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BUTLER MODULES OVER VALUATION DOMAINS

L. FUCHS AND E. MONARI-MARTINEZ

Let R be a commutative domain with 1, Q its field of quotients, and M a torsion-free R-module. By a balanced submodule of M is meant an RD-submodule N [i.e. $rN = N \cap rM$ for each $r \in R$] such that, for every R-submodule J of Q, every homomorphism $\eta: J \longrightarrow M/N$ can be lifted to a homomorphism $\chi: J \longrightarrow M$. This definition extends the notion of balancedness as introduced in abelian groups (see e.g. [10, p. 113]). The balanced-projective R-modules can be characterized as summands of completely decomposable R-modules (i.e. summands of direct sums of submodules of Q). If R is a valuation domain, then such summands are again completely decomposable; see [12, p. 275]. Recall that a torsion-free abelian group B of finite rank is called a Butler group [2] if it has one of the following equivalent properties:

(i) B is a pure (=RD-) subgroup of a completely decomposable group;

(ii) B is an epic image of a completely decomposable group of finite rank;

(iii) $\mathrm{Bext}^1(B, T) = 0$ holds for every torsion group T; here $\mathrm{Bext}^1(B, T)$ stands for the subgroup of $\mathrm{Ext}^1(B, T)$ consisting of the balanced-extensions of T by B.

Several definitions, not all equivalent, have been proposed for Butler groups of infinite rank. The countable case was studied by Bican-Salce [4]. Several interesting and deep results have been established for uncountable Butler groups (cf. Albrecht-Hill [1], Dugas [5], Dugas-Rangaswamy [7], Dugas-Hill-Rangaswamy [6]), but in view of the enormous difficulties, their study is far from being satisfactory.

In this paper, we investigate the analogues of Butler groups, the Butler modules, of finite and infinite ranks over valuation domains, with two major goals in mind: firstly, to learn more about balancedness, and secondly, to classify Butler modules as far as possible. Our task turns out to be more challenging than it looks at the first sight, since hardly any method developed in the study of Butler groups carries over to the valuation domain case without drastic change. An explanation for this might be sought for in the difference between the prime spectra of the underlying rings. Some of the techniques used here were introduced in the study of Baer and Whitehead modules; see [8], [9], and [3].

Our definition of Butler modules over valuation domains is based on the analogue of (iii). After settling the finite rank case, we concentrate on submodules of finite and then

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of countable rank in Butler modules. We prove that they ought to be completely decomposable [see (1.3) and (2.2)], the converse being an easy consequence of the definition. This is a relatively easy task compared to the case of modules of rank $\aleph_1$ where more sophisticated arguments are required to reach the same conclusion [see (3.7)]. We have not succeeded in obtaining a satisfactory classification for Butler modules whose rank exceeds $\aleph_1$—this did not come to us as a surprise being aware of the difficulties in the abelian group case. Fortunately, at least those Butler modules could be classified whose projective dimension is at most 1: they too are necessarily completely decomposable.

Throughout R will denote a valuation domain (i.e. a commutative domain in which the ideals form a chain under inclusion), and Q its field of quotients. We assume $Q \neq R$.

An exact sequence $0 \longrightarrow A \longrightarrow B \xrightarrow{\beta} C \longrightarrow 0$ of R-modules, C torsion-free, is called *balanced* if for any submodule J of Q, and any homomorphism $\eta: J \longrightarrow C$ there is a map $\chi: J \longrightarrow B$ such that $\beta\chi = \eta$. The extensions B of A by C which are represented by balanced-exact sequences form a submodule $\mathrm{Bext}^1_R(C,A)$ of $\mathrm{Ext}^1_R(C,A)$.

By the rank $\mathrm{rk}C$ of a torsion-free R-module C is meant $\dim Q \otimes_R C$ as a Q-vectorspace. The rank 1 modules are copies of submodules of Q; direct sums of rank 1 modules are called completely decomposable. The completely decomposable R-modules have the projective property relative to balanced-exact sequences.

For unexplained terminology and notation we refer to [12].

1. **Butler modules of finite rank.**

By a *uniserial* module is meant a module in which the submodules are totally ordered by inclusion. A *weakly polyserial* module is defined as a module M which has a finite chain $0 = M_0 < M_1 < \cdots < M_n = M$ of submodules such that all the factors $M_i/M_{i-1} (i = 1, \ldots, n)$ are uniserial. In [13] it has been shown that every weakly polyserial module [has finite Malcev rank, and therefore it] contains a finitely generated essential submodule. For a weakly polyserial module we now prove:

LEMMA 1.1. *Let $W = \bigoplus_{i \in I} R/Rr_i$ $(0 \neq r_i \in R)$ be a pure-projective torsion module and M a weakly polyserial submodule of W. Then $r_iM = 0$ for some $i \in I$.*

PROOF. Let H be a finitely generated essential submodule of M, and F a finite subset of I such that $H \subset V = \bigoplus_{i \in F} R/Rr_i$. The projection π of W onto V fixes H elementwise, so π is monic on the essential extension M of H. Hence $r_i \pi M = 0$ for some $i \in F$ implies $r_iM = 0$ for this i.

