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ON THE ZEROS OF SECOND ORDER LINEAR DIFFERENTIAL POLYNOMIALS

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We determine all functions $f(z)$ meromorphic in the plane such that $f'(z)/f(z)$ has finite order and $f(z)$ and $F(z)$ have only finitely many zeros, where $F(z)=f''(z)+Af(z)$ for some constant A.

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1. Introduction

Our starting point is the following result of Frank, Hennekemper and Polloczek [3]:

Theorem A *Suppose that $f(z)$ is meromorphic in the plane, and that $f(z)$ and $f^{(k)}(z)$ have only finitely many zeros, for some $k\geqq 3$. Then $f'(z)/f(z)$ is rational, that is $f(z)=R(z)\exp(P(z))$, where $R(z)$ is rational and $P(z)$ is a polynomial.*

A comparable result classifying functions $f(z)$ meromorphic in the plane such that $f(z)$ and $f''(z)$ have only finitely many zeros is not known. It is natural to extend the above problem to consideration of the zeros of a meromorphic function $f(z)$ and a linear differential polynomial F in $f(z)$, that is

$$F(z)=f^{(k)}(z)+\sum_{j=0}^{k-1} a_j(z)f^{(j)}(z) \tag{1.1}$$

where the a_j are, say, rational. Among other results in [2], Frank and Hellerstein classified completely those entire functions $f(z)$ such that $f(z)$ and $F(z)$ have only finitely many zeros, where F is given by (1.1) with $k\geqq 2$ and the a_j polynomials. They also showed that if f is meromorphic in the plane, and f and F have only finitely many zeros, where $k\geqq 3$ and the a_j are again polynomials, then f'/f has finite order determined by the degrees of the a_j. For constant coefficients, Steinmetz proved the following [15]:

Theorem B. *Suppose that $f(z)$ is meromorphic and non-constant in the plane and that $f(z)$ and $F(z)$ have no zeros, where $F(z)$ is given by* (1.1) *with $k\geqq 3$ and $a_0,\ldots,a_{k-1}$ constants. Then f satisfies one of the following:*

$$f(z)=\exp(az+b+e^{cz+d}); \tag{1.2}$$

$$f(z)=e^{az+b}(e^{cz+d}-1)^{-n}; \tag{1.3}$$

$$f(z)=e^{az+b}(z-c)^{-n}.$$

Here a,b,c,d are constants and n is a positive integer.

The following was proved in [**12**];

Theorem C. *Suppose that $f(z)$ is meromorphic of finite order in the plane, and that $f(z)$ and $F(z)$ have only finitely many zeros, where $F(z)=f''(z)-\alpha f(z)$ for some constant α. If $\alpha=0$, then f'/f is rational. If $\alpha\neq 0$, then either f'/f is rational, or f is given by* (1.3).

The case $\alpha=0$ in Theorem C represents a slight improvement of a result of Mues [**14**]. Now the hypothesis that f has finite order appears in Theorem C because the method of [**12**] uses asymptotic integration in sectors for f, with the Phragmén–Lindelöf principle used to "fill in the gaps". With a weaker assumption on f we shall prove here:

Theorem 1. *Suppose that $f(z)$ is meromorphic in $|z|\geqq R$, and that $a_0(z)$ and $a_1(z)$ are analytic there, with*

$$a_j(z)=0(|z|^{j-2}). \tag{1.4}$$

Suppose that $f(z)F(z)$ has no zeros in $|z|\geqq R$, where F is given by

$$F(z)=f''(z)+a_1(z)f'(z)+a_0(z)f(z).$$

Suppose finally that $\bar{N}(r,f)$ has finite lower order. Then f'/f has only finitely many poles in $|z|\geqq R$, with a pole or removable singularity at infinity.

Here $\bar{N}(r,f)$ counts the points at which f has poles, each counted just once (see Section 2).

Corollary. *If $f(z)$ is meromorphic in the plane, if $f(z)$ and $f''(z)$ have only finitely many zeros, and if f'/f has finite order, then f'/f is rational.*

Theorem 1 is sharp at least to the extent that the coefficients $a_j(z)$ cannot be made any larger. For example, $f(z)=\sec(\sqrt{z})$ has no zeros, and nor has

$$F(z)=f''(z)+(1/2z)f'(z)+(1/4z)f(z)=f^3(z)/2z.$$

It seems reasonable to believe that Theorem 1 would be true without any restriction on $\bar{N}(r,f)$. A comparable result is proved in **[13]** for k at least 3 and small rational coefficients in (1.1). Also **[11]** contains a result with extra hypotheses on f and F, but which allows slightly larger coefficients a_j. The assumption made on $\bar{N}(r,f)$ in Theorem 1 is however considerably weaker than that of Theorem C, making no restriction on the multiplicities of poles of f.

For the case $\alpha\neq 0$ of Theorem C, we have the following:

Theorem 2 *Suppose that $f(z)$ is meromorphic in the plane, and that $f(z)$ and $F(z)$ have only finitely many zeros, where $F(z)=f''(z)-\alpha f(z)$ for some non-zero constant α. If $\bar{N}(r,f)$ has finite lower order, then either f'/f is rational, or f is given by* (1.2) *or* (1.3).

We make the following remark about the proof of Theorems 1 and 2 above. Suppose for example that f and f'' have only finitely many zeros. Then, as in **[14]**, the function $z-(f/f')$ is a quotient of solutions of an equation $w''+bw=0$, where b has only finitely many poles. With our assumptions, b turns out to be rational. The proof then depends on demonstrating the existence of unbounded regions where f'/f has infinitely many poles whose residues are incompatible with arising from poles of f. This is the key to both proofs. It seems very difficult however to apply this in the general case where b may be transcendental.

