y \in \mathbf{S}$ is a morphism $Y \to X$ in $\mathbf{S}$ to an f-local object which is initial among all such morphisms; that is, given a map $Y \to Z$ to a f-local object there is a unique homotopy class of maps $X \to Z$ making the following triangle commute in the homotopy category:

$$
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow & \swarrow & \\
Z &
\end{array} \quad .
$$

One of the main results of the work cited above is that localizations exist and induce a functor $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$ which is augmented in the sense that there is a natural map $\eta_X : X \to L_f X$ and idempotent in the sense that the two natural maps

$$L_f \eta_X, \eta_{L_f X} : L_f X \longrightarrow (L_f)^2 X$$

are equal and isomorphisms in $\mathrm{Ho}(\mathbf{S})$.

The functor $L_f : \mathrm{Ho}(\mathbf{S}) \to \mathrm{Ho}(\mathbf{S})$ is induced by a functor $L = L_f : \mathbf{S} \to \mathbf{S}$. To describe L, consider the set $\mathbf{C}$ of cofibrations with elements

$$A \times \Delta^n \cup_{A \times \partial \Delta^n} B \times \partial \Delta^n \longrightarrow B \times \Delta^n, \quad n \geq 0.$$

Since $\mathbf{C}$ is a set, there is an infinite cardinal α so that the source and target of all morphisms in $\mathbf{C}$ are α-bounded.

The space LX is defined by a filtered colimit

$$LX = \varinjlim_{s < \kappa} E_s X,$$

where κ is some fixed cardinal with $\kappa > 2^\alpha$, and $E_s X$ is defined by transfinite induction. $E_0 X$ is a functorial choice of fibrant model for X—it is most convenient here to write $E_0 X = \mathrm{Ex}^\infty X$. If s is a limit ordinal $E_s = \varinjlim_{t < s} E_t$, and for successor ordinals $s + 1$, there is a push-out square

$$
\begin{array}{ccc}
\bigsqcup_{\mathbf{C}} C_i \times \mathbf{hom}(C_i, E_s X) & \longrightarrow & E_s X \\
\downarrow & & \downarrow \\
\bigsqcup_{\mathbf{C}} D_i \times \mathbf{hom}(C_i, E_s X) & \longrightarrow & \overline{E}_{s+1} X,
\end{array}
$$

and then $E_{s+1} X = \mathrm{Ex}^\infty \overline{E}_{s+1} X$ is a fibrant model for $\overline{E}_{s+1} X$. We shall say that κ is the *defining cardinal* for the functor L.

The slightly expanded definition of the stages in the factorization (using function spaces rather than merely sets of maps) is required for the resulting object to be continuous in the sense of **L7** below.

The canonical map $\eta_X : X \to LX$ is a cofibration and induces a weak equivalence

$$\eta_X^* : \mathbf{hom}(LX, Z) \longrightarrow \mathbf{hom}(X, Z)$$

for all f-local objects Z, by construction. Furthermore, LX is f-local: it is clearly fibrant and any map $C_i \rightarrow LX$ factors through some E_sX, for otherwise C_i would have too many subobjects.

The functor L also satisfies the following properties.

L1: L preserves weak equivalences.

L2: L preserves cofibrations.

L3: Let β be any cardinal with $\beta \geq \alpha$. Let $\{X_j\}$ be the filtered system of sub-objects of X which are β-bounded. Then the map

$$\varinjlim_j L(X_j) \longrightarrow L(X)$$

is an isomorphism.

L4: Let γ be an ordinal number of cardinality strictly greater than 2^α. Let $X\colon \gamma \rightarrow \mathbf{S}$ be a diagram of cofibrations so that for all limit ordinals $s < \gamma$ the induced map

$$\varinjlim_{t<s} X(t) \longrightarrow X(s)$$

is an isomorphism. Then $\varinjlim_{t<\gamma} L\big(X(t)\big) \cong L\big(\varinjlim_{t<\gamma} X(t)\big)$.

L5: Suppose that $\lambda = 2^\kappa$, where κ is the defining cardinal for the functor L. If X is λ-bounded, then $L(X)$ is λ-bounded.

L6: Let Y, Z be two subobjects of X. Then

$$L(Y) \cap L(Z) = L(Y \cap Z)$$

in $L(X)$.

