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RESEARCH ARTICLE

Global Asymptotics of the Sixth Painlevé Equation in Okamoto's Space

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Abstract
We study dynamics of solutions in the initial value space of the sixth Painlevé equation as the independent variable approaches zero. Our main results describe the repeller set, show that the number of poles and zeroes of general solutions is unbounded and that the complex limit set of each solution exists and is compact and connected.

Contents

1. Introduction

In this paper, we consider the celebrated equation

$$
\begin{aligned}
y'' = {} & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' \\
& + \frac{y(y-1)(y-x)}{2x^2(x-1)^2}\left(\theta_\infty^2 - \frac{\theta_0^2 x}{y^2} + \frac{\theta_1^2(x-1)}{(y-1)^2} + \frac{(1-\theta_x^2)x(x-1)}{(y-x)^2}\right),
\end{aligned} \tag{1.1}
$$

for $x \in \mathbb{C}$, $(\theta_0, \theta_1, \theta_x, \theta_\infty) \in \mathbb{C}^4$, in its initial value space, where initial values are given at a point $x_0 \in \mathbb{C}$, for small $|x_0|$. The equation is the sixth Painlevé equation, first derived in [6] from deformations of a linear system with four regular singular points, a generalisation of Gauss' hypergeometric equation; we will refer to it as $\mathrm{P_{VI}}$. Subsequently, it was recognised as the most general equation in the study of second-order ordinary differential equations (ODEs) whose movable singularities are poles [8, 28]. It has been studied widely because of its relation to mathematical physics and algebraic geometry (see [15, 22]). For special values of the parameters $(\theta_0, \theta_1, \theta_x, \theta_\infty)$, $\mathrm{P_{VI}}$ has algebraic and elliptic solutions that turn out to be related to a broad range of mathematical structures (see [2, 20] and references therein). For generic parameters, the solutions are higher transcendental functions that cannot be expressed in terms of algebraic or classical functions [32].

A large amount of work has been devoted to the description of these higher transcendental solutions. In this paper, we study global properties of such solutions of $\mathrm{P_{VI}}$ in the limit as $x \to 0$ in its initial value space (see Okamoto [25]). Under appropriate Möbius transformations of the variables [26], our results also apply in the limit as x approaches 1 or ∞. Further information and properties of $\mathrm{P_{VI}}$ are given in

Section 1.1 below. Our starting point is the equivalent nonautonomous Hamiltonian system

$$y' = \frac{\partial H}{\partial z}, \tag{1.2a}$$

$$z' = -\frac{\partial H}{\partial y}, \tag{1.2b}$$

with Hamiltonian

$$H = \frac{y(y-1)(y-x)}{x(x-1)}\left(z^2 - z\left(\frac{\theta_0}{y} + \frac{\theta_1}{y-1} + \frac{\theta_x - 1}{y-x}\right) + \frac{\theta\overline{\theta}}{y(y-1)}\right).$$

We will refer to the right side of Equations (1.2) as the Painlevé vector field and use the terminology

$$\theta := \frac{\theta_0 + \theta_x + \theta_1 + \theta_\infty - 1}{2}, \qquad \overline{\theta} := \theta - \theta_\infty.$$

To see that the system (1.2) is equivalent to P_{VI} (as shown by [27]), note that z is given by Equation (1.2a) as

$$2z := \left(\frac{x-1}{y} - \frac{x}{y-1} + \frac{1}{y-x}\right)y' + \frac{\theta_0}{y} + \frac{\theta_1}{y-1} + \frac{\theta_x - 1}{y-x}.$$

Substituting this into Equation (1.2b) gives P_{VI}. The Painlevé vector field becomes undefined at certain points in $\mathbb{C}^2$. Those points correspond to the following initial values of the system (1.2): $y = 0$ or $y = 1$ or $y = x$. Okamoto [25] showed how to regularise the system at such points. For each $x_0 \in \mathbb{C} \setminus \{0, 1\}$, he compactified the space of initial values $(y, z) \in (\mathbb{C} \setminus \{0, 1, x_0\}) \times \mathbb{C}$ to a smooth, complex surface $S(x_0)$. The flow of the Painlevé vector field is well-defined in $\mathcal{S} := \bigcup_{x_0 \in \mathbb{C}\setminus\{0,1\}} \mathcal{S}(x_0)$, which we refer to as Okamoto's space of initial values.

