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PERTURBATION THEOREMS FOR RELATIVE SPECTRAL PROBLEMS

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Eigenvalue problems of the form $Af = \lambda Bf$, where λ is a complex parameter and A and B are operators on a Hilbert Space, have been considered by a number of authors (e.g., [**1**; **3**; **5**; **7**; **10**]). In this paper, we shall be concerned with the existence and nature of eigenfunction expansions associated with such problems, with no assumptions of self-adjointness. The form of the theorems to be given here is: if the system (A, B) is spectral and complete (definitions below), and F and G are operators satisfying certain "smallness" conditions, then $(A + F, B + G)$ is also spectral and complete. The hypotheses for these theorems are chosen with an eye to applying the results to boundary-value problems on a compact interval. Such applications, together with an examination of circumstances under which the system (D^n, D^m) (D denoting differentiation) is spectral and complete under a broad class of boundary conditions, will be made in a later paper.

The spirit of this paper is closest to those of Schwartz [**11**] and Clark [**2**]; the theorems are proved by a contour-integral argument similar to that of Clark.

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Part I. Elementary theory of relative problems. Throughout this paper, the symbols H_0 and H will denote Hilbert spaces, A and B closed linear operators from H_0 to H, and λ a complex parameter. Whenever T is a linear operator, $D(T)$, $R(T)$ and $N(T)$ denote its domain, range, and null-space, respectively. The conditions $D(A) \subseteq D(B) \subseteq H_0$ and $D(A)$ dense in H_0 will be assumed throughout; we also assume that $H_0 \subseteq H$ with a dense continuous embedding. The identity operator in any space is denoted I. If X is a subset of a Hilbert space, and T is a linear operator, the closures of X and T will be denoted X^- and T^-. Much of the elementary theory of relative problems has been developed by Birnbaum [**1**].

Definition. Let $\rho(A, B)$ be the set of complex λ such that $(A - \lambda B)^{-1}$ exists as a bounded, everywhere defined map from H to H_0, and such that

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$(A - \lambda B)^{-1}B$ is a bounded, densely defined map of H_0 to itself. Let $\sigma(A, B)$ be the complement of $\rho(A, B)$. Let

$$P_\lambda = P_\lambda(A, B) = (A - \lambda B)^{-1},$$
$$R_\lambda = R_\lambda(A, B) = (P_\lambda B)^{-}.$$

Birnbaum proves that $\rho(A, B)$ is open, that P_λ and R_λ are analytic on $\rho(A, B)$, and that for $\lambda, \mu \in \rho(A, B)$,

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu = (\lambda - \mu)R_\mu R_\lambda. \tag{1}$$

We shall assume henceforth that $\rho(A, B)$ is not empty (it is easy to construct pairs, even with self-adjoint A and B, for which $\rho(A, B)$ is empty). Let $\lambda_0 \in \rho(A, B)$. By considering the equivalent problem $(A - \lambda_0 B)f = \mu Bf$, where $\mu = \lambda - \lambda_0$, we may assume without loss of generality that $0 \in \rho(A, B)$. For such problems we define the operator V as $R_0 = (A^{-1}B)^{-}$.

If δ is a bounded component of $\sigma(A, B)$, let Γ be a closed path in $\rho(A, B)$ enclosing δ but no other point of $\sigma(A, B)$. Define

$$E_\delta = (2\pi i)^{-1} \int_\Gamma R_\lambda d\lambda. \tag{2}$$

As in the usual spectral theory [**4**, p. 566], it can be shown [**1**] that E_δ is a projection on H_0, independent of the choice of Γ, and that the map $\delta \to E_\delta$ is a Boolean homomorphism of the algebra of bounded components of $\sigma(A, B)$. The following is similar to a lemma of Birnbaum:

LEMMA 1. *If $f \in D(A)$, then $E_\delta f \in D(A)$.*

Outline of proof. It is first shown that for any $g \in H$, $AP_\lambda g$ is continuous on $\rho(A, B)$ and, hence,

$$\int_\Gamma AP_\lambda g d\lambda \tag{3}$$

exists. We let $g = Bf$, and take a sequence of partitions of Γ such that the Riemann sums of the integral (3) and of

$$\int_\Gamma P_\lambda g d\lambda \tag{4}$$

converge. The conclusion follows from the facts that A is closed and (4) is $2\pi i E_\delta f$.