The following lemma provides us with an important tool for constructing balanced-exact sequences; it utilizes a crucial difference between abelian groups and modules over valuation domains.

LEMMA 1.2. *Let $\psi: N \longrightarrow C$ be a map from a torsion-free R-module N into an R-module C such that image $\psi(J)$ of any rank 1 (pure) submodule J of N is properly contained in a uniserial submodule U of C where the elements of U have principal ideal*

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annihilators. If the bottom row is a pure-exact sequence, then the top row obtained as a pullback is balanced-exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & M & \xrightarrow{\eta} & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \end{array}$$

PROOF. Let $\phi : J \longrightarrow N$ be a homomorphism, J torsion-free of rank 1. Ignoring the trivial case, we may assume J is a submodule of N and ϕ is the inclusion map. By hypothesis, there is a $c \in C$ such that $\psi\phi J \leq Rc$. Pure-exactness guarantees the existence of a $b \in B$ with $\operatorname{Ann} b = \operatorname{Ann} c$ and $\beta b = c$. Let χ be the isomorphism $Rc \longrightarrow Rb$ mapping c onto b, and set $\gamma = \chi\psi\phi : J \longrightarrow B$. As $\beta\gamma = \beta\chi\psi\phi = \psi\phi$, the pullback property of M implies the existence of a map $\delta : J \longrightarrow M$ satisfying $\eta\delta = \phi$. Thus the top row is balanced-exact.

We are now ready to concentrate on our first result. By a *Butler module* we mean a torsion-free R-module B such that

$$\operatorname{Bext}^1_R(B, T) = 0 \text{ for all torsion modules } T.$$

Evidently, all completely decomposable modules are Butler modules.

THEOREM 1.3. *Finite rank Butler modules over valuation domains are completely decomposable.*

PROOF. It suffices to show that the existence of an indecomposable Butler module B of rank $n > 1$ leads to a contradiction.

Let N be a pure submodule of rank $n - 1$ (≥ 1) in B, and set $B/N \cong J \leq Q$. In view of our hypothesis, N cannot be a summand of B, so J is not finitely generated, and moreover, every rank 1 submodule of B maps onto a proper submodule of B/N under the canonical map. These properties are preserved under passage from J to J/R (i.e. when a cyclic submodule is factored out).

Starting from a pure-projective resolution of J/R (bottom row), and using the composite map $B \longrightarrow B/N \cong J \longrightarrow J/R$ (which will be denoted by ψ), we form the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & T & \longrightarrow & W & \xrightarrow{\alpha} & J/R & \longrightarrow & 0 \end{array}$$

In view of (1.2), the top row is balanced-exact, and therefore it splits. This means there is a $\gamma : B \longrightarrow W$ such that $\alpha\gamma = \psi$. As epimorphic images of finite rank torsion-free modules are weakly polyserial, (1.1) implies that γB is annihilated by some $r \in R$ such that R/Rr is one of the summands of W; in other words, $r^{-1} \in J$. But then this r would annihilate the module $\alpha\gamma B = \psi B = J/R$; this is impossible unless $J = r^{-1}R$, i.e. J is finitely generated, a contradiction.

The last theorem shows that over arbitrary valuation domains Butler modules of finite rank can be defined in three equivalent ways, just as for abelian groups mentioned in the introduction.

2. **The countable rank case.**

Our next purpose is to characterize Butler modules of countable rank. We accomplish somewhat more in (2.2); this plus will be needed later on.

We start off with an easy lemma. For a module M, $\widehat{M}$ will denote its pure-injective hull.

LEMMA 2.1. *Let J be an R-module such that $R < J \leq Q$. The set of annihilator ideals of elements of $(J/R)^\wedge$ is the same as the set of annihilator ideals of elements of J/R.*

PROOF. Note that $(J/R)^\wedge \cong JS/S$ where S denotes a maximal immediate extension of R; see [12, p. 229]. An element of JS/S is of the form $r^{-1}\varepsilon+S$ with $r \in R$, $r^{-1} \in J$ and ε a unit in S, thus $\mathrm{Ann}_R(r^{-1}\varepsilon+S) = Rr$. But this is exactly the annihilator of $r^{-1}+R \in J/R$.

The following result generalizes our earlier (1.3).

THEOREM 2.2. *Countable rank pure submodules of Butler modules are completely decomposable. In particular, countable rank Butler modules are completely decomposable.*

PROOF. Let B be a Butler module, and by way of contradiction, suppose it contains an indecomposable pure submodule N of finite rank $n > 1$. We can argue as in the proof of (1.3) to conclude that there is an epimorphism $\psi: N \longrightarrow J/R$ with an infinitely generated J ($R < J \leq Q$) such that no rank 1 submodule of N is mapped onto J/R by ψ. Since N is pure in B, the homomorphism $\nu\psi: N \longrightarrow (J/R)^\wedge$ [with the canonical map $\nu: J/R \longrightarrow (J/R)^\wedge$] extends to a homomorphism $\psi_0: B \longrightarrow (J/R)^\wedge$.