Our main purpose is to describe the significant features of the flow in the singular limit $x \to 0$. In similar studies of the first, second and fourth Painlevé equations [4, 16, 17] in singular limits, we showed that successive resolutions of the Painlevé vector field at base points terminate after nine blow-ups of $\mathbb{CP}^2$, while for the fifth and third Painlevé equations, we showed that the construction consists of 11 blow-ups and two blow-downs [18, 19]. The initial value space in each case is then obtained by removing the infinity set, denoted $\mathcal{I}$, which is blow-ups of points not reached by any solution.

Our main results fall into three parts:

(a) *Existence of a repeller set:* Corollary 6.10 in Section 6 shows that $\mathcal{I}$ is a repeller for the flow. Theorem 6.9 provides the range of the independent variable for which a solution may remain in the vicinity of $\mathcal{I}$.

(b) *Numbers of poles and zeroes:* In Corollary 6.10, we prove that each solution that is sufficiently close to $\mathcal{I}$ has a pole in a neighbourhood of the corresponding value of the independent variable. Moreover, Theorem 7.4 shows that each solution with essential singularity at $x = 0$ has infinitely many poles and infinitely many zeroes in each neighbourhood of that point.

(c) *The complex limit set:* We prove in Theorem 7.2 that the limit set for each solution is nonempty, compact, connected and invariant under the flow of the autonomous equation obtained as $x \to 0$.

1.1. Background

P_{VI} is the top equation in the well-known list of six Painlevé equations. Each of the remaining Painlevé equations can be obtained as a limiting form of P_{VI}.

To describe the complex analytic properties of their solutions, we recall that a normalised differential equation of the form $y'' = \mathcal{R}(y', y, x)$ gives rise to two types of singularities, that is, where the solution is not holomorphic. A solution may have a *fixed* singularity where $\mathcal{R}(\cdot, \cdot, x)$ fails to be holomorphic;

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in the case of P_{VI}, these lie at $x = 0, 1, \infty$. The solutions may also have *movable* singularities. A movable singularity is a singularity whose location changes in a continuous fashion when going from one solution to a neighbouring solution under small changes in the initial conditions. We note that this informal definition, which is somewhat difficult to make more precise, dates back to Fuchs [5, p.699].

P_{VI} was discovered by R. Fuchs in 1905 [6] in his study of deformations of a linear system of differential equations with four regular singularities, generalising Gauss' hypergeometric equation. The latter has three regular singularities, placed at 0, 1 and ∞ by convention, and Fuchs took the fourth one to be at a location which is deformable. The compatibility of the linear system with the deformation equation gives rise to P_{VI}. It is well-known that P_{VI} also has an elliptic form, which arises when we introduce an incomplete elliptic integral on a curve parametrised by $y(x)$. P_{VI} then becomes expressible in terms of the Picard-Fuchs equation for the corresponding elliptic curve. This form has been used for the investigation of its special solutions, which exist for special parameter values. This fact was rediscovered by Manin [22] in his study of the mirror symmetries of the projective plane.

Given a Painlevé equation and x not equal to a fixed singularity of the equation, Okamoto showed [25] that the space of initial values forms a connected, compactified and regularised space corresponding to a nine-point blow-up of the two–complex-dimensional projective space $\mathbb{CP}^2$. For each given x, this is recognisable as an elliptic surface. These elliptic surfaces form fibres of a vector bundle as x varies, with $\mathbb{C}$ as the base space. Starting with a point (initial value) on such a fibre, a solution of the Painlevé equation follows a trajectory that pierces each successive fibre, forming leaves of a foliated vector bundle [23].

1.2. Outline of the paper

The plan of the paper is as follows. In Section 3, we construct the surface $\mathcal{S}(x_0)$. We define the notation and describe the results, with detailed calculations being provided in Appendix B. In Section 4, we describe the corresponding vector field for the limit $x \to 0$. The movable singularities of P_{VI} correspond to points x_0 where the Painlevé vector field becomes unbounded. In Section 5, we consider neighbourhoods of exceptional lines where this occurs. Estimates of the Painlevé vector field as x approaches 0 are deduced in Section 6. In Section 7, we consider the limit set. Finally, we give concluding remarks in Section 8.