Since $D(A)$ is dense in H_0, it follows that $E_\delta D(A)$ is dense in $R(E_\delta)$, and Lemma 1 implies that if $R(E_\delta)$ is finite-dimensional, then $R(E_\delta) \subseteq D(A)$.

LEMMA 2. *Let λ_0 be a simple pole of R_λ, and let $f \in R(E_{\lambda_0}) \cap D(A)$. Then $Af = \lambda_0 Bf$.*

Proof. If Γ is a suitably chosen contour, then

$$\begin{aligned}(A - \lambda_0 B)f &= (A - \lambda_0 B)E_{\lambda_0} f \\ &= (2\pi i)^{-1}(A - \lambda_0 B)\int_\Gamma R_\lambda f d\lambda \\ &= (2\pi i)^{-1}\left[\int_\Gamma AP_\lambda Bf d\lambda - \lambda_0 B \int_\Gamma P_\lambda Bf d\lambda\right].\end{aligned}$$

Using the identity $AP_\lambda Bf = Bf + \lambda BP_\lambda Bf$, the above becomes

$$(2\pi i)^{-1}[\int_\Gamma Bfd\lambda + B\int_\Gamma(\lambda - \lambda_0)R_\lambda fd\lambda] = 0,$$

since λ_0 is a simple pole of R_λ. Thus, $(A - \lambda_0 B)f = 0$.

Definition. We shall call the system (A, B) *discrete* if R_λ is a compact operator on H_0 for some (equivalently, for all) $\lambda \in \rho(A, B)$.

LEMMA 3. *Let (A, B) be discrete. Then the following four sets are the same:*
(a) $\sigma(A, B)$.
(b) *The poles $\{\lambda_i\}$ of R_λ.*
(c) *The reciprocals of the non-zero eigenvalues $\{\mu_i\}$ of V.*
(d) *The singularities of P_λ.*
For each i, E_{λ_i} is the spectral projection of V at $\mu_i = \lambda_i^{-1}$, and $R(E_{\lambda_i}) \subseteq D(A)$.

Proof. All the above assertions follow easily once we establish (a) = (c). Let τ be the set described in (c). Since τ is discrete and $\rho(A, B)$ is open and non-empty, it follows that $\rho(A, B) - \tau$ is open and non-empty. If $\lambda \neq 0$, $\lambda \in \rho(A, B) - \tau$, then both R_λ and $(\lambda^{-1}I - V)^{-1} = \lambda(I - \lambda V)^{-1}$ exist. Thus,

$$R_\lambda = ((I - \lambda A^{-1}B)^{-1}A^{-1}B)^- = \lambda^{-1}(\lambda^{-1}I - V)^{-1}V, \tag{5}$$

$$P_\lambda = ((I - \lambda A^{-1}B)^{-1}A^{-1})^- = \lambda^{-1}(\lambda^{-1}I - V)^{-1}A^{-1}. \tag{6}$$

These relations hold on the natural domains of each side, so if λ_0 is a singularity of R_λ, then λ_0^{-1} is an eigenvalue of V (since V is compact) and, thus, a pole of $(\mu I - V)^{-1}$, and by (5) a pole of R_λ. Thus, $\sigma(A, B) \subseteq \tau$, and $\sigma(A, B)$ consists of poles of R_λ. We now show that $\tau \subseteq \sigma(A, B)$. Let μ_0 be a non-zero eigenvalue of V, γ a circle around μ_0 enclosing no other point of $\sigma(V)$ and of small enough radius that the circle Γ formed of the reciprocals of points of γ encloses at most one singularity of R_λ.