2. Preliminaries

We use the following notation. For $r\geqq 0$, and α,β real, then

$$S(r,\alpha,\beta)=\{z:|z|>r,\ \alpha<\arg z<\beta\}.$$

We use $W(u,v,w)$ to denote the Wronskian of u,v,w.

We need the Nevanlinna theory for functions meromorphic in $0<R\leqq|z|<+\infty$. (See for example **[1,** p. 98].) For such a function $f(z)$, we have a representation $f(z)=z^n h(z)G(z)$, where n is an integer, $h(z)$ is analytic in $|z|\geqq R$ with a removable singularity at infinity, and $G(z)$ is meromorphic in the plane. (See **[16,** p. 15] for a proof of this fact.) We can thus define Nevanlinna functionals $m(r,f), N(r,f)$, etc. for $r\geqq R$, where

$$N(r,f)=\int_R^r n(t,f)\,dt/t,$$

and $n(t,f)$ is the number of poles of f, counting multiplicities, in $R\leqq|z|\leqq t$. $\bar{N}(r,f)$ is defined similarly, with multiple poles counted just once. The first fundamental theorem, in this setting, becomes, for finite a,

$$T(r,1/(f-a))=T(r,f)+O(\log r).$$

Now f has a pole or removable singularity at infinity if and only if $T(r,f)=O(\log r)$

through a sequence tending to infinity. Moreover, $T(r,f)$ differs from a non-decreasing function by a term which is $O(\log r)$. Denoting by $S(r,f)$ any quantity which is $O(\log^+ T(r,f)+\log r)$, possibly outside a set of finite linear measure, we have $m(r,f'/f)=S(r,f)$. We can define the order and lower order of such terms just as in [**6**, p. 16]. Finally we remark that Clunie's lemma is valid in this context. If $P(f)f^n=Q(f)$, where $P(f)$ and $Q(f)$ are differential polynomials in f whose coefficients $a(z)$ satisfy $m(r,a)=S(r,f)$, and if $Q(f)$ has total degree at most n in f and its derivatives, then $m(r,P(f))=S(r,f)$. The proof is identical to that in [**6**, p. 68].

3. Lemmas needed for the proofs of Theorems 1 and 2

Our proofs make extensive use of Hille's method of asymptotic integration of the equation

$$y''+a_1y'+a_0y=0. \tag{3.1}$$

(See [**8**, Chapter 7], or [**9**]). Suppose that a_1 and a_0 are analytic in a sector $S_1=S(r_1,\theta_0-\alpha,\theta_0+\alpha)$, such that $a_1(z)=0(|z|^{-1})$ and $a_1'(z)=0(|z|^{-2})$ and

$$a_0(z)=\gamma z^n(1+0(1/|z|)) \tag{3.2}$$

there, where $n\geqq -1$, γ is a non-zero constant and $0\leqq\alpha\leqq 2\pi/(n+2)$, and $\alpha\leqq\pi$ if $n=-1$. We first make a change of variable $y=uv$, where $v'/v=-a_1/2$, so that u satisfies

$$u''+b(z)u=0 \tag{3.3}$$

and $b(z)=\gamma z^n(1+o(1))$ as z tends to infinity in S_1. Now take a large z_0 in S_1: setting

$$Z=\int_{z_0}^{z} b(t)^{1/2}\,dt=\beta z^{(n+2)/2}(1+o(1)), \tag{3.4}$$

where $\beta=\gamma^{1/2}2/(n+2)$, makes Z analytic and one–one in a sector

$$S_2=S(r_2,\theta_0-\alpha+\varepsilon,\theta_0+\alpha-\varepsilon).$$

Here r_2 depends on ε, which may be chosen arbitrarily small and positive. Now (3.1) has linearly independent solutions

$$v(z)\,b(z)^{-1/4}(1+o(1))\exp((-1)^k iZ) \tag{3.5}$$

in S_2, for $k=1,2$, and any solution y of (3.1) satisfies

$$\log^+|y(z)|=0(|z|^{(n+2)/2}) \tag{3.6}$$

there. If $\arg z = \theta_0$ is a critical ray for (3.3), that is

$$\operatorname{Arg}\gamma + (n+2)\theta_0 = 0 \pmod{2\pi} \tag{3.7}$$

then $Re(i\beta z^{(n+2)/2}) = 0$ on this ray, and if $|\theta - \theta_0|$ is small and positive, then for large z on $\arg z = \theta$, one of the solutions (3.5) is large and the other is small. We need:

Lemma 1. *Suppose that a_1 and a_0 are rational, such that a_1 vanishes at infinity and a_0 satisfies* (3.2) *with $\gamma \neq 0$ and $n \geqq -1$. Suppose further that* (3.1) *has linearly independent solutions which are meromorphic in the plane. Then $n \neq -1$. Also if $n = 0$ then there exist a non-zero constant A and rational functions R_1, R_2 such that $R_1(z)e^{Az}$ and $R_2(z)e^{-Az}$ are solutions of* (3.1).

Proof. The first part is essentially Theorem 2 of [7]. If $n = -1$, we can choose θ_0 satisfying (3.7), and for a small positive ε, determine a non-trivial solution y of (3.1) such that for some positive c_1,

$$\log|y(z)| \leqq -c_1|z|^{1/2} \tag{3.8}$$

in a sector $S(r_2, \theta_0 + \varepsilon, \theta_0 + 2\pi - \varepsilon)$. But $y(z)$ has only finitely many poles and order at most 1/2, and so must be rational, by (3.8) and the Phragmén–Lindelöf principle [4, p. 104]. This is a contradiction, as no rational function can satisfy (3.8) without vanishing identically.