In order to assure that the extension ψ_0 has the same property as the one mentioned above for ψ, we have to choose ψ_0 more carefully. Select a maximal family $\{C_i\}$ of rank 1 pure submodules of B such that the set $\{N, C_i (i \in I)\}$ is purely independent (see [12, p. 202]). Then $B^* = N \oplus (\oplus C_i)$ is a pure-essential submodule of B, thus no rank 1 submodule of B maps upon a non-zero pure submodule of B/B^*. If ψ_0 is chosen so as to act trivially on the C_i but agree with ψ on N, then ψ_0 will have the property needed to apply (1.2). (Another proof can be given by using $*$-maps and applying (4.2).) From

(1.2) we conclude that the top row in the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & \overset{\gamma}{\swarrow} & \downarrow{\scriptstyle \psi_0} & & \\ 0 & \longrightarrow & T & \longrightarrow & W & \xrightarrow{\alpha} & (J/R)^{\wedge} & \longrightarrow & 0 \end{array}$$

is balanced-exact where the bottom row is chosen to be a pure-projective resolution of $(J/R)^{\wedge}$. As B is a Butler module, the top row splits. Thus there exists a map $\gamma: B \longrightarrow W$ making the lower triangle commute. By (1.1), there is an $r \in R$ such that R/Rr is a summand of W and $r\gamma N = 0$. By virtue of (2.1), we have $r^{-1} \in J$. But such an r cannot annihilate $J/R = \psi_0 N = \alpha\gamma N$ whenever J is infinitely generated. This contradiction shows that all finite rank pure submodules of B are completely decomposable.

Let M be a pure submodule of countable rank in the Butler module B. We represent M as the union of an ascending chain $M_0 = 0 < M_1 < \cdots < M_n < \cdots$ of pure submodules with M_n of rank n. What we have already proved above implies that each M_n is completely decomposable. Therefore, M_{n-1} as a pure submodule of M_n is a summand of M_n (cf. [12, p. 274]), i.e. $M_n = M_{n-1} \oplus J_n$ for a suitable rank 1 submodule J_n. We conclude that $M = \oplus_{n<\omega} J_n$, as claimed.

3. **Butler modules of rank $\aleph_1$.** We now turn our attention to Butler modules of uncountable rank. Several preliminary lemmas are required for the proof of the main result (3.7).

LEMMA 3.1. *Let* $0 \longrightarrow T_\nu \longrightarrow M_\nu \xrightarrow{\alpha_\nu} A_\nu \longrightarrow 0$ *(*$\nu < \kappa$*) be a direct system of balanced-exact sequences where the* A_ν *are reduced torsion-free modules. Assume that the connecting maps* $\phi_{\nu\mu}: A_\nu \longrightarrow A_\mu$ *(for all* $\nu \leq \mu < \kappa$*) are monomorphisms with* $\operatorname{Im}\phi_{\nu\mu}$ *pure in* A_μ*. Then the direct limit of the system,*

$$0 \longrightarrow T \longrightarrow M \xrightarrow{\alpha} A \longrightarrow 0,$$

is likewise a balanced-exact sequence.

PROOF. Let J be an R-module, $R \leq J < Q$, and $\eta: J \longrightarrow A$.

$$\begin{array}{ccccccccc} & & & & & & & & J \\ & & & & \overset{\chi_\nu}{\swarrow} & & \overset{\eta_\nu}{\swarrow} & \overset{\eta}{\swarrow} & \\ 0 & \longrightarrow & T_\nu & \longrightarrow & M_\nu & \xrightarrow{\alpha_\nu} & A_\nu & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \rho_\nu} & & \downarrow{\scriptstyle \phi_\nu} & & \\ 0 & \longrightarrow & T & \longrightarrow & M & \xrightarrow{\alpha} & A & \longrightarrow & 0 \end{array}$$

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Here ρ_ν, ϕ_ν stand for the canonical maps. Given $0 \neq x \in J$, there is a $\nu < \kappa$ such that $\phi_\nu x_\nu = \eta x$ for some $x_\nu \in A_\nu$. As $\operatorname{Im}\phi_\nu$ has to be pure in A, the correspondence $x \mapsto x_\nu$ extends to a homomorphism $\eta_\nu : J \longrightarrow A_\nu$. Evidently, $\phi_\nu \eta_\nu = \eta$. By the balanced-exactness of the top row, there is a map $\chi_\nu : J \longrightarrow M_\nu$ such that $\alpha_\nu \chi_\nu = \eta_\nu$. Thus $\phi_\nu \alpha_\nu \chi_\nu = \phi_\nu \eta_\nu = \eta$. Now $\rho_\nu \chi_\nu : J \longrightarrow M$ satisfies $\alpha(\rho_\nu \chi_\nu) = (\alpha \rho_\nu)\chi_\nu = (\phi_\nu \alpha_\nu)\chi_\nu = \phi_\nu \eta_\nu = \eta$, as desired.