Assume that $\gamma \subseteq \rho(V)$ and $\Gamma \subseteq \rho(A, B)$. All this is possible by discreteness. Using (1) and computing, we find

$$\int_\Gamma R_\lambda d\lambda = \int_\gamma \mu^{-1}R_\mu(V)d\mu[\int_\gamma \mu R_\mu(V)d\mu + \int_\delta \mu R_\mu(V)d\mu],$$

where δ is a contour enclosing all of $\sigma(V)$ except μ. The above is equal to $\int_\gamma R_\mu(V)d\mu$, by the functional calculus for V, keeping in mind that γ and δ enclose disjoint parts of $\sigma(V)$. This expression is $(2\pi i)$ times the spectral projection of V at μ_0 and is, thus, not zero. Consequently, $\int_\Gamma R_\lambda d\lambda$ is not zero, so R_λ must have a singularity inside Γ. By shrinking the radius to zero we see that the singularity is at μ_0^{-1}. The rest of the assertions follow.

We assume, henceforth, that (A, B) is discrete. Let $\{\lambda_i\}$ be its eigenvalues and $\{E_i\}$ the associated projections on H_0. The $\{E_i\}$ commute, are pairwise disjoint (i.e., $E_iE_j = \delta_{ij}E_i$), and generate a Boolean algebra of projections on H_0.

Definition. If the Boolean algebra generated by $\{E_i\}$ is uniformly bounded in the operator norm on H_0, we say that the pair (A, B) is *spectral*. If the

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subspace of H_0 generated by the ranges of the $\{E_i\}$ is all of H_0, then (A, B) is *complete.*

LEMMA 4. *Let λ_0 be a simple pole of R_λ. Then*

$$R_\lambda E_{\lambda_0} = (\lambda_0 - \lambda)^{-1} E_{\lambda_0}.$$

If γ_i is a small circle enclosing λ_i, define

$$Q_{ij} = (2\pi i)^{-1} \int_{\gamma_i} (\mu - \lambda_i)^j R_\mu d\mu.$$

LEMMA 5. *Let (A, B) be discrete, spectral, and complete. Let p_i be the order of the pole of R_λ at λ_i, and suppose that for $i > N$, $p_i = 1$. Let $\lambda \in \rho(A, B)$; then*

$$R_\lambda = \sum_{i=1}^{N} \sum_{j=0}^{p_i - 1} (-1)^j (\lambda_i - \lambda)^{-(j+1)} Q_{ij} + \sum_{N+1}^{\infty} (\lambda_i - \lambda)^{-1} E_i,$$

with convergence in the strong operator topology of H_0.

The proofs of Lemmas 4 and 5 are straightforward and will be omitted.

COROLLARY. *Under the hypotheses of Lemma 5, let $d(\lambda) = \inf|\lambda_i - \lambda|$. Then $||R_\lambda|| \leqq M_\epsilon d(\lambda)^{-1}$, where M_ϵ is a constant, provided that λ stays a distance $\epsilon > 0$ away from all non-simple poles of R_λ.*

Part II. Perturbation theorems. We will say that a pair (A, B) is *admissible* if it has the following properties:

(1) (A, B) is discrete, spectral and complete.

(2) For some N, $i \geqq N$ implies E_i has a 1-dimensional range.

(3) For some $k \geqq 1$, the eigenvalues $\{\lambda_j\}$ satisfy $\alpha j^k \leqq |\lambda_j| \leqq \alpha' j^k$, with α, α' positive constants, and

$$||\lambda_j| - |\lambda_{j-1}||^{-1} = O(j^{-(k-1)}).$$

If (A, B) is admissible and $0 \in \rho(A, B)$, then V is a compact spectral operator on H_0 of which 0 is not an eigenvalue. Denote the eigenvalues of V by $\{\mu_i\}$. We list without proof two simple facts that will be needed later.

LEMMA 6. *If S and T are bounded operators from H_1 to H_2 (two Hilbert spaces), then $T = SK$, with K bounded on H_1, if and only if $R(T) \subseteq R(S)$.*

LEMMA 7. *There are constants p and p' depending only on $\{E_i\}$ such that for any finite sequence $\{\alpha_i\}$ of scalars, we have*

$$p||\sum \alpha_i E_i f||^2 \leqq \sum |\alpha_i|^2 ||E_i f||^2 \leqq p' ||\sum \alpha_i E_i f||^2$$

for all $f \in H_0$.