For the second part, we can again take θ_0 satisfying (3.7). Now (3.4) becomes

$$Z = iAz + b_1 \log z + 0(1/|z|)$$

for some constants b_1 and A with $A \neq 0$. Thus (3.1) has a solution satisfying

$$y(z) = z^{b_2}(1 + o(1))e^{A_1 z}$$

in $S_3 = S(r_3, \theta_0 + \varepsilon, \theta_0 + \pi - \varepsilon)$, say, with b_2 a constant and $A_1 = \pm A$, and such that $\exp(A_1 z)$ tends to zero in S_3. Determining a representation for y in $S_4 = S(r_3, \theta_0 - \pi + \varepsilon, \theta_0 - \varepsilon)$, we see that since y has only finitely many poles and order at most 1, then $y(z)\exp(-A_1 z)$ must be rational, using the Phragmén–Lindelöf principle again. We can make a second such solution similarly.

Lemma 2. *Suppose that $f(z)$ is meromorphic in $|z| \geqq R$, and that f and F have no zeros there, where*

$$F(z) = f''(z) + a_1(z)f'(z) + a_0(z)f(z)$$

and a_0, a_1 are analytic in $|z| \geqq R$ and satisfy (1.4). *Then for all r outside a set of finite linear measure we have*

$$T(r, f'/f) = O(\bar{N}(r, f) + \log r).$$

Proof. Obviously we may assume that $H=f'/f$ has an essential singularity at infinity, for otherwise there is nothing to prove. We follow the now standard Tumura–Clunie method [**6**, pp. 69–73].

Setting $G=F/f$, and $A=H'-G'H/2G$, we obtain $m(r,A)=S(r,H)$ from Clunie's lemma. We define h formally by $h^2=G$, so that setting $g=a_1+h'/h$ we have

$$G=(H+g/2)^2+B$$

where $B=A+a_0-g^2/4$ satisfies $m(r,B)=S(r,H)$. Writing $U=H+g/2$, we obtain $UP=C$, where $m(r,C)=S(r,H)$ and $P=2U'-(G'/G)U$. Thus $m(r,P)=S(r,H)$ by Clunie's lemma. If P is not identically zero, we get

$$m(r,U)\leqq m(r,PU)+m(r,1/P)\leqq N(r,P)+S(r,H)$$

and the conclusion follows. If P vanishes identically, then $G=U^2+B=cU^2$ for some constant c. If $c\neq 1$, then $m(r,U)=S(r,H)$ so that $m(r,H)=S(r,H)$. Finally if $c=1$ then substituting $H=U-g/2$ into $G=H^2+H'+a_1H+a_0$ and using the fact that $h'/h=U'/U$ we obtain

$$g^2/4-g'/2-ga_1/2+a_0=0,$$

which implies that g is analytic in $|z|>R$ and has a removable singularity at infinity, with $g(\infty)=0$. Hence G is analytic in $|z|>R$ with at most a pole at infinity, and the same is true for H, by Clunie's lemma.

4. Proof of Theorem 1

We assume that there exists a function $f(z)$ meromorphic in $|z|\geqq R>0$ such that fF has no zeros there, where

$$F(z)=f''(z)+a_1(z)f'(z)+a_0(z)f(z) \tag{4.1}$$

and $a_1(z), a_0(z)$ are analytic in $|z|\geqq R$ with

$$a_j(z)=O(|z|^{j-2}). \tag{4.2}$$

We assume that $\bar{N}(r,f)$ has finite lower order, and will show that f'/f has only finitely many poles in $|z|\geqq R$, with at most a pole at infinity.

We take a sector $S_0=S(R,\alpha,\alpha+2\pi)$ for some real α, and define a function g analytic on S_0 by

$$g^2=f/F. \tag{4.3}$$

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We remark that g might not be meromorphic on $|z| \geqq R$, but certainly g^2 is, and so is g'/g.

Now the equation

$$y'' + a_1 y' + a_0 y = 0 \tag{4.4}$$

has by (4.2) a regular singular point at infinity, so that by [**10**, Chapter 15], (4.4) has a solution $f_1(z)$ analytic in S_0 such that for some constant γ_1, the function $h_1(z) = f_1(z) z^{-\gamma_1}$ has a removable singularity at infinity, with $h_1(\infty) = 1$. We can also define a function W analytic in S_0 by

$$W'/W = -a_1 \tag{4.5}$$

so that for some constant γ_2, the function $h_2(z) = W(z) z^{-\gamma_2}$ is again analytic in $|z| \geqq R$, with a removable singularity at infinity, and with $h_2(\infty) = 1$. Finally we can define a second solution f_2 of (4.4) by

$$W(f_1, f_2) = W \tag{4.6}$$

so that f_2 is analytic in S_0 with

$$\log^+ |f_2(z)| = O(\log |z|). \tag{4.7}$$

Now (4.6) implies that

$$W(f_1, f_2, f) = WF = Wf/g^2$$

so that, proceeding as in [**2**],

$$W(w_1, w_2) = W \tag{4.8}$$

where for $j = 1, 2$,

$$w_j = f'_j g + f_j h \tag{4.9}$$

and

$$h = -(f'/f)g. \tag{4.10}$$

Now g and h are analytic in S_0, and therefore so is each w_j, so that (4.8) implies that w_1, w_2 are linearly independent solutions of an equation

$$w'' + a_1 w' + b_1 w = 0 \tag{4.11}$$

where b_1 is analytic in S_0. Here we used the fact that W is non-vanishing in S_0. We claim that b_1 is in fact analytic in $|z|>R$, with at most a pole at infinity.