2. Resolution of singularities

In this section, we explain how to construct the space of initial values for the system (1.2). The notion of initial value spaces described in Definition 2.2 is based on foliation theory, and we start by first motivating the reason for this construction. We then explain how to construct such a space by carrying out resolutions or blow-ups, based on the process described in Definition 2.3.

The system (1.2) is a system of two first-order ODEs for $(y(x), z(x))$. Given initial values (y_0, z_0) at x_0, local existence and uniqueness theorems provide a solution that is defined on a local polydisk $U \times V$ in $\mathbb{C} \times \mathbb{C}^2$, where $x_0 \in U \subset \mathbb{C} \setminus \{0, 1\}$ and $(y_0, z_0) \in V \subset (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$. Our interest lies in global extensions of these local solutions. However, the occurrence of movable poles in the Painlevé transcendents acts as a barrier to the extension of $U \times V$ to the whole domain of (1.2). The first step to overcome this obstruction is to compactify the space $\mathbb{C}^2$, in order to include the poles. We carry this out by embedding $\mathbb{C}^2$ into the first Hirzebruch surface $\mathbb{F}_1$ [1, 14]. $\mathbb{F}_1$ is a projective space covered by four affine coordinate charts (given in Section 3).

The next step in this process results from the occurrence of singularities in the Painlevé vector field (1.2) in V. By the term *singularity,* we mean points where (y', z') becomes either unbounded or undefined because at least one component approaches the undefined limit $0/0$. We are led, therefore, to construct a space in which the points where the singularities appear are regularised. The process of regularisation is called "blowing up" or *resolving a singularity*.

The appearance of these singularities is related to the irreducibility of the solutions of Painlevé equations, originally due to Painlevé [28], which we have restated below in modern terminology.

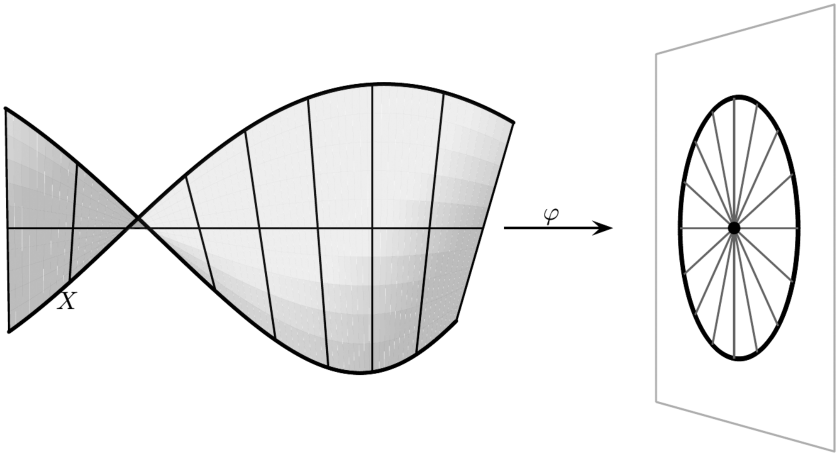

Figure 1. *The blow-up of the plane at a point.*

A function is said to be reducible to another function if it is related to it through a series of allowable operations (described by Painlevé and itemised as (O), (P1)–(P5) in [30, p.33]).

Theorem 2.1. *If the space of initial values for a differential equation is a compact rational variety, then the equation can be reduced either to a linear differential equation of higher order or to an equation governing elliptic functions.*

Modern proofs of the irreducibility of the Painlevé equations have been developed by many authors, including Malgrange [21], Umemura [30, 31] and Watanabe [32]. Since the Hirzebruch surface is a compact rational variety, the above theorem implies that it cannot be the space of initial values for (1.2). We are now in a position to define the notion of initial value space.

