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EXISTENCE OF WEIGHT SPACE DECOMPOSITIONS FOR IRREDUCIBLE REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

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Let L denote a finite-dimensional simple Lie algebra over an algebraically closed field K of characteristic zero. It is well known that every finite-dimension l, irreducible representation of L admits a weight space decomposition(1); moreover every irreducible representation of L having at least one weight space admits a weight space decomposition. Also, to the best of my knowledge, all detailed studies of infinite-dimensional irreducible representations of L have been predicated on the existence of a weight space decomposition (cf. [2], [3], [4], [5]). In this note we present a necessary and sufficient condition for a given irreducible representation of L to have a weight space decomposition and provide an example to show that not all irreducible representations of L have weight space decompositions.

1. **Existence of weight space decompositions.** Let U denote the universal enveloping algebra of L, let $\mathscr{H}$ be a Cartan subalgebra of L, and let $k[\mathscr{H}]$ denote the subalgebra of U generated by 1 and $\mathscr{H}$. Further for any $H \in \mathscr{H}$, let $k[H]$ denote the subalgebra of U generated by 1 and H. Finally if $\{\rho, V\}$ is a representation of L and $\lambda \in \mathscr{H}^* = \mathrm{Hom}_K(\mathscr{H}, K)$ let

$$V_\lambda = \{v \in V \mid \rho(H)v = \lambda(H)v \text{ for all } H \in \mathscr{H}\}.$$

THEOREM. *If $\{\rho, V\}$ is an irreducible representation of L, the following statements are equivalent*:

(a) $V = \Sigma \oplus V_\lambda$, i.e. *$V$ admits a weight space decomposition.*
(b) $(\forall v \in V)$ $\rho(k[\mathscr{H}])v$ *is finite dimensional.*
(b′) $(\forall v \in V)(\forall H \in \mathscr{H})$ $\rho(k[H])v$ *is finite dimensional.*
(c) $(\exists v \in V, v \neq 0)$ $\rho(k[\mathscr{H}])v$ *is finite dimensional.*
(c′) $(\exists v \in V, v \neq 0)(\forall H \in \mathscr{H})$ $\rho(k[H])v$ *is finite dimensional.*

Proof. The implications (a)⇒(b)⇒(b′) and (b)⇒(c)⇒(c′) are immediate. Since $\mathscr{H}$ is commutative and finite dimensional we also have that (b′)⇒(b) and (c′)⇒(c). Thus it suffices to prove that (c)⇒(a). Again since $\mathscr{H}$ is commutative and

(1) For basic definitions and properties of Lie algebras and their representations (cf. [1], [6]).

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$W=\rho(k[\mathscr{H}])v$ is finite dimensional there exists a nonzero element $w\in W$ such that

$$(\forall H\in\mathscr{H})\rho(H)w = \lambda(H)w$$

where $\lambda(H)\in K$.

Since ρ is a representation we have $\lambda\in\mathscr{H}^*$ and hence $w\in V_\lambda$. Then $\Sigma\oplus V_\lambda$ is nonempty $\rho(L)$-invariant subspace of V and as $\{\rho, V\}$ is assumed irreducible we have (a). Q.E.D

If C denotes the centralizer of the Cartan subalgebra $\mathscr{H}$ in U then the above theorem implies that for any maximal left ideal M of U with $\dim_K(C/C\cap M) < +\infty$, the left regular representation of L in U/M admits a weight space decomposition. Various forms of converses to this statement are still open. For example, if M is a maximal left ideal of U for which $M\cap C$ is a maximal left ideal of C, does U/M admit a weight space decomposition?

2. **An irreducible representation of A_1 having no weight space decomposition.** We shall now make use of the criteria established in the first section to construct an irreducible representation of a simple Lie algebra which does not admit a weight space decomposition. Let A_1 denote the usual three-dimensional Lie algebra over the complex numbers $\mathscr{C}$ with basis $\{X, Y, H\}$ and Lie multiplication given by $[X, Y]=H$, $[H, X]=2X$, and $[H, Y]=-2Y$. Let M denote a maximal left ideal of $U(A_1)$ containing $X-1$. There exists at least one such maximal left ideal as $X-1$ is not invertible in $U(A_1)$. We claim that $k[H]+M$ is infinite dimensional—more precisely,

$$\{1+M, H+M, H^2+M,\dots\}$$

is linearly independent in $U(A_1)/M$. In fact suppose we have

$$\lambda_0 1+\lambda_1 H+\cdots+\lambda_n H^n\in M$$

where $\lambda_i\in\mathscr{C}$ with $\lambda_n\neq 0$. Note that

$$(X-1)^n H^m = \begin{cases} 0 & \text{mod } M \quad \text{if } n>m \\ (-2)^n\cdot n!1 & \text{mod } M \quad \text{if } n=m. \end{cases}$$

Then

$$(X-1)^n(\lambda_0 1+\cdots+\lambda_n H^n)\in M$$

implies

$$\lambda_n(X-1)^n H^n\in M$$

and hence

$$0\neq\lambda_n(-2)^n\cdot n!1\in M.$$

This contradicts the maximality of M in $U(A_1)$ and hence, by the theorem in §1, we conclude that $U(A_1)/M$ does not admit a weight space decomposition.

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