L7: The functor L is continuous; that is, it extends to a natural morphism of simplicial sets

$$L\colon \mathbf{hom}(X, Y) \longrightarrow \mathbf{hom}(LX, LY)$$

compatible with composition.

The idea of proof for **L1** is to show inductively that any weak equivalence $g\colon X \rightarrow Y$ induces weak equivalences $g_*\colon E_sX \rightarrow E_sY$. This is a patching lemma argument, given that g induces a weak equivalence

$$g_*\colon \mathbf{hom}(C_i, E_sX) \longrightarrow \mathbf{hom}(C_i, E_sY)$$

since E_sX and E_sY are fibrant.

For **L2**, one shows inductively that if $X \rightarrow Y$ is a cofibration, then $E_sX \rightarrow E_sY$ is a cofibration and, for successor ordinals,

$$E_s(Y) \cup_{E_s(X)} \overline{E}_{s+1}(X) \longrightarrow \overline{E}_{s+1}(Y)$$

is a cofibration. This last ultimately relies on the fact that for all $C_i \rightarrow D_i$ in $\mathbf{C}$, the map

$$C_i \times \mathbf{hom}(C_i, E_sY) \cup_{C_i \times \mathbf{hom}(C_i, E_sX)} D_i \times \mathbf{hom}(C_i, E_sX) \longrightarrow D_i \times \mathbf{hom}(C_i, E_sY)$$

is an inclusion, hence a cofibration. In particular, one needs that

$$\mathbf{hom}(C_i, E_s X) \longrightarrow \mathbf{hom}(C_i, E_s Y)$$

is an inclusion. Note further that Kan's Ex^∞ construction preserves cofibrations.

The statement **L3** is again verified by proving the corresponding statement for each of the functors E_s. Suppose that $g: C_i \times \Delta^n \to E_s X$ is a map, and take a simplex (x, θ) of $C_i \times \Delta^n$. Then $g(x, \theta)$ lies in some $E_s Y$ such that Y is a β-bounded subcomplex of X by the inductive assumption. But then all images $f(x, \theta)$ lie in some $E_s W$ where W is a β-bounded subcomplex of X, since $|C_i \times \Delta^n| \leq \alpha$, and the collection of all β-bounded subcomplexes of X is closed under unions of size at most β. We have therefore seen that

$$\varinjlim_j \mathbf{hom}(C_i, E_s X_j) \longrightarrow \mathbf{hom}(C_i, E_s X)$$

is an isomorphism for all $C_i \to D_i$ in $\mathbf{C}$. The rest of the verification is formal, since the Ex^∞ functor preserves filtered colimits.

The statement **L4** has a similar but easier argument: there is an isomorphism

$$\varinjlim_{t < \gamma} \mathbf{hom}\big(C_i, E_s X(t)\big) \cong \mathbf{hom}\big(C_i, \varinjlim_{t < \gamma} E_s X(t)\big),$$

since otherwise C_i would have too many subobjects.

For **L5**, one shows that $E_s(X)$ is λ-bounded, and uses the observation that

$$\lambda = 2^\kappa > \kappa > 2^\alpha > \alpha.$$

The fact that $E_{s+1}(X)$ is λ-bounded ultimately relies on the fact that

$$\big|D_i \times \mathbf{hom}(C_i, E_s X)\big| \leq \alpha \cdot \lambda^\alpha = \alpha \cdot (2^\kappa)^\alpha = \alpha \cdot 2^{\kappa \cdot \alpha} = \lambda.$$

Note that if Y is λ-bounded, then $\mathrm{Ex}^\infty Y$ is λ-bounded, since λ is infinite.

Note that by **L2** we may assume we have $L(Y) \cap L(Z) \subseteq L(Y \cap Z) \subseteq L(X)$. One shows that $E_s(Y) \cap E_s(Z) = E_s(Y \cap Z)$. The limit ordinal case follows from the successor ordinal case because filtered colimits commute with pullbacks. The successor ordinal case follows from fact that, degreewise, $E_{s+1} X_n$ has the form

$$E_{s+1} X_n = \Big(\bigsqcup_C (D_i - C_i) \times \mathbf{hom}(C_i, E_s X)_n\Big) \sqcup E_s X_n,$$

and the image of the inclusion $E_s Y_n \hookrightarrow E_s X_n$ associated to any subcomplex $Y \subset X$ has the form

$$\Big(\bigsqcup_C (D_i - C_i) \times \mathbf{hom}(C_i, E_s Y)_n\Big) \sqcup E_s Y_n.$$

The Ex^∞ functor preserves pullbacks, giving **L6**.