Definition 2.2 [9–11, 25]. Let $(\mathcal{E}, \pi, \mathcal{B})$ be a complex analytic fibration, Φ a foliation of $\mathcal{E}$ and Δ a holomorphic differential system on $\mathcal{E}$, such that:

- the leaves of Φ correspond to the solutions of Δ;
- the leaves of Φ are transversal to the fibres of $(\mathcal{E}, \pi, \mathcal{B})$;
- for each path p in the base $\mathcal{B}$ and each point $X \in \mathcal{E}$, such that $\pi(X) \in p$, the path p can be lifted into the leaf of Φ containing point X.

Then each fibre of the fibration is called *a space of initial values* for the system Δ.

The properties listed in Definition 2.2 imply that each leaf of the foliation is isomorphic to the base $\mathcal{B}$. Since the transcendental solutions of the sixth Painlevé equation can be globally extended as meromorphic functions of $x \in \mathbb{C} \setminus \{0, 1\}$, we search for the fibration with the base equal to $\mathbb{C} \setminus \{0, 1\}$.

In order to construct the fibration, we apply the blow-up procedure defined below [3, 12, 13] to the singularities of the system (1.2) that occur where at least one component becomes undefined of the form $0/0$. Okamoto [25] showed that such singular points are contained in the closure of infinitely many leaves. Moreover, these leaves are holomorphically extended at such a point.

Definition 2.3. *The blow-up* of the plane $\mathbb{C}^2$ at point $(0, 0)$ is the closed subset X of $\mathbb{C}^2 \times \mathbb{CP}^1$ defined by the equation $u_1 t_2 = u_2 t_1$, where $(u_1, u_2) \in \mathbb{C}^2$ and $[t_1 : t_2] \in \mathbb{CP}^1$ (see Figure 1). There is a natural morphism $\varphi : X \to \mathbb{C}^2$, which is the restriction of the projection from $\mathbb{C}^2 \times \mathbb{CP}^1$ to the first factor. $\varphi^{-1}(0, 0)$ is the projective line $\{(0, 0)\} \times \mathbb{CP}^1$, called *the exceptional line*.

Remark 2.4. Notice that the points of the exceptional line $\varphi^{-1}(0, 0)$ are in bijective correspondence with the lines containing $(0, 0)$. On the other hand, φ is an isomorphism between $X \setminus \varphi^{-1}(0, 0)$ and $\mathbb{C}^2 \setminus \{(0, 0)\}$. More generally, any complex two-dimensional surface can be blown up at a point [3, 12, 13].

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In a local chart around that point, the construction will look the same as described for the case of the plane.

Notice that the blow-up construction separates the lines containing the point $(0, 0)$ in Definition 2.3, as shown in Figure 1. In this way, the solutions of (1.2) containing the same point can be separated. Additional blow-ups may be required if the solutions have a common tangent line or a tangency of higher order at such a point. The explicit resolution of the vector field (1.2) is carried out in Appendix B. Okamoto described so-called *singular points of the first class* that are not contained in the closure of any leaf of the foliation given by the system of differential equations. At such points, the corresponding vector field is infinite.

3. The construction of Okamoto's space

In this section, we construct Okamoto's space of initial values in such a way as to ensure that the process yields a well-defined compact surface if we set $x_0 = 0$. We start by defining a new time coordinate $t = \ln x$, or $x = \exp(t)$, suitable for taking the limit $x \to 0$, and rewrite the dependent variables as

$$u(t) = y(x), \ v(t) = z(x).$$

For conciseness, we continue to use the notation $x = e^t$ where needed.

Denoting t-derivatives by dots, we get $\dot{u} = x\frac{\partial H}{\partial v}$, $\dot{v} = -x\frac{\partial H}{\partial u}$, or, equivalently

$$\dot{u} = \frac{\partial E}{\partial v}, \ \dot{v} = -\frac{\partial E}{\partial u}, \tag{3.1}$$

where

$$E = \frac{u(u-1)(u-e^t)}{e^t - 1}\left\{v^2 - v\left(\frac{\theta_0}{u} + \frac{\theta_1}{u-1} + \frac{\theta_x - 1}{u - e^t}\right) + \frac{\theta\overline{\theta}}{u(u-1)}\right\}.$$