LEMMA 3.2. *Let M be a torsion-free module of rank κ, κ a regular cardinal. For any homomorphism $\eta : M \longrightarrow \oplus_i C_i$,* $\operatorname{Im}\eta$ *is contained in a direct sum of* $\max\{\kappa, \aleph_0\}$ *many C_i's.*

PROOF. If κ is a finite cardinal, then M is a polyserial module and the claim is known (see e.g. [11]). If κ is infinite, then M is the set union of κ finite rank submodules M_j, $M = \bigcup_{j \in J} M_j$ where $|J| = \kappa$. The assertion follows from $\operatorname{Im}\eta = \bigcup_j \operatorname{Im}' M_j$ and from what has been said about the finite rank case.

LEMMA 3.3. *Let $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ be an exact sequence of torsion-free R-modules with αA pure-essential in B. Then there is an induced exact sequence*

$$\operatorname{Hom}_R(A, T) \longrightarrow \operatorname{Ext}^1_R(C, T) \longrightarrow \operatorname{Bext}^1_R(B, T) \longrightarrow \operatorname{Bext}^1_R(A, T)$$

for every R-module T.

PROOF. If αA is pure-essential in B, then for every rank 1 submodule J of B, βJ is properly contained in a rank 1 submodule of C. Therefore, if $0 \longrightarrow T \longrightarrow M \longrightarrow C \longrightarrow 0$ represents an element of $\operatorname{Ext}^1(C, T)$, then being pure-exact (C is torsion-free !) implies that the induced sequence in $\operatorname{Ext}^1(B, T)$ is balanced-exact, as is clear from (1.2).

It remains to show that under the map $\operatorname{Ext}^1(B, T) \longrightarrow \operatorname{Ext}^1(A, T)$ induced by α, $\operatorname{Bext}^1(B, T)$ is mapped into $\operatorname{Bext}^1(A, T)$. Let the bottom row represent an element of $\operatorname{Bext}^1(B, T)$ and M^* be defined via pullback in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M^* & \xrightarrow{\sigma} & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & T & \longrightarrow & M & \xrightarrow{\rho} & B & \longrightarrow & 0. \end{array}$$

For every submodule J of Q and every homomorphism $\eta : J \longrightarrow A$, there is a $\chi : J \longrightarrow M$ such that $\rho\chi = \alpha\eta$. The pullback property of M^* guarantees the existence of a map $\xi : J \longrightarrow M^*$ satisfying $\sigma\xi = \eta$. This proves the balancedness of the top row.

A submodule A of an R-module B is said to have the *Torsion Extension Property* if, for each torsion R-module T, the map $\operatorname{Hom}(B, T) \longrightarrow \operatorname{Hom}(A, T)$ induced by the inclusion $A \longrightarrow B$ is surjective. In this case, we say that A is a TEP-*submodule* of B. This property was introduced by Dugas-Rangaswamy [7] in investigating Butler groups.

The following is the analogue of a result by Dugas-Hill-Rangaswamy [6]; even their proof carries over to our case.

LEMMA 3.4. *B/A is a Butler module for every pure TEP-submodule A of a Butler module B.*

PROOF. Hypothesis implies that the induced map $\phi: \mathrm{Ext}^1(B/A, T) \longrightarrow \mathrm{Ext}^1(B, T)$ is monic for any torsion module T. The map $\mathrm{Bext}^1(B/A, T) \longrightarrow \mathrm{Bext}^1(B, T)$ is a restriction of ϕ, so it is injective. Hence B/A is a Butler module.

A simple technical lemma is the following.

LEMMA 3.5. *Let A be a pure TEP-submodule of the Butler module B such that B/A is of rank 1. Then A is a summand of B.*

PROOF. If A is not a summand of B, then A is not balanced in B, so it is pure-essential in B. In view of (3.3), for every torsion module T we have an exact sequence

$$\mathrm{Hom}(B, T) \longrightarrow \mathrm{Hom}(A, T) \longrightarrow \mathrm{Ext}^1(B/A, T) \longrightarrow \mathrm{Bext}^1(B, T) = 0.$$

By hypothesis, the first map in this sequence is surjective; thus $\mathrm{Ext}^1(B/A, T) = 0$ for all torsion modules T. In other words, B/A is a Baer module; so it is free (see [8]). This contradicts the hypothesis that A is not a summand.

We have come to our key lemma. This is a modified version of a lemma employed in [8]; the same idea occurs in [6].

LEMMA 3.6. *Let κ be an uncountable regular cardinal and $M_0 = 0 < M_1 < \cdots < M_\alpha < \cdots$ $(\alpha < \kappa)$ a well-ordered continuous ascending chain of submodules of M such that*

(i) $M = \bigcup_{\alpha<\kappa} M_\alpha$ is torsion-free;

(ii) M_α is a pure in $M_{\alpha+1}$ for all $\alpha < \kappa$;

(iii) M_α is a Butler module of rank $< \kappa$ for each $\alpha < \kappa$.

If M is a Butler module, then the set

$$E = \{\alpha < \kappa \mid \exists \beta > \alpha \text{ such that } M_\alpha \text{ is not TEP in } M_\beta\}$$

is not stationary in κ.