We will also need the following result of Kato [**6**]:

LEMMA 9. *Let $\{P_j\}$ and $\{F_j\}$ be two sequences of projections such that $P_j P_k = \delta_{jk} P_j$ and $F_j F_k = \delta_{jk} F_j$. Assume that the $\{F_j\}$ are self-adjoint and $\sum F_j = I$. If*

(1) $\dim R(P_0) = \dim R(F_0) < \infty$,

(2) *there is a* C *in* $[0, 1)$ *such that*

$$\sum_1^\infty ||F_j(P_j - F_j)f||^2 \leqq C^2||f||^2 \textit{ for all } f,$$

then there is a bicontinuous operator W *such that* $P_j = W^{-1}F_jW$, *for all* j.

Given an operator K, we define the *C-norm* of K to be the infimum of the numbers $||K_s||$, taken over all representations $K = K_s + K_c$, with K_s bounded and K_c compact. We now give two perturbation theorems with an outline of their proofs, followed by a discussion of their applicability.

THEOREM 1. *Let* (A, B) *be admissible, with* $0 \in \rho(A, B)$, *and let* F *be a closed operator from* H_0 *to* H *satisfying:*

(a) $D(A) \subseteq D(F)$;

(b) $(A^{-1}F)^- = V^{1/k}K$, *where the closure is in the norm of* H_0, *and* K *is a bounded operator on* H_0.

Then, if the C-norm of K *is sufficiently small, the system* $(A - F, B)$ *is admissible, and its eigenvalues can be enumerated in a sequence* $\{\nu_j\}$ *such that* $|\lambda_j - \nu_j| = O(j^{k-1})$.

Proof. By a well-known theorem of Lorch, Mackey, and Wermer [**12**], we may assume without loss of generality that the projections $\{E_i\}$ are self-adjoint. Let $\tilde{R}_\lambda = R_\lambda(A - F, B)$ and $\tilde{R}_\lambda^\epsilon = R_\lambda(A - \epsilon F, B)$, for ϵ in $[0, 1]$. Let Γ_i be a circle of radius $r_i = \alpha i^{k-1}$ and centre λ_i, where α is chosen so that the circles do not overlap, and for large i,

$$(|\lambda_i| - r_i - (|\lambda_{i-1}| + r_{i-1})) \geqq \delta i^{k-1},$$

for some $\delta > 0$. This is possible by condition (3) of admissibility. For any integer $J \geqq 1$, let $G_J = \sum_1^J E_i$, and $Q_J = I - G_J$. Elementary computations provide the identities

$$\tilde{R}_\lambda^\epsilon = R_\lambda + \epsilon(P_\lambda F)^-(I - \epsilon(P_\lambda F)^-)^{-1}R_\lambda, \tag{7}$$

$$(P_\lambda F)^- = \mu G_J(\mu I - V)^{-1}V^{1/k}G_JK + \sum_{J+1}^{\infty} \lambda_i^{\theta}(\lambda_i - \lambda)^{-1}E_iQ_JK, \tag{8}$$

where $J \geqq N$, $\mu = \lambda^{-1}$, and $\theta = (k-1)/k$. From (7), it is clear that if there is a $\lambda \in \rho(A, B)$ such that $||(P_\lambda F)^-|| < 1$, then $\tilde{R}_\lambda^\epsilon$ will be defined and compact and, hence, $(A - \epsilon F, B)$ will be discrete, for all ϵ. The first term in (8), denoted $U_J(\lambda)$, is analytic in λ^{-1} in a neighbourhood of 0, and has a zero at $\mu = 0$. Thus, choosing $|\lambda|$ sufficiently large (given J) makes $||U_J(\lambda)||$ arbitrarily small. Now $K = K_s + K_c$, with $||K_s||$ arbitrarily close to the C-norm of K. If λ lies outside all the circles Γ_i, then the function $|\lambda_i|^{2\theta}|\lambda - \lambda_i|^{-2}$ is bounded by a multiple of $i^{2k-2}r_i^{-2} = \alpha^{-1}$. We find, then, that

$$||P_\lambda Ef||^2 \leqq C(||U_J(\lambda)||^2 + (||K_s|| + \epsilon_J)^2\alpha^{-1})||f||^2,$$

where C is a constant and $\epsilon_J = ||Q_JK_c||$. Since $\epsilon_J \to 0$ as $J \to \infty$, by choosing

J and $||K_s||$ to make the second term small, and then choosing $|\lambda|$ large enough to make $||U_J(\lambda)||$ small, we can see that for suitable λ, $||P_\lambda F|| < 1$.