Suppose we are given $x_0 = e^{t_0} \in \mathbb{C} \setminus \{0, 1\}$. We compactify the space of initial values $(u(t_0), v(t_0)) \in \mathbb{C}^2$ to the first Hirzebruch surface $\mathbb{F}_1$ [14], which is covered by four affine charts in $\mathbb{C}^2$ [1]

$$(u_0, v_0) = (u, v), \qquad (u_1, v_1) = \left(u, \frac{1}{v}\right),$$
$$(u_2, v_2) = \left(\frac{1}{u}, \frac{1}{uv}\right), \qquad (u_3, v_3) = \left(\frac{1}{u}, uv\right).$$

Let L be the unique section of the natural projection $\mathbb{F}_1 \to \mathbb{P}^1$ defined by $(u, v) \mapsto u$. Then, L is given by $\{v_0 = 0\} \cup \{v_3 = 0\}$ and the self-intersection of its divisor class is -1. We identify four particular fibres of this projection:

$$\mathcal{V}_j := \{u_0 = j\} \cup \{u_1 = j\} \quad \forall j \in \{0, x, 1\}, \quad \mathcal{D}_\infty := \{u_2 = 0\} \cup \{u_3 = 0\}.$$

Note that as fibres of the projection, these lines all have self-intersection 0. Then $\mathbb{F}_1 \setminus \mathbb{C}^2$ is given by $\mathcal{D}_\infty \cup \mathcal{H}$, where

$$\mathcal{H} := \{v_1 = 0\} \cup \{v_2 = 0\}.$$

This section $\mathcal{H}$, called a 'horizontal line' in the following, by a small abuse of common terminology, is topologically equivalent to the formal sum $L + \mathcal{D}_\infty$ in $\mathrm{H}_2(\mathbb{F}_1, \mathbb{Z})$. In particular, its self-intersection number is given by $\mathcal{H} \cdot \mathcal{H} = L \cdot L + \mathcal{D}_\infty \cdot \mathcal{D}_\infty + 2L \cdot \mathcal{D}_\infty = -1 + 0 + 2 = +1$, where the dot $\cdot$ denotes

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Table 1. *Five base points and the charts in which they are visible. The chart (u_0, v_0) is omitted because no base points are visible in this chart.*

Points \ Charts	(u_1, v_1)	(u_2, v_2)	(u_3, v_3)
β_0	$(0, 0)$		
β_x	$(x, 0)$	$(1/x, 0)$	
β_1	$(1, 0)$	$(1, 0)$	
β_∞		$(0, 1/\theta)$	$(0, \theta)$
β_∞^-		$\left(0, 1/\overline{\theta}\right)$	$\left(0, \overline{\theta}\right)$

the intersection form of divisor classes in the Picard group of the surface. In each chart, the vector field, respectively, becomes

$$\begin{cases} \dot{u} &= \dfrac{u(u-1)(u-x)}{x-1}\left(2v - \dfrac{\theta_0}{u} - \dfrac{\theta_1}{u-1} - \dfrac{\theta_x - 1}{u-x}\right), \\ \dot{v} &= -\dfrac{3u^2 - 2(x+1)u + x}{x-1}v^2 + 2\dfrac{\theta + \overline{\theta}}{x-1}uv - \left(\dfrac{x\theta_0}{x-1} + \theta_1 + \dfrac{\theta+\overline{\theta}}{x-1}\right)v - \dfrac{\theta\overline{\theta}}{x-1}, \end{cases}$$

$$\begin{cases} \dot{u}_1 &= \dfrac{u_1(u_1-1)(u_1-x)}{x-1}\left(2\dfrac{1}{v_1} - \dfrac{\theta_0}{u_1} - \dfrac{\theta_1}{u_1-1} - \dfrac{\theta_x - 1}{u_1-x}\right), \\ \dot{v}_1 &= \dfrac{3u_1^2 - 2(x+1)u_1 + x}{x-1} - 2\dfrac{\theta + \overline{\theta}}{x-1}u_1v_1 + \left(\dfrac{x\theta_0}{x-1} + \theta_1 + \dfrac{\theta+\overline{\theta}}{x-1}\right)v_1 + \dfrac{\theta\overline{\theta}}{x-1}v_1^2, \end{cases}$$