The statement **L7** is equivalent to asserting that for all $X, K \in \mathbf{S}$, there is a natural map $L(X) \times K \to L(X \times K)$ so that the following triangle commutes

$$X \times K \xrightarrow{\eta_X \times 1} L(X) \times K$$
$$\eta_X \times 1 \searrow \quad \downarrow$$
$$L(X \times K)$$

subject to the requirements that $L(X) \times \Delta^0 \cong L(X \times \Delta^0)$, and the following two maps agree:

$$L(X) \times (K \times L) \to L\big(X \times (K \times L)\big) \cong L\big((X \times K) \times L\big)$$

and

$$L(X) \times (K \times L) \to \big(L(X) \times K\big) \times L \to L(X \times K) \times L \to L\big((X \times K) \times L\big).$$

Again, one shows the result holds for all E_s. The limit ordinal case follows from the successor ordinal case, which in turn follows by induction and the fact that, for sets, the push-out of

$$B \times K \xrightarrow{i \times K} A \times K \xrightarrow{j \times K} C \times K$$

is isomorphic to $(B \cup_A C) \times K$. Note as well that the Ex^∞ functor is continuous in the same sense.

Suppose now that $f : A \to B$ is a cofibration of simplicial presheaves on a small Grothendieck site $\mathcal{C}$, and say that a simplicial presheaf Z on $\mathcal{C}$ is f-*local* if Z is globally fibrant and if the map $Z \to *$ has the right lifting property with respect to all simplicial presheaf cofibrations

$$B \times Y \cup_{A \times Y} A \times L_U \Delta^n \xrightarrow{(f, j)_*} B \times L_U \Delta^n$$

arising jointly from $f : A \to B$ and inclusions $j : Y \subset L_U \Delta^n$.

REMARK 4.1. The presheaf theoretic definition of f-local object seems to differ from the corresponding definition for simplicial sets, but it doesn't. The ordinary function complex $\mathbf{hom}(X, Y)$ for simplicial sets X and Y is the internal complex for the simplicial set category, and a completely analogous definition of f-local object for simplicial presheaves would require the internal function complex for that setting. We have chosen not to involve that concept in this discussion; the equivalent adjoint formulation of f-local object is easier to write down.

If Z is f-local, then it has the right lifting property with respect of all inclusions

$$(4.2) \qquad\qquad B \times C \cup_{A \times C} A \times D \xrightarrow{(f, j)_*} B \times D.$$

The class of all maps j such that $Z \to *$ has the right lifting property with respect to $(f, j)_*$ is saturated, and contains the set of maps $Y \subset L_U \Delta^n$. It follows in particular that the induced map

$$f^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$$

is a trivial Kan fibration if Z is an f-local object. More generally, all maps

$$(f \times 1_X)^* : \mathbf{hom}(B \times X, Z) \longrightarrow \mathbf{hom}(A \times X, Z)$$

are trivial fibrations if Z is f-local, and so the canonical map

$$\mathbf{hom}(B \times D, Z) \xrightarrow{(f^*, j^*)} \mathbf{hom}(A \times D, Z) \times_{\mathbf{hom}(A \times C, Z)} \mathbf{hom}(B \times C, Z)$$

is a trivial Kan fibration if Z is f-local.

We construct a functor $L = L_f : \mathbf{S}\operatorname{Pre}(\mathcal{C}) \longrightarrow \mathbf{S}\operatorname{Pre}(\mathcal{C})$ by starting with an infinite cardinal α which is an upper bound for the cardinality of the set of morphisms of $\mathcal{C}$ and that of all sets of sections of B. The method of constructing the functor L is analogous to the simplicial set construction, starting with the set $\mathbf{C}$ of cofibrations having elements

$$A \times L_U \Delta^n \cup_{A \times Y} B \times Y \hookrightarrow B \times L_U \Delta^n,$$

indexed over the set of all cofibrations $Y \subset L_U \Delta^n$.