Thus, there is a circle ζ centred at the origin such that all singularities of $\tilde{R}_\lambda^\epsilon$ lie inside either ζ or one of the Γ_i, and ζ can be taken independent of ϵ in [0, 1]. From this and (7), it follows that $\tilde{R}_\lambda^\epsilon$ is a continuous function of λ and ϵ in $\zeta \times [0, 1]$. Choose ζ so that it intersects none of the Γ_i and encloses $\Gamma_1 \ldots \Gamma_J$, for J to be determined. Define

$$\tilde{G}_J = (2\pi i)^{-1} \int_\zeta \tilde{R}_\lambda d_\lambda.$$

By a simple homotopy argument, we see that for any sufficiently large J, the ranges of $\tilde{G}_J$ and G_J have the same dimension. Kato's first hypothesis is thus satisfied. Define

$$\tilde{E}_j = (2\pi i)^{-1} \int_{\Gamma_j} \tilde{R}_\lambda d_\lambda.$$

By a straightforward estimate based on (7), we find that $||E_j - \tilde{E}_j|| < 1$ for all sufficiently large j, so $\tilde{E}_j$ has a one-dimensional range, and thus $\tilde{R}_\lambda$ has a single simple pole ν_j inside Γ_j. From this follows the final assertion of the theorem, together with properties (2), (3) in the definition of admissibility.

It remains only to prove that $(A - F, B)$ satisfies the second hypothesis of Kato's lemma. It is enough to show that there are some N and $C < 1$ such that

$$\sum_N^\infty ||(E_n^* - \tilde{E}_n^*)f||^2 \leqq C^2||f||^2, \tag{9}$$

for all $f \in H_0$.

The reason for passing to adjoints will be clear shortly. Recall that $E_n^* = E_n$, and let γ_n be the circle consisting of the complex conjugates of points in Γ_n. Then

$$||(\tilde{E}_n^* - E_n)f||^2 = (2\pi i)^{-1}\left\| \int_{\gamma_n} R_\lambda^*(I - (P_\lambda F)^*)^{-1}(P_\lambda F)^* f d\lambda \right\|^2 \tag{10}$$

$$\leqq r_n^2 \sup_{\lambda \in \gamma_n} ||R_\lambda^*(I - (P_\lambda F)^*)^{-1}(P_\lambda F)^* f||^2 \tag{11}$$

$$\leqq M\, ||(P_{\eta_n} F)^* f||^2, \tag{12}$$

where M is a constant and the supremum in (11) is attained at $\eta_n \in \gamma_n$; the above follows from the corollary to Lemma 5 and the first part of the present proof. It is easy to see that the expression in (12) is bounded by a multiple of

$$||U_J^*(\eta_n)f||^2 + (||K_s^*|| + \epsilon_J)^2 \Bigg(\sum_{\substack{j \geqq J \\ j \neq n}} |\lambda_j|^{2\theta} |\lambda_j - \lambda_n|^{-2} ||E_j f||^2 + |\lambda_n|^{2\theta} r_n^{-2} ||E_n f||^2 \Bigg), \tag{13}$$

where J may be any sufficiently large value. We must, therefore, estimate the following three expressions:

(a) $$\sum_{n=N}^{\infty} ||U_J(\eta_n)f||^2,$$

(b) $$(||K_s^*|| + \epsilon_J)^2\left(\sum_{n=N}^{\infty} |\lambda_n|^{2\theta} r_n^{-2} ||E_n f||^2\right),$$