$$\begin{cases} \dot{u}_2 &= 2\dfrac{(u_2-1)(xu_2-1)}{(1-x)v_2} + \dfrac{(u_2-1)(\theta+\overline{\theta} - x\theta_0 u_2)}{1-x} - \theta_1 u_2, \\ \dot{v}_2 &= -\dfrac{(\theta v_2 - 1)(\overline{\theta}v_2 - 1)}{(1-x)u_2} - \dfrac{x}{1-x}(\theta_0 v_2 - 1)u_2, \end{cases}$$

$$\begin{cases} \dot{u}_3 &= -\dfrac{(u_3-1)(2v_3 - (\theta+\overline{\theta}))}{1-x} - \theta_1 u_3 + \dfrac{x}{1-x}u_3(u_3-1)(2v_3 - \theta_0), \\ \dot{v}_3 &= \dfrac{(v_3-\theta)(v_3-\overline{\theta})}{(1-x)u_3} - x\dfrac{(v_3-\theta_0)u_3v_3}{1-x}. \end{cases}$$

One realises that the vector field is infinite on $\mathcal{H} : \{v_1 = 0\} \cup \{v_2 = 0\}$. More precisely, it is infinite or undetermined precisely there. We use the term *base point* for points where the vector field becomes undetermined. For example, the point $(u_1, v_1) = (0, 0)$ in the coordinate chart $\mathbb{C}^2_{u_1,v_1}$ is a base point because the equation for $\dot{u}_1$ approaches $0/0$ as $(u_1, v_1) \to (0, 0)$. In total, we find the following five base points in $\mathbb{F}_1$, possibly visible in several charts. This initial situation is summarised in Table 1 and Figure 2. Where needed in figures, we indicate the self-intersection number n of an exceptional divisor by annotating it by (n).

Okamoto's procedure consists in resolving the vector field by successively blowing up the base points until the vector field becomes determined. Since later on we need a well-defined compact surface if we set $x = 0$, we may not blow up β_0 and β_x simultaneously. As detailed in Appendix B.1.1, the blow-up of $\beta_0, \beta_x, \beta_1$ with β_x after β_0 consists of replacing the charts $\mathbb{C}^2_{u_1,v_1}$ and $\mathbb{C}^2_{u_2,v_2}$ by the following five $\mathbb{C}^2$-charts, endowed with the obvious rational transition maps,

$$\begin{aligned} (\tilde{u}_1, \tilde{v}_1) &:= \left(u_1, \tfrac{v_1}{u_1(u_1-1)(u_1-x)}\right) & (\tilde{u}_2, \tilde{v}_2) &:= \left(u_2, \tfrac{v_2}{(1-u_2)(1-xu_2)}\right) \\ (u_{02}, v_{02}) &:= \left(\tfrac{u_1}{v_1}, v_1\right) & (u_{12}, v_{12}) &:= \left(\tfrac{u_1-1}{v_1}, v_1\right) \\ (u_{x2}, v_{x2}) &:= \left(\tfrac{u_1(u_1-x)}{v_1}, \tfrac{v_1}{u_1}\right) \end{aligned}$$

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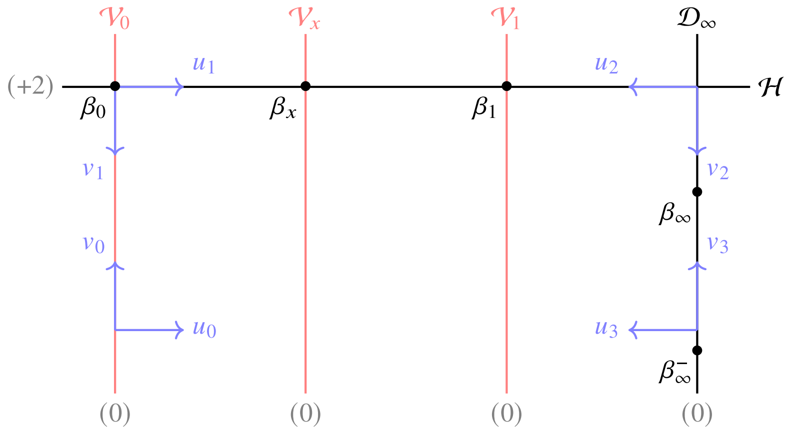

Figure 2. *The surface $\mathbb{F}_1$ with its coordinates and the base point configuration. The numbers in parentheses indicate self-intersection numbers.*