The construction of L for simplicial sets depends on the existence of a continuous functorial fibrant model construction. For simplicial presheaves, we require a continuous functorial globally fibrant model $j_X : X \longrightarrow GX$ in order to carry out the analogous argument. The Ex^∞ functor does not produce globally fibrant models for simplicial presheaves, so we have to do something more interesting:

LEMMA 4.3. *Suppose that the infinite cardinal α is an upper bound for the cardinality of the set of morphisms of a small Grothendieck site $\mathcal{C}$. Then there is a functorial natural map $j_X : X \longrightarrow GX$ such that the map j_X is a trivial cofibration, GX is globally fibrant, and the following properties hold:*

G1*: G preserves weak equivalences.*

G2*: G preserves cofibrations.*

G3*: Let β be any cardinal with $\beta \geq \alpha$. Let $\{X_j\}$ be the filtered system of sub-objects of X which are β-bounded. Then the map*

$$\varinjlim_j GX_j \longrightarrow GX$$

is an isomorphism.

G4*: Let γ be an ordinal number of cardinality strictly greater than 2^α. Let $X : \gamma \longrightarrow \mathbf{S}\operatorname{Pre}(\mathcal{C})$ be a diagram of cofibrations so that for all limit ordinals $s < \gamma$ the induced map*

$$\varinjlim_{t < s} X(t) \longrightarrow X(s)$$

is an isomorphism. Then $\varinjlim_{t < \gamma} GX(t) \cong G\left(\varinjlim_{t < \gamma} X(t)\right)$.

G5*: Suppose that $\lambda = 2^\kappa$, where κ is the defining cardinal for the functor G. If X is λ-bounded, then GX is λ-bounded.*

G6*: Let Y and Z be two subobjects of X. Then*

$$G(Y) \cap G(Z) = G(Y \cap Z)$$

in $G(X)$.

G7*: The functor G extends to a natural morphism of simplicial sets*

$$G: \mathbf{hom}(X, Y) \longrightarrow \mathbf{hom}(GX, GY)$$

compatible with composition.

PROOF. As before, the defining cardinal κ for the functor G is a cardinal $\kappa > 2^\alpha$. We begin by writing $C_i \to D_i$ for the set of trivial cofibrations $Y \subset L_U \Delta^n$, where U varies through the set of objects of $\mathcal{C}$. Note that there are at most 2^α trivial cofibrations $Y \subset L_U \Delta^n$, and that each such Y is α-bounded. Then $GX = \varinjlim_{s < \kappa} G_s X$ where $G_s X = \varinjlim_{t < s} G_t X$ at limit ordinals $t < \kappa$, and $G_{s+1} X$ is constructed from $G_s X$ by requiring that the following diagram is a pushout

$$
\begin{array}{ccc}
\bigsqcup_i C_i \times \mathbf{hom}(C_i, G_s X) & \longrightarrow & G_s X \\
\downarrow & & \downarrow \\
\bigsqcup_i D_i \times \mathbf{hom}(C_i, G_s X) & \longrightarrow & G_{s+1} X
\end{array}
$$

Each of the maps

$$\bigsqcup_i C_i \times \mathbf{hom}(C_i, G_s X) \to \bigsqcup_i D_i \times \mathbf{hom}(C_i, G_s X)$$

is a trivial cofibration of simplicial presheaves, so that the canonical map $j_X: X \to GX$ is a trivial cofibration. GX is globally fibrant, since any map $C_i \to GX$ factors through some $G_s X$ with $s < \kappa$. In particular, **G1** is a triviality. The statements **G2**–**G7** are proved by analogy with **L2**–**L7** above. ∎

To construct LX for a simplicial presheaf X, take κ to be a cardinal greater than 2^α, and let $E_0 X = GX$. Define $E_s X = G(\varinjlim_{t < s} E_t X)$ at limit ordinals $s < \kappa$, where G is the globally fibrant model construction of Lemma 4.3. At successor ordinals, $E_{s+1} X = G(\overline{E}_{s+1} X)$, where $\overline{E}_{s+1} X$ is defined by the pushout

$$
\begin{array}{ccc}
\bigsqcup_{\mathbf{C}} C_i \times \mathbf{hom}(C_i, E_s X) & \longrightarrow & E_s X \\
\downarrow & & \downarrow \\
\bigsqcup_{\mathbf{C}} D_i \times \mathbf{hom}(C_i, E_s X) & \longrightarrow & \overline{E}_{s+1} X.
\end{array}
$$