(c) $$(||K_s^*|| + \epsilon_J)^2\left(\sum_{n=N}^{\infty} \sum_{\substack{j \geqq J \\ j \neq n}} |\lambda_j|^{2\theta} |\lambda_j - \lambda_n|^{-2} ||E_j f||^2\right).$$

A simple computation shows that, by taking N and J large and $||K_s||$ small, the expressions (a) and (b) can be made arbitrarily small. There remains (c): the double series in (c) is bounded by a multiple of

$$\sum_n \sum_j j^{2k-2} |n^k - j^k|^{-2} ||E_j f||^2 = \sum_{j=J}^{\infty} j^{-2} ||E_j f||^2 \sum_{\substack{n \geqq N \\ n \neq j}} |n^k j^{-k} - 1|^2.$$

A short computation shows that this is bounded by

$$C\left(C' + \sup_j j^{-1} \int_{|t-1| \geqq j^{-1}} |t^k - 1|^{-2} dt\right) ||f||^2.$$

The supremum is finite, so taking N and J large and $||K_s||$ small makes (c) arbitrarily small. From Kato's Lemma, then, it follows that $\{\tilde{E}_i^*\}$ and, hence, $\{\tilde{E}_i\}$ generate uniformly bounded algebras of projections, and $\sum \tilde{E}_i = I$. This completes the proof.

The proof of the following theorem is sufficiently like that of Theorem 1 that we omit it:

THEOREM 2. *Let (A, B) and F satisfy the hypotheses of Theorem* 1, *and suppose that B is invertible as a map from H_0 to H. Let G be a closed operator from H_0 to H such that:*

(a) $D(B) \subseteq D(G)$;

(b) *$B^{-1}G$ is bounded on H_0;*

(c) $(A^{-1}G)^- = V^{(k+1)/k}L$, *with L bounded on H_0.*

Let η be a complex number. Then, if $|\eta|$ is sufficiently small, the system $(A - F, B + \eta G)$ is admissible.

In order to discuss the applicability of Theorems 1 and 2 we need some further definitions. We recall that H_0 is densely and continuously embedded in H, so we may regard A and B either as operators from H_0 to H, or as densely defined operators on H. We assume, henceforth, that A is closed in the latter sense.

Definition. Let (A, B) be admissible, and assume that $0 \in \rho(A, B)$. Let $\Lambda = (A^*A)^{\frac{1}{2}}$, a positive, self-adjoint operator on H. Then $D(A) = D(\Lambda)$;

assume that $H_0 = D(\Lambda^\alpha)$, for some $\alpha \in [0, 1]$, with the norm $\|f\|_{H_0} = \|\Lambda^\alpha f\|_H$. We shall say that (A, B) is *regular* if there is an integer $n \geqq k$ such that

$$D(\Lambda^{\alpha+1/n}) \subseteq R(V^{1/k}).$$

Here, V is considered as an operator on $H_0 = D(\Lambda^\alpha)$. The system (A, B) is *super-regular* if

$$D(\Lambda^{(\alpha+(k+1)/n)}) \subseteq R(V^{(k+1)/k}).$$

If (A, B) is regular and F is a closed operator on H satisfying:

(1) $D(A) \subseteq D(F) \subseteq H_0$,

(2) $F = \Lambda^{(n-1)/n}K$, with K bounded on H_0,

then $(A^{-1}F)^- = (A^{-1}\Lambda)^-\Lambda^{-1/n}K$. Since $A^{-1}\Lambda$ extends to a bounded operator on H which takes $D(A)(=D(\Lambda))$ to itself, a standard interpolation theorem [**8**, p. 31] shows that $(A^{-1}\Lambda)^-$ takes $D(\Lambda^{\alpha+1/n})$ to itself, and regularity plus Lemma 6 imply that $(A^{-1}F)^- = V^{1/k}L$, with L bounded on H_0. The C-norm of L is a multiple of the C-norm of K, the multiple depending on (A, B) but not on F. Thus, if the C-norm of K is small, the hypotheses of Theorem 1 are satisfied, and we have:

THEOREM 3. *Let (A, B) and F satisfy the hypotheses of Theorem* 1 *with* (b) *replaced by*