PROOF. Without loss of generality we may assume that in the definition of E, $\beta = \alpha + 1$.

For each $\alpha \in E$, choose a torsion module T_α and a homomorphism $\eta_\alpha: M_\alpha \longrightarrow T_\alpha$ which cannot be extended to $M_{\alpha+1} \longrightarrow T_\alpha$. For $\alpha \in \kappa \setminus E$, let $T_\alpha = 0$ and $\eta_\alpha = 0$. Setting $S_\alpha = \oplus_{\gamma<\alpha} T_\gamma$, form the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_\alpha & \longrightarrow & S_\alpha \oplus M_\alpha & \xrightarrow{\phi_\alpha} & M_\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \chi_\alpha & & \downarrow & & \\
0 & \longrightarrow & S_{\alpha+1} & \longrightarrow & S_{\alpha+1} \oplus M_{\alpha+1} & \xrightarrow{\phi_{\alpha+1}} & M_{\alpha+1} & \longrightarrow & 0
\end{array}$$

where both exact sequences split with ϕ_α, $\phi_{\alpha+1}$ as projections; the extremal vertical arrows are the obvious embeddings. Define $\chi_\alpha : S_\alpha \oplus M_\alpha \longrightarrow S_{\alpha+1} \oplus M_{\alpha+1}$ via

$$\chi_\alpha : (s_\alpha, m_\alpha) \longmapsto (s_\alpha + \eta_\alpha m_\alpha, m_\alpha).$$

Note that because of (3.1) and (iii), the splitting exact sequence at a limit ordinal $\alpha < \kappa$ will be the direct limit of the splitting exact sequences for $\beta < \alpha$.

Let $0 \longrightarrow S \longrightarrow X \overset{\phi}{\longrightarrow} M \longrightarrow 0$ be the direct limit of the split exact sequences with the indicated maps, for $\alpha < \kappa$. In view of (3.1), this sequence is balanced-exact. Therefore, since M is a Butler module, there is a splitting map $\psi : M \longrightarrow X$. Because of (iii) and (3.2), ψ maps M_α into the direct sum of M_α and a set of less than κ many T_γ's. We infer that the set

$$C = \{\alpha < \kappa \mid \psi M_\alpha \leq S_\alpha \oplus M_\alpha\}$$

is a club (closed and unbounded) in κ. We have commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S_\alpha & \longrightarrow & S_\alpha \oplus M_\alpha & \underset{\psi_\alpha}{\overset{\phi_\alpha}{\rightleftarrows}} & M_\alpha & \longrightarrow & 0 \quad (\alpha \in C) \\
 & & \downarrow & & \downarrow{\scriptstyle \bar{\chi}_\alpha} & & \downarrow & & \\
0 & \longrightarrow & S & \longrightarrow & X = S \oplus M & \underset{\psi}{\overset{\phi}{\rightleftarrows}} & M & \longrightarrow & 0
\end{array}$$

where the vertical maps are the canonical inclusions in the direct limits and $\psi_\alpha = \psi | M_\alpha$. Evidently, since $\bar{\chi}_\alpha | S_\alpha$ is the identity map, $\bar{\chi}_\alpha$ must act like this: $\bar{\chi}_\alpha(s_\alpha, m_\alpha) = (s_\alpha - \xi_\alpha m_\alpha, m_\alpha)$ for $s_\alpha \in S_\alpha$, $m_\alpha \in M_\alpha$ where $\xi_\alpha : M_\alpha \longrightarrow S$ is such that $\xi_\alpha(m_\alpha)$ is the projection of $-\bar{\chi}_\alpha(0, m_\alpha)$ on S. The choice of $\alpha \in C$ implies $\operatorname{Im} \xi_\alpha \leq S_\alpha$. Since $\bar{\chi}_\alpha = \bar{\chi}_{\alpha+1}\chi_\alpha$,

$$s_\alpha - \xi_\alpha m_\alpha = s_\alpha + \eta_\alpha m_\alpha - \xi_{\alpha+1} m_\alpha \quad (m_\alpha \in M_\alpha)$$

whence $\eta_\alpha m_\alpha = \xi_{\alpha+1} m_\alpha - \xi_\alpha m_\alpha$ in S. Both sides belong to $S_{\alpha+1} = S_\alpha \oplus T_\alpha$. If $\pi_\alpha : S_{\alpha+1} \longrightarrow T_\alpha$ denotes the obvious projection, then $\eta_\alpha m_\alpha = \pi_\alpha \xi_{\alpha+1} m_\alpha$ $(m_\alpha \in M_\alpha)$ follows. This shows that η_α is the restriction of $\pi_\alpha \xi_{\alpha+1} : M_{\alpha+1} \longrightarrow T_\alpha$. Therefore, $\alpha \notin E$, and so $C \cap E = \emptyset$, i.e. E is not stationary.