The notation $C_i \to D_i$ refers to all simplicial presheaf morphisms

$$A \times L_U \Delta^n \cup_{A \times Y} B \times Y \hookrightarrow B \times L_U \Delta^n.$$

Finally,

$$LX = \varinjlim_{s < \kappa} E_s X.$$

Note that LX is globally fibrant: any map $Y \to LX$ must factor through some $E_s X$ since $\kappa > 2^\alpha$, and all $E_s X$ are globally fibrant by construction. A similar argument shows that LX is f-local. In the presence of Lemma 4.3, arguments for the simplicial presheaf analogues of statements **L1**–**L7** go through just as before. In particular, we have proved

THEOREM 4.4. *Let $f: A \to B$ be a cofibration in $\mathbf{S}\operatorname{Pre}(\mathcal{C})$, and suppose that α is an infinite cardinal which is an upper bound for the cardinalities of both B and the set of morphisms of $\mathcal{C}$. Then there is a functor $L = L_f: \mathbf{S}\operatorname{Pre}(\mathcal{C}) \to \mathbf{S}\operatorname{Pre}(\mathcal{C})$ and a natural map $\eta_X: X \to L(X)$ so that $L(X)$ is f-local and η_X is a cofibration which induces weak equivalences*

$$\eta_X^*: \mathbf{hom}\big(L(X), Z\big) \to \mathbf{hom}(X, Z)$$

for all f-local simplicial presheaves Z.

As above, this yields a localization on the homotopy category.

We now use Theorem 1.1 to produce the f-local category structure on $\mathbf{S}\operatorname{Pre}(\mathcal{C})$. The following result allows us to identify the class $\mathbf{E}$ of f-local equivalences.

LEMMA 4.5. *Let $g: X \to Y$ be a morphism of simplicial pre-sheaves in $\mathbf{S}\operatorname{Pre}(\mathcal{C})$. The following statements are equivalent.*

(1) $Lg: LX \to LY$ is a weak equivalence
(2) $g^*: \mathbf{hom}(Y, Z) \to \mathbf{hom}(X, Z)$ is a weak equivalence for all f-local objects Z in $\mathbf{S}\operatorname{Pre}(\mathcal{C})$.
(3) $[Y, Z] \to [X, Z]$ is an isomorphism for all f-local Z in $\mathbf{S}\operatorname{Pre}(\mathcal{C})$.

PROOF. Examine the following diagram:

$$
\begin{array}{ccc}
\mathbf{hom}(LY, Z) & \xrightarrow{\;\eta^*\;} & \mathbf{hom}(Y, Z) \\
{\scriptstyle Lg^*}\big\downarrow & & \big\downarrow{\scriptstyle g^*} \\
\mathbf{hom}(LX, Z) & \xrightarrow[\;\eta^*\;]{} & \mathbf{hom}(X, Z).
\end{array}
$$

Since η_X induces a trivial Kan fibration $\eta_X^*: \mathbf{hom}(LX, Z) \to \mathbf{hom}(X, Z)$ for all X and all f-local Z, the horizontal maps labelled η^* are weak equivalences. Thus if $Lg: LX \to LY$ is a weak equivalence, so is g^*. Thus (1) implies (2). We have that (2) implies (3) because any f-local object is globally fibrant. For (3) implies (1) note that for all X and all f-local Z,

$$[X, Z] \cong [LX, Z]$$

since L induces the localization functor L_f on the homotopy category. Thus (3) says $[LY, Z] \to [LX, Z]$ is an isomorphism for all f-local Z. Since LX and LY are f-local, this implies that $Lg: LX \to LY$ is an isomorphism in the homotopy category; Lg is therefore a weak equivalence. ∎

We now define **E**, the class of f-local equivalences, by the three equivalent conditions of Lemma 4.3. We next want to show **E** satisfies the seven axioms required by Theorem 1.1. Axioms **E1**–**E3** are obvious, while Lemmas 4.6 and 4.7 handle axioms **E4**–**E6** and **E7**, respectively.