To establish this claim, first note that

$$w_1=g f_1((f'_1/f_1)-(f'/f)) \tag{4.12}$$

so that w'_1/w_1 is meromorphic in $|z|>R$, and therefore so is b_1. Now by Lemma 2 there is a sequence $r_m \to +\infty$ such that

$$T(r_m, f'/f)=O(r_m^{M_1})$$

for some positive M_1, so that writing (4.12) in the form $w_1=g f_1 \phi$, and using (4.3) we obtain

$$T(s_m, g^2)+T(s_m, \phi)=O(s_m^{M_2}) \tag{4.13}$$

for some sequence $s_m \to +\infty$. Now (4.12), (4.13) and the lemma of the logarithmic derivative imply that

$$m(r, b_1)=O(\log r) \tag{4.14}$$

through a sequence of r tending to infinity. To complete the proof of the claim, we note that we are free to define, in a sector $S_1=S(R, \alpha-\pi, \alpha+\pi)$, functions g^*, h^*, W^*, f_j^* and w_j^* in exactly the same way as g, h, W, f_j and w_j were defined above. Thus w_1^* and w_2^* are analytic solutions of some equation

$$w''+a_1 w'+b_1^* w=0$$

where b_1^* is analytic in S_1. But it is quite clear that in the intersection of S_0 and S_1, w_1^* and w_2^* are linear combinations of w_1 and w_2, so that $b_1^*=b_1$ and b_1 is analytic in S_1. Now (4.14) implies that b_1 has at most a pole at infinity.

Suppose now that b_1 satisfies $b_1(z)=O(|z|^{-2})$ as $z\to\infty$. Then the equation (4.11) has a regular singular point at infinity so that

$$\log^+|w_j(z)|=O(\log|z|)$$

as $z\to\infty$ in S_0. But

$$g=(w_2 f_1-w_1 f_2)/W \tag{4.15}$$

so that using (4.7) and that fact that $W(z)z^{-\gamma_2}$ has a removable singularity at infinity we obtain

$$\log^+|g^2(z)|=O(\log|z|)$$

in S_0. A similar estimate holds in S_1, so that $T(r, f/F) = O(\log r)$ and the conclusion of Theorem 1 now follows from Lemma 2.

We consider now the case where $b_1(z) \neq O(|z|^{-2})$ as $z \to \infty$, and may write

$$b_1(z) = \sum_{k=-\infty}^{n} a_k^* z^k \tag{4.16}$$

in $|z| > R$ where $a_n^* \neq 0$ and $n \geqq -1$, and we set $N = (n+2)/2 \geqq 1/2$. We shall eventually show that this case is impossible. We assert first that

$$T(r, f/F) = O(r^N). \tag{4.17}$$

For the region $|z| > R$ may be divided up into overlapping sectors in which using the method described in Section 3 we have

$$\log^+ |w(z)| = O(|z|^N)$$

for any solution of (4.11). Defining g, h, W, f_j and w_j in such sectors as above, and applying (4.15) we obtain (4.17). We now choose a real θ_0 satisfying

$$\operatorname{Arg}(a_n^*) + (n+2)\theta_0 = 0 \pmod{2\pi}, \tag{4.18}$$

and take a small positive δ_1. We take analytic solutions f_1 and f_2 of (4.4) in a sector $S_2 = S(R_1, \theta_0 - \delta_1, \theta_0 + \delta_1)$ such that $W(f_1, f_2) = W$, and functions g, h, w_1, w_2 as in (4.3), (4.9) and (4.10), all defined exactly as in S_1, keeping the same notation for convenience. We define additional analytic functions U_1, U_2, G_1 and G_2 in S_2 by

$$U_j = W^{-1/2} w_j \quad \text{and} \quad G_j = W^{-1/2} f_j \tag{4.19}$$

for $j = 1, 2$. We have, by (4.6) and (4.8),

$$W(U_1, U_2) = W(G_1, G_2) = 1 \tag{4.20}$$

and G_1, G_2 solve an equation

$$y'' + A^* y = 0, \tag{4.21}$$

where A^* is analytic in $|z| > R$ with

$$A^*(z) = O(|z|^{-2}), \tag{4.22}$$

using (4.5). Also U_1, U_2 solve

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$$u''+Bu=0 \tag{4.23}$$

where

$$B(z)=a_n^* z^n(1+o(1)) \tag{4.24}$$

is analytic in $|z|>R$. Now provided R_1 is sufficiently large and δ_1 is sufficiently small, the equation (4.23) has in S_2 linearly independent solutions u_1, u_2 given by

$$u_k(z)=B(z)^{-1/4}v_k(z)=B(z)^{-1/4}(1+o(1))\exp((-1)^{k+1}iZ) \tag{4.25}$$

where for a suitable point z_0,

$$Z=\int_{z_0}^{z} B(t)^{1/2}\,dt=-i\beta z^N+O(|z|^{N-1/2}). \tag{4.26}$$

Here β is a non-zero constant, and by (4.18), $Re(\beta z^N)=0$ on $\arg z=\theta_0$. We can write

$$U_k(z)=B(z)^{-1/4}(C_k v_1+D_k v_2) \tag{4.27}$$

for constants C_1, C_2, D_1 and D_2, where C_1 and C_2 cannot both vanish, and nor can D_1 and D_2. Finally we set

$$\phi_1=C_2G_1-C_1G_2 \quad \text{and} \quad \phi_2=D_2G_1-D_1G_2 \tag{4.28}$$

and note that by (4.15), (4.19) and (4.27),

$$g=u_1\phi_1+u_2\phi_2. \tag{4.29}$$

We make some observations about G_1 and G_2. Now G_1 is determined from f_1, which is chosen so that $f_1 z^{-\gamma_1}$ has a removable singularity at infinity. Further, G_2 is determined from f_2, which is required only to satisfy $W(f_1,f_2)=W$. Thus we are free to choose G_2 subject only to (4.20). We can assume therefore that in S_2,

$$G_1(z)=z^{\nu}(1+O(1/|z|)) \tag{4.30}$$

and that since

$$(G_2/G_1)'=G_1^{-2}=z^{-2\nu}(1+O(1/|z|)),$$

we have

$$G_2/G_1=(1-2\nu)^{-1}z^{1-2\nu}(1+o(1)) \tag{4.31}$$

or, if $v=1/2$,

$$G_2/G_1=\log z+o(1) \tag{4.32}$$

in S_2. We thus have

$$G_1G_2=z(1-2v)^{-1}(1+o(1)) \quad \text{or} \quad G_1G_2=(z\log z)(1+o(1)) \tag{4.33}$$

in S_2. We shall establish the following Claim:

Claim. *There exist constants $\lambda>1$ and $K>0$ with the following property. There exist arbitrarily large positive r such that in*

$$\lambda^{-1}r<|z|<\lambda r, \quad |\arg z-\theta_0|<\delta_1,$$

$f(z)$ has no poles, while for $j=1,2$,

$$|\log|\phi_j(z)||\leqq K\log|z|. \tag{4.34}$$

To prove this Claim we set

$$H=(U_1/U_2)(f_2/f_1)=(U_1/U_2)(G_2/G_1). \tag{4.35}$$

By (4.9) and (4.19),

$$f'/f=(1-H)^{-1}((f'_1/f_1)-(Hf'_2/f_2)),$$

and thus at a large pole of f in S_2 we must have $H=1$, since f_1 and f_2 are non-zero for large z in S_2, by (4.30) and (4.31) or (4.32). Also a pole of f is a zero of g, but not of h, so that U_1 and U_2 cannot vanish at a large pole of f, and H cannot have pole there. Also

$$(f'_1/f_1)-(f'_2/f_2)=-W/(f_1f_2)=-1/(G_1G_2).$$

Thus at a large pole of f in S_2, we have $H=1$ and f'/f has a simple pole with residue equal to

$$(-1/(G_1G_2))/(-H')=((H'/H)(G_1G_2))^{-1}=(1-(G_1G_2)/(U_1U_2))^{-1} \tag{4.36}$$

by (4.20) and (4.35).

Assume for the time being that $Re(1-2v)\neq 0$ and that $G_1/G_2\to 0$ in S_2, that is that $Re(1-2v)>0$. At a large pole of f in $S_2, H=1$ gives

$$G_1/G_2=(C_1v_1+D_1v_2)/(C_2v_1+D_2v_2) \tag{4.37}$$

using (4.27). We note that by (4.25),

$$v_1 v_2 \to 1 \tag{4.38}$$

as $z \to \infty$ in S_2. Recalling that U_1, U_2 are non-zero at a large pole ζ of f in S_2, (4.37) gives

$$(C_1 + o(1))v_1(\zeta) + (D_1 + o(1))v_2(\zeta) = 0. \tag{4.39}$$

If $C_1 = 0$, then $D_1 \neq 0$ and (4.38) and (4.39) yield

$$(C_1 v_1 + D_1 v_2)(C_2 v_1 + D_2 v_2) = O(1) \tag{4.40}$$

at ζ. The same conclusion (4.40) holds if $D_1 = 0$ and if $C_1 D_1 \neq 0$. Now (4.40) implies that

$$U_1(\zeta) U_2(\zeta) = O(|B(\zeta)|^{-1/2}) = O(|\zeta|^{1/2}), \tag{4.41}$$

using (4.27). Now (4.33) and (4.41) yield

$$(U_1(\zeta) U_2(\zeta))/(G_1(\zeta) G_2(\zeta)) = O(|\zeta|^{-1/2}) \tag{4.42}$$

which on substitution into (4.36) gives a contradiction, since the residue (4.36) is required to be a negative integer. In this case therefore f can have only finitely many poles in S_2. By the obvious symmetry of (4.37) and the fact that we are only concerned with the products $G_1 G_2$ and $U_1 U_2$ in (4.36), the same conclusion holds if $Re(1 - 2\nu) < 0$ so that $G_2/G_1 \to 0$ in S_2.

Now suppose that $1 - 2\nu = 0$. In this case, by (4.32), $G_1/G_2 \to 0$ in S_2 again, so that as above we obtain (4.39) and (4.40) at a large pole ζ of f in S_2. By (4.33), (4.42) now becomes

$$(U_1(\zeta) U_2(\zeta))/(G_1(\zeta) G_2(\zeta)) = O((|\zeta|^{1/2} \log|\zeta|)^{-1})$$

and we have a contradiction as before.

We have thus proved that if $Re(1 - 2\nu) \neq 0$ or if $\nu = 1/2$ then f has only finitely many poles in S_2. Moreover, in this case (4.34) holds for all sufficiently large z in S_2, by (4.30) and (4.31) or (4.32).

We still need to establish the Claim in the case where $1 - 2\nu = i\mu$, where μ is real and non-zero. In this case, $\log|G_1/G_2| = O(1)$ in S_2. For a large pole ζ of f in S_2 we again have (4.37), which gives, at ζ,

$$(C_1 - C_2 G_1/G_2)v_1 + (D_1 - D_2 G_1/G_2)v_2 = 0. \tag{4.43}$$

Using (4.38), this equation clearly gives

$$|v_1(\zeta)|+|v_2(\zeta)|=O(1) \tag{4.44}$$

unless one of the coefficients of v_1, v_2 in (4.43) is small. Now (4.44) leads to (4.41) again, and we obtain (4.42) and a contradiction as before. Therefore in this case, for any $\varepsilon_1>0$, f has only finitely many poles in S_2 outside the set where

$$|C_1-C_2G_1/G_2|<\varepsilon_1 \quad \text{or} \quad |D_1-D_2G_1/G_2|<\varepsilon_1.$$

By (4.31) it is clear that there are regions free of such points as described in the Claim. In such regions (4.34) follows from (4.30) and (4.31), and the Claim is proved in all cases.