We can now prove:

THEOREM 3.7. *Over a valuation domain, Butler modules of rank $\leq \aleph_1$ are completely decomposable.*

PROOF. Let $M = \bigcup_{\alpha<\omega_1} M_\alpha$ be a filtration of the Butler module M of rank $\aleph_1$ with M_α pure and of countable rank. (2.2) implies that M_α is a Butler module. By virtue of (3.6), there is a club C in ω_1 such that for each $\alpha \in C$, M_α is a TEP-submodule in M_β for every $\beta > \alpha$. Keeping only the M_α with $\alpha \in C$ and renaming them by the ordinals $< \omega_1$, we may assume that every M_α is TEP in $M_{\alpha+1}$.

From (3.4) we infer that each M/M_α is a Butler module. Therefore, from (2.2) it follows that, for each $\alpha, M_{\alpha+1}/M_\alpha$ is completely decomposable. (2.2) and (3.5) show that M_α is a summand in N for every rank 1 summand N/M_α of $M_{\alpha+1}/M_\alpha$ for each $\alpha < \omega_1$. By a well-known Kaplansky lemma, M_α is a summand of $M_{\alpha+1}$. Hence $M \cong \oplus(M_{\alpha+1}/M_\alpha)$ is completely decomposable, in fact.

4. **$*$-maps and balanced extensions.** In our study of Butler modules of large ranks, we require the notion of $*$-homomorphism. The concept of h-map introduced by Dugas-Hill-Rangaswamy [6] suggested the study of this notion and its connection with balanced extensions.

A homomorphism $\phi : A \longrightarrow B$ between torsion-free R-modules A, B is called a $*$-*homomorphism* if, for each rank one pure submodule J of A, either $\phi J = 0$ or ϕJ is not pure in B. In other words, ϕJ is always properly contained in a cyclic submodule of B.

LEMMA 4.1. *The $*$-homomorphisms $A \longrightarrow B$ for torsion-free R-modules A, B form a submodule* $\mathrm{Hom}_R^*(A, B)$ *of* $\mathrm{Hom}_R(A, B)$.

PROOF. If $\phi_1, \phi_2 : A \longrightarrow B$ are $*$-homomorphisms, then for a rank 1 pure submodule J of A, $\phi_i J < Rb_i$ for suitable $b_i \in B$. Pick any $0 \neq a \in J$; there are $r_i \in R$ such that $\phi_i a = r_i b_i$ $(i = 1, 2)$. If $r_1 | r_2$ in R, then $(\phi_1 + \phi_2)a = r_1(b_1 + r_2 r_1^{-1} b_2)$ implies $(\phi_1 + \phi_2)J < R(b_1 + r_2 r_1^{-1} b)$ (observe that r_1 does not divide a in J). Hence $\phi_1 + \phi_2$ is likewise a $*$-homomorphism. That $r\phi_1$ is also a $*$-homomorphism for each $r \in R$ is obvious.

LEMMA 4.2. *Let A and B be torsion-free R-modules and C a pure-essential submodule of A. A homomorphism $\phi : A \longrightarrow B$ is a $*$-map if and only if its restriction $\phi|_C : C \longrightarrow B$ is a $*$-map.*

PROOF. It suffices to verify the 'if' part. Let $\phi|_C$ be a $*$-homomorphism, and assume, by way of contradiction, that there is a rank 1 pure submodule J of A such that ϕJ is pure in B. Clearly, $J \cap C = 0$, so by pure-essentialness, for any $0 \neq x \in J$ (which we keep fixed), there exist $c \in C$, $a \in A$ and $r \in R$ such that $ra = x + c$, but r divides neither x nor c. From the purity of C and J in A we conclude that, for every divisor s of r, s divides either both x and c, or none of them. $\phi|_C$ being a $*$-map implies there is a $t \in R$ which does not divide c in C but divides $\phi|_C$ in B; there is no loss of generality in choosing t among the divisors of r. Thus t does not divide ϕx, contradicting $\phi x = r\phi a - \phi c$ in B.

From [12, p. 245] we know that every R-module M has a cotorsion hull $M^\bullet$ which is unique up to isomorphism over M. We shall need the following simple observation.

LEMMA 4.3. *For a torsion R-module T, T is pure-essential in $T^\bullet$.*

PROOF. By the definition of cotorsion hull, for a submodule H of $T^\bullet$, $H \cap T = 0$ and $T^\bullet/(H + T)$ torsion-free imply $H = 0$. As $T^\bullet/T$ is torsion-free, $T^\bullet/(H + T)$ being torsion-free is equivalent to $(H+T)/T$ being pure ($=RD$) in $T^\bullet/T$. Hence the claim is evident.

Let $0 \longrightarrow T \longrightarrow M \xrightarrow{\beta} B \longrightarrow$ be an exact sequence where T is torsion and B is torsion-free. For every homomorphism $\eta\colon B \longrightarrow T^\bullet/T$ there is a commutative diagram

$$(1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M & \xrightarrow{\beta} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\chi} & & \downarrow{\scriptstyle\eta} & & \\ 0 & \longrightarrow & T & \longrightarrow & T^\bullet & \xrightarrow{\alpha} & T^\bullet/T & \longrightarrow & 0. \end{array}$$

In fact, the induced exact sequence $\mathrm{Hom}(B,T^\bullet) \longrightarrow \mathrm{Hom}(B,T^\bullet/T) \longrightarrow \mathrm{Ext}^1(B,T) \longrightarrow \mathrm{Ext}^1(B,T^\bullet)=0$ shows that every extension of T by B comes from a homomorphism $\eta\colon B \longrightarrow T^\bullet/T$. Thus $\chi\colon M \longrightarrow T^\bullet$ must exist.