For each $i \in \{0, 1, x\}$, what formerly was the point β_i is now replaced by an exceptional line

$$\mathcal{D}_i : \{\tilde{u}_1 = i\} \cup \{v_{i2} = 0\}$$

of self-intersection -1. The strict transform of $\mathcal{H}$, that is the closure of $\mathcal{H} \setminus \{\beta_0, \beta_x, \beta_1\}$ after blow-up is given by

$$\mathcal{H}^* := \{\tilde{v}_1 = 0\} \cup \{\tilde{v}_2 = 0\}\,.$$

As a general fact, each time we blow up a point on a curve, the self-intersection number of the strict transformation of the curve is the former self-intersection number decreased by unity. Since here we have blown up three points, $\mathcal{H}^*$ has self-intersection number (-2). The blow-up of β_∞ consists of removing the point $(0, 1/\theta)$ (corresponding to β_∞) from the chart $\mathbb{C}^2_{\tilde{u}_2,\tilde{v}_2}$ and replacing the chart $\mathbb{C}^2_{u_3,v_3}$ by the following pair of $\mathbb{C}^2$-charts.

$$(\tilde{u}_3, \tilde{v}_3) := \left(u_3, -\tfrac{v_3-\theta}{u_3}\right) \quad (u_{\infty 2}, v_{\infty 2}) := \left(\tfrac{u_3}{v_3-\theta}, v_3 - \theta\right)$$

Again, we obtain an exceptional line $\mathcal{E}_\infty$ and a strict transform $\mathcal{D}^*_\infty$, such that $\mathcal{D}_\infty = \mathcal{E}_\infty \cup \mathcal{D}^*_\infty$, where

$$\mathcal{E}_\infty := \{\tilde{u}_3 = 0\} \cup \{v_{\infty 2} = 0\}\,, \quad \mathcal{D}^*_\infty = \{\tilde{u}_2 = 0\} \cup \{u_{\infty 2} = 0\}\,.$$

In each of the seven new charts that we have to add to $\mathbb{C}^2_{u,v}$ in order to fully describe the surface resulting of $\mathbb{F}_1$ after this first sequence of blow-ups, we again look at the resulting vector field (see Section B.1.2) and find the following base points, including the still unresolved β^-_∞. The situation is summarised in Table 2.

In Figure 3, the notation '(n)' again indicates 'self-intersection number equal to n'. Moreover, as a visual guideline, we again included the strict transforms $\mathcal{V}^*_i := \{u_{i2} = 0\} \cup \{u_0 = i\}$ of the former vertical lines $\mathcal{V}_i$. Those have self-intersection (-1).

We blow-up the remaining base points by replacing each chart $\mathbb{C}^2_{u_{i2},v_{i2}}$ by a pair of $\mathbb{C}^2$-charts with corresponding index as follows, and then removing the already blown up base points that are still visible

Table 2. *Base points remaining after blowing up $\beta_0, \beta_x, \beta_1$ and β_∞. The chart $(\tilde{u}_3, \tilde{v}_3)$ is ommitted as there is no base point remaining in this chart.*

Charts \ Points	γ_0	γ_x	γ_1	β_∞^-
$(\tilde{u}_1, \tilde{v}_1)$	$\left(0, \frac{1}{x\theta_0}\right)$	$\left(x, \frac{1}{x(x-1)\theta_x}\right)$	$\left(1, \frac{1}{(1-x)\theta_1}\right)$	
$(\tilde{u}_2, \tilde{v}_2)$		$\left(\frac{1}{x}, \frac{x}{(x-1)\theta_x}\right)$	$\left(1, \frac{1}{(1-x)\theta_1}\right)$	$\left(0, \frac{1}{\theta}\right)$
(u_{02}, v_{02})	$(\theta_0, 0)$			
(u_{x2}, v_{x2})	$\left(-x\theta_0, \frac{1}{\theta_0}\right)$	$(x\theta_x, 0)$		
(u_{12}, v_{12})			$(\theta_1, 0)$	
$(u_{\infty 2}, v_{\infty 2})$				$(0, -\theta_\infty)$