We are now in a position to obtain a contradiction, and show that the case $b_1(z)\neq O(|z|^{-2})$ is impossible. We take a region

$$\lambda^{-1}r<|z|<\lambda r, \quad |\arg z-\theta_0|<\delta_1,$$

with r large, on which f has no poles and hence g has no zeros, and on which (4.34) holds. Setting $\rho=\lambda^{1/2}$, standard estimates yield a positive constant K_2 such that

$$|g'(z)/g(z)|\leqq K_2r^{N-1} \tag{4.45}$$

on

$$\rho^{-1}r\leqq|z|\leqq\rho r, \quad |\arg z-\theta_0|\leqq\delta_1/2.$$

For g^2 is analytic on $|z|\geqq R$, and by (4.17) can be represented as the product of a power of z, a function with a removable singularity at infinity, and an entire function h_3 with $T(s,h_3)=O(s^N)$. We recall that $g=u_1\phi_1+u_2\phi_2$, and that u_1 and u_2 are given by (4.25) and (4.26), where $Re(\beta z^N)=0$ on $\arg z=\theta_0$. We assume without loss of generality that $Re(\beta z^N)>0$ on $\theta_0<\arg z<\theta_0+\delta_1$. For any fixed δ_2 in $(0,\delta_1/2)$, we have, if r is large enough,

$$|\log g(z)-\log u_k(z)|\leqq K_3\log|z| \tag{4.46}$$

on $\rho^{-1}r\leqq|z|\leqq\rho r$, $\arg z=\theta_0+(-1)^{k+1}\delta_2$. Here K_3 is a positive constant, and (4.46) follows from (4.25) and (4.34). Let Γ be the contour consisting of the arcs

$$|z|=\rho^{-1}r, \quad |z|=\rho r, \quad |\arg z-\theta_0|\leqq\delta_2,$$

and the straight line segments joining their end-points, and let the vertices of Γ be X_1, Y_1, Y_2, X_2, ordered anti-clockwise, with $X_1=\rho^{-1}r\exp(i(\theta_0-\delta_2))$. By (4.45) there is a positive constant K_4 such that the integral of g'/g around the curved parts of Γ, described in the positive sense, has modulus at most $\delta_2K_4r^N$. Choose δ_2 so small that

$$|\delta_2 K_4| < |\beta e^{iN\theta_0}|(\rho^N - \rho^{-N})\cos(N\delta_2).$$

By (4.25), (4.26) and (4.46), if r is large enough,

$$\log g(Y_1) - \log g(X_1) + \log g(X_2) - \log g(Y_2) = -2\beta e^{iN\theta_0} r^N(\rho^N - \rho^{-N})\cos(N\delta_2) + \eta$$

where, for some positive constant K_5,

$$|\eta| < K_5(r^{N-1/2} + \log r).$$

If r is large enough, this contradicts the argument principle.

5. Proof of Theorem 2

Suppose that $f(z)$ is meromorphic in the plane and that f and F have only finitely many zeros, where

$$F(z) = f''(z) - \alpha f(z) \tag{5.1}$$

and α is a non-zero constant. Suppose further that $\bar{N}(r, f)$ has finite lower order. By results from [2], where we apply Theorem 1 if f is entire and Lemma 8 if f is not entire, f'/f has finite lower order. We shall prove that either f is given by (1.2), or f has finite order, so that in the latter case we can appeal to Theorem C. Clearly we may assume that $\alpha = 1$, that f is transcendental, and that F is non-constant.

We begin by defining a rational function R and an entire function g by

$$g(z)^2 = R(z)f(z)/F(z) \tag{5.2}$$

Now g has finite lower order, and so has

$$h(z) = -(f'(z)/f(z))g(z) \tag{5.3}$$

which has only finitely many poles. Indeed there is a sequence s_n tending to infinity such that

$$T(s_n, g) + T(s_n, h) = O(s_n^{M_1}) \tag{5.4}$$

for some positive M_1. To see this, we need only take a corresponding sequence s_n' for $T(r, f'/f)$ and if necessary make s_n slightly smaller. Now

$$W(e^z, e^{-z}, f) = -2F = -2Rf/g^2$$

so that

$$W(w_1, w_2) = -2R \tag{5.5}$$

where for $j = 1, 2$,

$$w_j = f_j h + f'_j g \tag{5.6}$$

with

$$f_1(z)=e^z, \quad f_2(z)=e^{-z}. \tag{5.7}$$

Each w_j is meromorphic in the plane with only finitely many poles and with finite lower order, by (5.4). Now (5.5) implies that w_1, w_2 are linearly independent solutions of an equation

$$w''-(R'/R)w'+b_1w=0 \tag{5.8}$$

where $b_1(z)$ is meromorphic in the plane and has only finitely many poles, since poles of b_1 can only arise from poles of the w_j and zeros of their Wronskian. Since w_1 has finite lower order the lemma of the logarithmic derivative implies that $b_1(z)$ is rational. We shall show that $b_1(\infty)\neq 0,\infty$ by considering the two contrary cases $b_1(\infty)=0, b_1(\infty)=\infty$ separately.

Suppose that $b_1(\infty)=0$. Then by Lemma 1 we must have $b_1(z)=O(|z|^{-2})$ as $z\to\infty$, so that (5.8) has a regular singular point at infinity. Thus each w_j is rational, and by (5.6) and (5.7) we obtain

$$f'/f=(v_1+1)/(v_1-1)$$

where $v_1=T_1e^{2z}$ and T_1 is rational. Thus f'/f has infinitely many poles at points where $v_1=1$, and at a large such pole f'/f has residue

$$2/(v_1')=2v_1/v_1'=1+o(1).$$

But this implies that f has infinitely many zeros, which is a contradiction.