We can now verify:

THEOREM 4.4. *Let T be a torsion and B a torsion-free module. The top row in (1) is balanced-exact if and only if there is a $*$-homomorphism η of B into $T^\bullet/T$ making the diagram commute.*

PROOF. Suppose that such an η exists. By (4.3), in the bottom row T is pure-essential in $T^\bullet$. Because of (1.2) the top row in (1) is balanced-exact.

Conversely, suppose that the top row in (1) is balanced-exact. The induced exact sequence $\mathrm{Hom}_R(M,T^\bullet) \longrightarrow \mathrm{Hom}_R(T,T^\bullet) \longrightarrow \mathrm{Ext}^1_R(B,T^\bullet)=0$ shows that the natural embedding $T \longrightarrow T^\bullet$ extends to a homomorphism $\chi\colon M \longrightarrow T^\bullet$. This induces a map $\eta\colon B \longrightarrow T^\bullet/T$ making (1) commute. Let J be a rank 1 submodule of B and $\delta\colon J \longrightarrow B$ the inclusion map. By balancedness, there is a $\gamma\colon J \longrightarrow M$ such tha $\beta\gamma=\delta$. Visibly, $\alpha\chi\gamma=\eta\beta\gamma=\eta\delta$ maps J into $T^\bullet/T$. If $\mathrm{Im}\,\eta\delta \neq 0$, then $\mathrm{Im}\,\eta\delta$ cannot be RD in $T^\bullet/T$ as is clear from (4.3). Thus η is a $*$-homomorphism.

From (4.3) it follows that every homomorphism $B \longrightarrow T^\bullet/T$ which lifts to a homomorphism $B \longrightarrow T^\bullet$ must be a $*$-map. Hence we arrive at an exact sequence

$$(2)\qquad \mathrm{Hom}_R(B,T^\bullet) \longrightarrow \mathrm{Hom}^*_R(B,T^\bullet/T) \longrightarrow \mathrm{Bext}^1_R(B,T) \longrightarrow 0.$$

5. **Butler modules of projective dimension ≤ 1.** Though we have been unable to classify Butler modules in general, we can give a full characterization of those whose projective dimension is at most 1; see (5.4) below.

Recall that a submodule A of an R-module C of projective dimension ≤ 1 is called *tight* if p.d. $C/A \leq 1$ (and hence p.d. $A \leq 1$ as well). The following easy result will be needed.

LEMMA 5.1. *For a torsion-free R-module C of projective dimension 1, there is a continuous well-ordered ascending chain of pure submodules:*

$$(3)\qquad 0=C_0 < C_1 < \cdots < C_\tau = C$$

for some ordinal τ *such that for each* $\nu < \tau$, *both* rk $C_{\nu+1}/C_\nu \leq 1$ *and* p.d. $C_{\nu+1}/C_\nu \leq 1$.

PROOF. If rk $C \leq \aleph_0$, then C is countably generated, and so are its finite rank pure submodules (see [12, p. 80]). Representing C as the union of an ascending chain of finite rank pure submodules of ranks $n = 1, 2, \ldots$, this chain is as desired.

If rk $C > \aleph_0$, then the existence of a chain like (3) with rk $C_{\nu+1}/C_\nu \leq \aleph_0$ and p.d. $C_{\nu+1}/C_\nu \leq 1$ follows at once from the existence of tight systems (see [12, p. 84]). Such a chain can be refined by inserting submodules between consecutive members in the way described in the preceding paragraph, so as to have always rank 1 factors.

The following lemma is crucial in the proof of (5.3).

LEMMA 5.2. *Let N be an epic image of a cotorsion R-module and* $0 \longrightarrow A \longrightarrow B \overset{\beta}{\longrightarrow} C \longrightarrow 0$ *an exact sequence with C torsion-free of p.d.* ≤ 1. *Every homomorphism* $\eta: A \longrightarrow N$ *extends to a homomorphism* $\chi: B \longrightarrow N$. *If η is a $*$-homomorphism, χ can also be chosen as a $*$-homomorphism.*

PROOF. Let $\eta: M \longrightarrow N$ be an epimorphism with cotorsion M. The induced exact sequence $0 = \mathrm{Ext}^1_R(C, M) \longrightarrow \mathrm{Ext}^1_R(C, N) \longrightarrow \mathrm{Ext}^2_R(C, \mathrm{Ker}\,\eta) = 0$ shows that $\mathrm{Ext}^1_R(C, N) = 0$ for all torsion-free C of p.d. 1. There is another induced sequence, viz. $\mathrm{Hom}_R(B, N) \longrightarrow \mathrm{Hom}_R(A, N) \longrightarrow \mathrm{Ext}^1_R(C, N) = 0$ which establishes the first claim.