LEMMA 4.6.   *The class of f-local trivial cofibrations is closed under cobase change, colimits over ordinal numbers, and coproducts.*

PROOF.   Use (2) of Lemma 4.5, and the fact that trivial fibrations in simplicial sets are closed under base change, limits over ordinal numbers, and products.   ∎

LEMMA 4.7.   *Let $\lambda = 2^\kappa$, where κ is the defining cardinal for the functor L (and $\kappa > 2^\alpha$). Then the class of f-local trivial cofibrations satisfies the bounded cofibration condition for the cardinal λ.*

PROOF.   Let $X \longrightarrow Y$ be an f-local equivalence and a cofibration, and let $A \subseteq Y$ be a λ-bounded sub-object. We inductively define a chain of λ-bounded sub-objects $A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq Y$ over λ, and a chain of sub-objects

$$L(A) = L(A_0) \subseteq X_1 \subseteq L(A_1) \subseteq X_2 \subseteq L(A_2) \subseteq \cdots L(Y),$$

also over λ, with the property that

$$L(X) \cap X_s \longrightarrow X_s$$

is a weak equivalence. Then we set $B = \varinjlim_{s < \kappa} A_s$ and, by **L6**,

$$L(X \cap B) = L(X) \cap L(B) = \varinjlim_{s < \kappa} L(X) \cap X_s$$
$$\longrightarrow \varinjlim_{s < \kappa} X_s \cong L(B)$$

is a weak equivalence as required.

The A_s and X_s are defined recursively. Suppose $s + 1$ is a successor ordinal and A_s has been defined. Then, since A_s is λ-bounded, LA_s is λ-bounded by **L5**. Hence there is a λ-bounded sub-object $X_{s+1} \subseteq L(Y)$ so that $L(A_s) \subseteq X_{s+1}$ and $L(X) \cap X_{s+1} \longrightarrow X_{s+1}$ is a weak equivalence (see the proof of Theorem 2.7 with f the identity functor, or argue directly). Since $L(Y) = \varinjlim_j L(Y_j)$ where $Y_j \subseteq Y$ runs over the λ-bounded sub-objects of Y, there is a λ-bounded sub-object A'_{s+1} so that $X_{s+1} \subseteq L(A'_{s+1})$. Let $A_{s+1} = A_s \cup A'_{s+1}$. Finally, suppose s is a limit ordinal. Then set $X_s = \varinjlim_{t < s} L(A_t) \cong \varinjlim_{t < s} X_t$. The object X_s is λ-bounded and $L(X) \cap X_s \longrightarrow X_s$ is a weak equivalence. Choose $A'_s \subseteq Y$ so that A'_s is λ-bounded and $X_s \subseteq L(A'_s)$ and set $A_s = \varinjlim_{t < s} A_t \cup A'_s$.   ∎

THEOREM 4.8.   *The category $\mathbf{S}\,\mathrm{Pre}(\mathcal{C})$ acquires the structure of a simplicial model category with ordinary simplicial presheaf cofibrations and f-local equivalences.*

PROOF.   First of all, if $Z \in \mathbf{S}\,\mathrm{Pre}(\mathcal{C})$ is f-local and $K \in \mathbf{S}$, then X^K is f-local. To see this, recall that $Z \longrightarrow *$ has the right lifting property with respect to all cofibrations $(f, j)^*$ as in (4.2); we can in particular assume that j has the form $Y \times K \hookrightarrow L_U\Delta^n \times K$, and then use adjointness.

Let $K \longrightarrow L$ be a cofibration in $\mathbf{S}$ and $C \longrightarrow D$ a cofibration in $\mathbf{S}\,\mathrm{Pre}(\mathcal{C})$. Consider the cofibration in $\mathbf{S}\,\mathrm{Pre}(\mathcal{C})$

$$(4.9) \qquad D \times K \cup_{C \times K} C \times L \longrightarrow D \times L.$$

Let Z be f-local and apply $\mathbf{hom}(\cdot, Z)$. Then there is an augmented pull-back square in $\mathbf{S}$

$$
\begin{array}{ccccc}
\mathbf{hom}(D \times L, Z) & \longrightarrow & \mathbf{hom}(D \times K \cup_{C \times K} C \times L, Z) & \longrightarrow & \mathbf{hom}(C \times L, Z) \\
 & & \downarrow & & \downarrow \\
 & & \mathbf{hom}(D \times K, Z) & \longrightarrow & \mathbf{hom}(C \times K, Z).
\end{array}
$$