Suppose that $b_1(\infty)=\infty$. Then we can write

$$b_1(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots \tag{5.9}$$

for large z, where n is at least 1 and $a_n\neq 0$. Set $N=(n+2)/2$. We observe first that the plane can be divided up into overlapping sectors in which we have (see Section 3)

$$\log^+|w(z)|=O(|z|^N)$$

for every solution of (5.8). Solving (5.6) for g we obtain a similar estimate for $\log^+|g(z)|$ and deduce that

$$T(r,g)=O(r^N). \tag{5.10}$$

We now take θ_0 in $(-\pi/2,\pi/2)$ satisfying

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$$\operatorname{Arg}(a_n)+(n+2)\theta_0=0 \pmod{2\pi} \tag{5.11}$$

and will proceed to a contradiction. We shall first establish the following Claim.

Claim 1. *For any positive* ε_1 *and* r_1, f *has infinitely many poles in the sector* $S_1=S(r_1,\theta_0-\varepsilon_1,\theta_0+\varepsilon_1)$.

We again use the method of asymptotic integration as described in Section 3, and a fairly standard application of the argument principle. Obviously we may assume that r_1 is large and that ε_1 is small. Now the equation (5.8) has linearly independent solutions u_1 and u_2 in S_1 given by

$$u_j(z)=R(z)^{1/2}b(z)^{-1/4}(1+o(1))\exp((-1)^{j+1}iZ) \tag{5.12}$$

where

$$b(z)=b_1(z)+R''/2R-3R'^2/4R^2$$

and

$$iZ=\beta_1 z^N+O(|z|^{N-1/2}). \tag{5.13}$$

Here β_1 is a non-zero constant. Also $Re(\beta_1 z^N)=0$ on $\arg z=\theta_0$, and we may assume that $Re(\beta_1 z^N)>0$ on $S(r_1,\theta_0,\theta_0+\varepsilon_1)$. Now we can write

$$u_j=F_j'g+F_jh$$

for $j=1,2$, where F_1, F_2 are linearly independent solutions of $y''-y=0$, and so

$$g=(u_1F_2-u_2F_1)/W(F_2,F_1). \tag{5.14}$$

Now (5.12), (5.13), (5.14) and obvious estimates for F_1, F_2 imply that for any sufficiently small positive δ_1 there exists $r_2>0$ such that for $|z|\geqq r_2$, $\arg z=\theta_0+(-1)^j\delta_1$, we have

$$\log g(z)=(-1)^j\beta_1 z^N+O(|z|^{N-1/2}). \tag{5.15}$$

Here we have used the fact that F_1 and F_2 cannot vanish in S_1, by the choice of θ_0. Now if f has only finitely many poles in S_1, (5.10) and routine estimates yield a positive constant K_1 such that

$$|g'(z)/g(z)|\leqq K_1|z|^{N-1} \tag{5.16}$$

in $S(r_3,\theta_0-\varepsilon_1/2,\theta_0+\varepsilon_1/2)$ if r_3 is large. Take a large r_4 and r_5 with r_5/r_4 large, and let Γ

be the contour consisting of the circular arcs from $r_k \exp(i(\theta_0+\delta_1))$ to $r_k \exp(i(\theta_0-\delta_1))$, for $k=4,5$, and the straight line segments joining them, Γ described anti-clockwise. Now the integral of g'/g around the two circular arcs in the positive sense has, by (5.16), modulus at most $K_2\delta_1 r_5^N$ for some positive constant K_2. Choose δ_1 so small that $K_2\delta_1<|\beta_1 e^{iN\theta_0}\cos N\delta_1|$. By (5.15), if r_4 is large enough, then the integral of g'/g along the two straight line segments, again in the positive sense, is equal to

$$-2\beta_1(r_5^N-r_4^N)\,e^{iN\theta_0}\cos N\delta_1+O(r_5^{N-1/2}).$$

This is clearly impossible, and Claim 1 is proved.

We now examine the residues of the poles of f'/f in S_1, to obtain a contradiction and thus show that $b_1(\infty)=\infty$ is impossible. By the choice of θ_0, we may assume that

$$|e^{2z}|>|z|^{2n} \tag{5.17}$$

in S_1. Now (5.6) yields

$$f'/f=(1+H)/(1-H) \tag{5.18}$$

where

$$H=e^{-2z}w_1/w_2. \tag{5.19}$$

At a pole of f in S_1 we have $H=1$, and f'/f has a simple pole with residue $-2/H'$. At a 1-point of H,

$$H'=H'/H=-2+W(w_2,w_1)/(w_1w_2)=-2+2R/(w_1w_2)$$

using (5.5). So the multiplicity of a large pole of f in S_1 is

$$(-1+R/(w_1w_2))^{-1}. \tag{5.20}$$

Set

$$W_j(z)=R(z)^{-1/2}\,b(z)^{1/4}\,w_j(z). \tag{5.21}$$

At a large pole of f in S_1, ζ say, we have

$$e^{2\zeta}=W_1(\zeta)/W_2(\zeta). \tag{5.22}$$

We may write, by (5.12),

$$W_j(z)=C_j(1+o(1))e^{iZ}+D_j(1+o(1))e^{-iZ}$$

in S_1, for some constants C_1, D_1, C_2, D_2, and observe that there are positive constants

ε_2 and K_3 such that if ζ is large and $|W_2(\zeta)|<\varepsilon_2$ then $|W_1(\zeta)W_2(\zeta)|\leqq K_3$. But this last estimate gives, using (5.21),

$$|w_1(\zeta)w_2(\zeta)|=O(|R(\zeta)|\,|b(\zeta)|^{-1/2})=O(|R(\zeta)|\,|\zeta|^{-1/2})$$

which on substitution into (5.20) yields a contradiction. On the other hand, $|W_2(\zeta)|\geqq\varepsilon_2$ leads to

$$|W_2(\zeta)W_1(\zeta)|\geqq|\zeta|^{2n}\varepsilon_2^2$$

by (5.17) and (5.22), so that in this case

$$|w_2(\zeta)w_1(\zeta)|\geqq|R(\zeta)|\,|\zeta|^n$$

if ζ is large enough, and again (5.20) gives a contradiction.