Let the sequence (3) be chosen as stated in (5.1), and set $B_\nu = \beta^{-1} C_\nu$. If η is a $*$-homomorphism, we will extend it stepwise to $*$-homomorphisms $\chi_\nu: B_\nu \longrightarrow N$ such that $\chi = \bigcup_{\nu < \mu} \chi_\nu$ will be a $*$-homomorphism $B \longrightarrow N$. Start with $\chi_0 = \eta$. Let $\mu \leq \tau$ and suppose that $*$-maps $\chi_\nu: B_\nu \longrightarrow N$ have already been defined for each $\nu < \mu$ such that $\chi_\lambda \subset \chi_\nu$ if $\lambda < \nu$.

If μ is a limit ordinal, then define $\chi_\mu = \bigcup_{\nu < \tau} \chi_\nu: B_\nu \longrightarrow N$. If $\mu - 1$ exists, then we distinguish two cases. If $B_{\mu-1}$ is balanced in B_μ, then $B_\mu = B_{\mu-1} \oplus J_\mu$ for some rank 1 submodule J_μ of B_μ, and we can extend $\chi_{\mu-1}$ by choosing an arbitrary $*$-homomorphism $J_\mu \longrightarrow N$ (e.g. the zero map). If $B_{\mu-1}$ is not balanced in B_μ, then it is pure-essential in B_μ, and (4.2) guarantees that any extension $\chi_\mu: B_\mu \longrightarrow N$ of $\chi_{\mu-1}$ will be a $*$-map.

Our final preparatory result is the most relevant.

LEMMA 5.3. *A pure tight submodule of a Butler module of projective dimension 1 is again a Butler module.*

PROOF. Let A be a pure tight submodule of the Butler module B of p.d. 1. Suppose $0 \longrightarrow T \longrightarrow M \longrightarrow A \longrightarrow 0$ is a balanced-exact sequence with T a torsion module. In view of (4.4) there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & A & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle \eta} & & \\
0 & \longrightarrow & T & \longrightarrow & T^\bullet & \overset{\gamma}{\longrightarrow} & T^\bullet/T & \longrightarrow & 0
\end{array}$$

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with a $*$-homomorphism η. By making use of (5.2), we infer that η extends to a $*$-homomorphism $\chi: B \longrightarrow T^\bullet/T$. Now (2) implies the existence of a homomorphism $\xi: B \longrightarrow T^\bullet$ satisfying $\gamma\xi = \chi$. The restriction of ξ to A yields a map $A \longrightarrow T^\bullet$ whose existence assures the splitting of the given balanced-exact sequence.

We have come to the main result of this section which reads as follows.

THEOREM 5.4. *Butler modules of projective dimension ≤ 1 are completely decomposable.*

PROOF. By induction on the rank. Suppose B is a Butler module of p.d. 1 and of rank κ.

If $\kappa \leq \aleph_0$, then (2.2) implies the claim. Thus assume $\kappa > \aleph_0$ and the assertion holds for Butler modules of rank $< \kappa$.

Case 1. κ is a regular cardinal. Let

$$0 = B_0 < B_1 < \cdots < B_\alpha < \cdots \quad (\alpha < \kappa) \quad [\cup B_\alpha = B]$$

be a filtration of B such that all the B_α's are pure and tight submodules of rank $< \kappa$; such a filtration exists because B has a tight system as shown in [12, p. 88]. Observe that by (5.3) each B_α is a Butler module. (3.6) implies that we can drop to a club in κ, such that, for $\alpha < \beta$, B_α is a TEP-submodule of B_β. Relabeling the B_α, we may assume that, for $\alpha < \kappa$, B_α is TEP in $B_{\alpha+1}$. An appeal to (3.4) shows that all the factors $B_{\alpha+1}/B_\alpha$ are Butler modules. As they are of p.d. ≤ 1, induction hypothesis implies that all the factors $B_{\alpha+1}/B_\alpha$ are completely decomposable. Application of (3,5), along with a well-known Kaplansky lemma, leads us to the conclusion that for each α, B_α is a summand of $B_{\alpha+1}$, $B_{\alpha+1} = B_\alpha \oplus C_\alpha$ where C_α is completely decomposable. Hence $B = \oplus C_\alpha$ is completely decomposable.

Case 2. κ is a singular cardinal. We need a strengthened version of Shelah's Singular Compactness Theorem which applies also to cardinals below the cardinality of R. In order to avoid lengthy quotations of definitions and results, as well as long proofs, let us just refer to Theorem 10 in [9] and to the second example following it which applies to the class of direct sums of countably generated modules. The Butler modules under consideration belong to this class, hence [9, Theorem 10] along with our induction hypothesis implies that Butler modules of p.d. 1 and of cardinality κ are completely decomposable.

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Department of Mathematics
Tulane University
New Orleans, Louisiana 70118
U. S. A.

Dipartimento di Matematica
Università di Padova, via Belzoni 7
35131 Padova
Italy