(b_1) *(A, B) is regular, $D(A) \subseteq D(F) \subseteq H_0$;*

(b_2) *$\Lambda^{-(n-1)/n}F$ extends to a bounded operator on H_0, of sufficiently small C-norm.*

Then the conclusions of Theorem 1 *hold.*

We leave it to the reader to carry out a similar analysis and prove:

THEOREM 4. *Let (A, B) be super-regular and assume the hypotheses of Theorem* 2, *with* (c) *replaced by*

$$\Lambda^{(n-k-1)/n}G \text{ is bounded on } H_0.$$

Then the conclusions of Theorem 2 *hold.*

Regularity is a rather mild restriction on a system (A, B), while super-regularity is quite a stringent one. For this reason, Theorems 2 and 4 are of limited usefulness. In a subsequent paper, we intend to develop constructive methods for determining regularity and super-regularity, for quite a general class of problems. For now, we remark that if $R(B)$ is all of H, then $D(A) = R(A^{-1}B) \subseteq R(V)$, and by interpolation (A, B) is regular.

We shall close by examining a very simple example of the problems treated here to illustrate the application of Theorem 1. Let H^n be the space of $(n-1)$-fold absolutely continuous functions on $[0, 1]$ with nth derivatives in $L^2 = L^2[0, 1]$, with the usual norm [**4**, p. 1,296], and H_0^n the closure in H^n of the set of functions vanishing on a neighbourhood of 0 and 1. Let $p \geqq 0$ be a smooth function and let A be the operator determined by the formal

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expression $(-D^2 + p)^2$ on the domain $H^4 \cap H_0^2$. Let B be determined by the expression $(-D^2 + p)$ and F by $q_2D^2 + q_1D + q_0$, with the $\{q_i\}$ smooth functions; the domains of B and F will be specified below. We know that A is a positive self-adjoint operator on $H = L^2$. Among the problems in the class described is the Orr-Summerfeld equation [**3**]. There are three natural choices for H_0, as follows:

(a) Let $H_0 = D(A^{1/4}) = H_0^1; D(B) = \{f \in H^2 | f(0) = f(1) = 0\}; D(F) = D(A^{1/2}) = H_0^2$. Then B is a positive operator on L^2, and $D(B^{\frac{1}{2}}) = H_0^1 = H_0$. The norm $|B^{\frac{1}{2}}f|$ is equivalent to the norm of H_0^1, and in the former norm, V is easily seen to be a self-adjoint operator on H_0. It follows that (A, B) is spectral and complete, and since B is invertible, (A, B) is also regular. It is not difficult to show that (A, B) is admissible, and that the hypotheses of Theorem 3 are satisfied. Thus, $(A + F, B)$ is admissible, and so if $f \in H_0^1$, then the eigenfunction series

$$(14) \qquad \sum_1^\infty E_i f$$

converges to f in the topology of H_0^1, i.e., uniformly and with L^2 convergence of derivatives. This result was obtained by Di-Prima and Habetler [**3**].

(b) Let $H_0 = D(A^{\frac{1}{2}}) = H_0^2; D(B) = D(F) = H_0$. It can be shown that this problem is regular, and that the hypotheses of Theorem 3 are satisfied, so once again $(A - F, B)$ is spectral and complete. Here, the expansion theorem states that if $f \in H_0^2$, the eigenfunction series (14) converges uniformly, is termwise differentiable in the sense of uniform convergence, and is twice termwise differentiable in the sense of L^2-convergence. This improves a result of Schensted, mentioned in [**3**].

(c) Let $H_0 = L^2$. Then $D(B)$ can be any subspace of H^2 (containing H_0^2) without changing $V = (A^{-1}B)^-$. By (a) or (b), the problem $(A + F, B)$ is complete, but since (letting $\lambda_0 \in \rho(A + F, B)$) the operator

$$((A + F - \lambda_0 B)^{-1}B)^-$$

has a non-trivial null-space, it follows that $(A + F, B)$ cannot be spectral. Thus, there are L^2-functions whose eigenfunction series (14) fail to converge in L^2.

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