Now, there is a natural isomorphism of this diagram to

$$
\begin{array}{ccccc}
\mathbf{hom}(D, Z^L) & \longrightarrow & \mathbf{hom}(D, Z^K) \times_{\mathbf{hom}(C, Z^K)} \mathbf{hom}(C, Z^L) & \overset{q}{\longrightarrow} & \mathbf{hom}(C, Z^L) \\
 & & {\scriptstyle i}\downarrow & & \downarrow \\
 & & \mathbf{hom}(D, Z^K) & \longrightarrow & \mathbf{hom}(C, Z^K).
\end{array}
$$

Now suppose $C \longrightarrow D$ is an f-local equivalence. Then by Lemma 4.3(2),

$$\mathbf{hom}(D, Z^K) \longrightarrow \mathbf{hom}(C, Z^K)$$

is trivial fibration, and so q is a trivial fibration. Since

$$\mathbf{hom}(D, Z^L) \longrightarrow \mathbf{hom}(C, Z^L)$$

is a weak equivalence, (4.9) is an f-local equivalence. Similarly, if $K \longrightarrow L$ is a weak equivalence, $\mathbf{hom}(C, Z^L) \longrightarrow \mathbf{hom}(C, Z^K)$ is a weak-equivalence, since Z is fibrant. Then i is a weak equivalence and the result follows. ∎

One final remark. While the fibrations in the f-local model category structure of $\mathbf{S}\,\mathrm{Pre}(\mathcal{C})$ are cloaked in mystery, we do have the following congruence.

PROPOSITION 4.10. *An object $X \in \mathbf{S}\,\mathrm{Pre}(\mathcal{C})$ is fibrant in the f-local model category structure if and only if it is f-local.*

PROOF.   First suppose that X is fibrant. We know that the map

$$(f, j)_* \colon B \times Y \cup_{A \times Y} A \times L_U\Delta^n \hookrightarrow B \times L_U\Delta^n$$

is a cofibration and an f-local equivalence, so that the induced map

$$(f, j)^* \colon \mathbf{hom}(B \times L_U\Delta^n, X) \longrightarrow \mathbf{hom}(B \times Y \cup_{A \times Y} A \times L_U\Delta^n, X)$$

is a trivial Kan fibration. But then $X \longrightarrow *$ has the right lifting property with respect to all maps $(f,j)_*$, and so X is f-local.

Now suppose X is f-local. Consider a lifting problem

$$
\begin{array}{ccc}
C & \longrightarrow & X \\
\downarrow & \nearrow & \\
D & &
\end{array}
$$

where $C \to D$ is a f-local trivial cofibration. Since X is f-local there is a map $L(C) \to X$ making the following diagram commute

$$
\begin{array}{ccc}
C & \longrightarrow & X \\
\downarrow & \nearrow & \\
L(C) & &
\end{array} \quad .
$$

Since $L(C) \to L(D)$ is a trivial cofibration, there is a map $L(D) \to X$ making

$$
\begin{array}{ccccc}
C & \longrightarrow & L(C) & \longrightarrow & X \\
\downarrow & & \downarrow & \nearrow & \\
D & \longrightarrow & L(D) & &
\end{array}
$$

commute. This solves the original lifting problem. ∎

REMARK 4.11. Suppose that S is a Noetherian scheme of finite dimension, and let $(Sm/S)_{\mathrm{Nis}}$ denote the site of smooth schemes over S, equipped with the Nisnevich topology. The affine line $\mathbb{A}^1$ over S represents a sheaf of the same name on $(Sm/S)_{\mathrm{Nis}}$. The Morel-Voevodsky $\mathbb{A}^1$-local theory [17], [18] is the closed model structure on the category of simplicial presheaves on $(Sm/S)_{\mathrm{Nis}}$ which arises from Theorem 4.8 by localizing at a cofibration associated to the constant simplicial presheaf map $\mathbb{A}^1 \longrightarrow *$. Note that the same theory arises from localizing at any rational point $* \longrightarrow \mathbb{A}^1$ of the affine line.

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