We are therefore left only to consider the case where $b_1(\infty)\neq 0,\infty$, the alternative cases having each led to a contradiction. Now by Lemma 1, the equation (5.8) has linearly independent solutions V_1e^{Az}, V_2e^{-Az}, where V_1 and V_2 are rational, and A is a non-zero constant. Now (5.3), (5.6) and (5.7) give

$$\frac{f'-f}{f'+f}=e^{-2z}\,\frac{A_1V_1e^{Az}+B_1V_2e^{-Az}}{A_2V_1e^{Az}+B_2V_2e^{-Az}} \tag{5.23}$$

where A_1, A_2, B_1, B_2 are constants. We note that A_1 and A_2 cannot both vanish, and nor can B_1 and B_2, since w_1 and w_2 are linearly independent. We shall complete the proof of Theorem 2 by dividing this case up into subcases.

Subcase 1. Suppose that $A_1=B_2=0$, or $B_1=A_2=0$.
Then (5.23) yields

$$f'/f=-1+2/(1-v_2)$$

where $v_2=V_3e^{d_1z}$ with V_3 rational, and d_1 a constant. If $d_1=0$, then obviously f'/f is rational. If $d_1\neq 0$ then f'/f has infinitely many poles at points where $v_2=1$, with residue

$$-2/v_2'=-2/((V_3'+d_1V_3)e^{d_1z})=-2/d_1+o(1).$$

So $2/d_1$ must be a positive integer, and V_3 must in fact be constant, and f is given by (1.3).

Subcase 2. Suppose that $A_1B_1A_2B_2 \neq 0$.

We shall show that in this case f has finite order. From (5.23) we see that there exist a real θ_1 and positive r_6, K_4 and K_5 such that for large z outside the semi-infinite "logarithmic" strips

$$\Omega_j = \{z: |z| > r_6, |\arg z + (-1)^j \theta_1| < K_4 \log|z|/|z|\} \tag{5.24}$$

we have

$$(f'-f)/(f'+f) = e^{-2z}\phi(z)$$

where $|\log|\phi(z)|| \leqq K_5$. This clearly gives

$$f'(z)/f(z) = O(1) \tag{5.25}$$

for large z outside Ω_1, Ω_2 and the semi-infinite strips Ω_3, Ω_4 given by $|z| > r_6$, $|Re(z)| < K_6$, say. For a large z outside the Ω_j this leads to

$$\log^+ |1/f(z)| = O(|z|). \tag{5.26}$$

Now setting $G = f'/f$, (5.23) yields $T(r, G) = O(r)$. By Theorem 3 of [**5**], we can certainly obtain

$$|G'(z)/G(z)| \leqq (\log r)^4 \tag{5.27}$$

for all z on $|z| = r$, where r lies outside a set of finite logarithmic measure. For such r, (5.25) and (5.27) lead to

$$G(z) = O(\exp(\log r)^Q)$$

on $|z| = r$, for some positive Q, and hence using (5.26) we obtain

$$\log^+ |1/f(z)| = O(\exp(\log r)^{2Q}) \tag{5.28}$$

on the same circle, if r is large. On the other hand, if Ω is one of Ω_1, Ω_2, Ω_3, Ω_4, and A_3, A_4 are suitably chosen constants, then by (5.26) the subharmonic function

$$\psi(z) = \log^+ |e^{A_3 z + A_4} f(z)^{-1}|$$

vanishes on the boundary of Ω. Estimates for harmonic measure [**4**, p. 104] now show that either ψ vanishes for all z in Ω, or

$$\liminf_{s\to\infty} m(s)\exp\left(-\pi\int_{r_6}^{s} dt/t\theta(t)\right)>0 \tag{5.29}$$

where $m(s)=\sup\{\psi(z):|z|=s,\ z \text{ in } \Omega\}$ and $\theta(t)$ is the angular measure of the intersection of Ω with $|z|=r$. But $\theta(t)=O((\log t)/t)$ so that (5.29) yields $\log m(s)>K_7 s/(\log s)$, say, which contradicts (5.28). Thus $\psi(z)\equiv 0$ on Ω and (5.26) holds for all large z.

Subcase 3. Suppose that exactly one of A_1, A_2, B_1, B_2 is zero.
Then without loss of generality (5.23) gives

$$(f'-f)/(f'+f)=A_1e^{-2z}+B_1Ve^{Bz} \tag{5.30}$$

or

$$(f'-f)/(f'+f)=(A_2e^{2z}+B_2Ve^{Bz})^{-1} \tag{5.31}$$

with B constant and V rational. We may assume that (5.30) holds for otherwise we need only set $f_3(z)=f(-z)$ and apply the following argument to f_3. Now (5.30) gives

$$f'/f=-1+2/(1-A_1e^{-2z}-B_1Ve^{Bz}). \tag{5.32}$$

If $B=-2$, then (5.32) shows that f'/f is bounded at least for large z outside a pair of logarithmic strips, so that we can apply the argument of the previous section to conclude that f has finite order. The same conclusion holds if $B\neq 0, -2$, except that we may need more than two, but at most six, logarithmic strips if B_1V does not vanish identically. Finally if $B=0$, we obtain

$$f'/f=-1+2/(V_4-A_1e^{-2z})$$

with V_4 rational. In this case either V_4 is identically zero, in which case f is given by (1.2) or f'/f has infinitely many poles at points where $V_4=A_1e^{-2z}$, with residue $2/(V_4'+2V_4)$. Thus a large pole ζ of f has multiplicity bounded by a power of $|\zeta|$, so that $N(r,f)$ has finite order. We can thus write $f=e^{g_0}/H_1$, where H_1 has finite order and g_0 is entire. But Lemma 8 of [2] gives $m(r,f'/f)=O(\log r)$, and g_0 is a polynomial. Now f has finite order and Theorem 2 is proved.

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