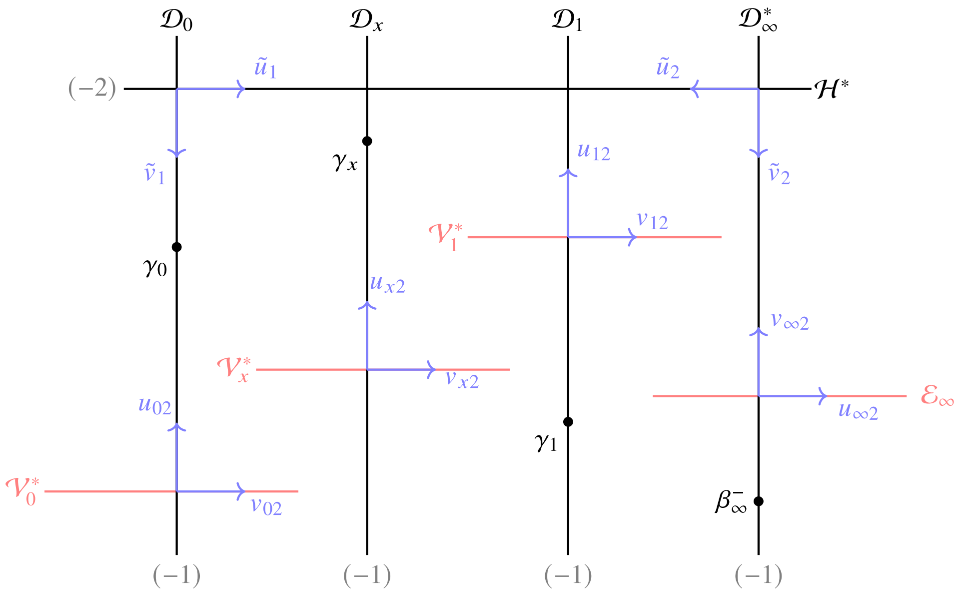

Figure 3. *The surface $\mathbb{F}_1$ after the first sequence of blow-ups and the new base point configuration.*

from other charts.

$$
\begin{aligned}
(u_{03}, v_{03}) &:= \left(u_{02} - \theta_0, \frac{v_{02}}{u_{02} - \theta_0}\right) & (u_{04}, v_{04}) &:= \left(\frac{u_{02} - \theta_0}{v_{02}}, v_{02}\right)\\
(u_{x3}, v_{x3}) &:= \left(u_{x2} - x(\theta_x - 1), \frac{v_{x2}}{u_{x2} - x(\theta_x - 1)}\right) & (u_{x4}, v_{x4}) &:= \left(\frac{u_{x2} - x(\theta_x - 1)}{v_{x2}}, v_{x2}\right)\\
(u_{13}, v_{13}) &:= \left(u_{12} - \theta_1, \frac{v_{12}}{u_{12} - \theta_1}\right) & (u_{14}, v_{14}) &:= \left(\frac{u_{12} - \theta_1}{v_{12}}, v_{12}\right)\\
(u_{\infty 3}, v_{\infty 3}) &:= \left(u_{\infty 2}, \frac{v_{\infty 2} + \theta_\infty}{u_{\infty 2}}\right) & (u_{\infty 4}, v_{\infty 4}) &:= \left(\frac{u_{\infty 2}}{v_{\infty 2} + \theta_\infty}, v_{\infty 2}\right)
\end{aligned}
$$

We obtain the following new exceptional lines, for $i \in \{0, 1, x\}$.

$$
\mathcal{E}_i := \{u_{i3} = 0\} \cup \{v_{i4} = 0\}, \quad \mathcal{E}_\infty^- := \{u_{\infty 3} = 0\} \cup \{v_{\infty 4} = 0\}.
$$

Moreover, we have the following new strict transforms, for $i \in \{0, 1, x\}$.

$$
\mathcal{D}_i^* := \{v_{i3} = 0\} \cup \{\tilde{u}_1 = i\}, \quad \mathcal{D}_\infty^{**} := \{\tilde{u}_2 = 0\} \cup \{u_{\infty 4} = 0\}.
$$

The above charts of the Hirzebruch surface blown up in our eight base points are detailed in the Appendix Section B.2. As we can see from the equations there, the vector field is now free of base points. We say

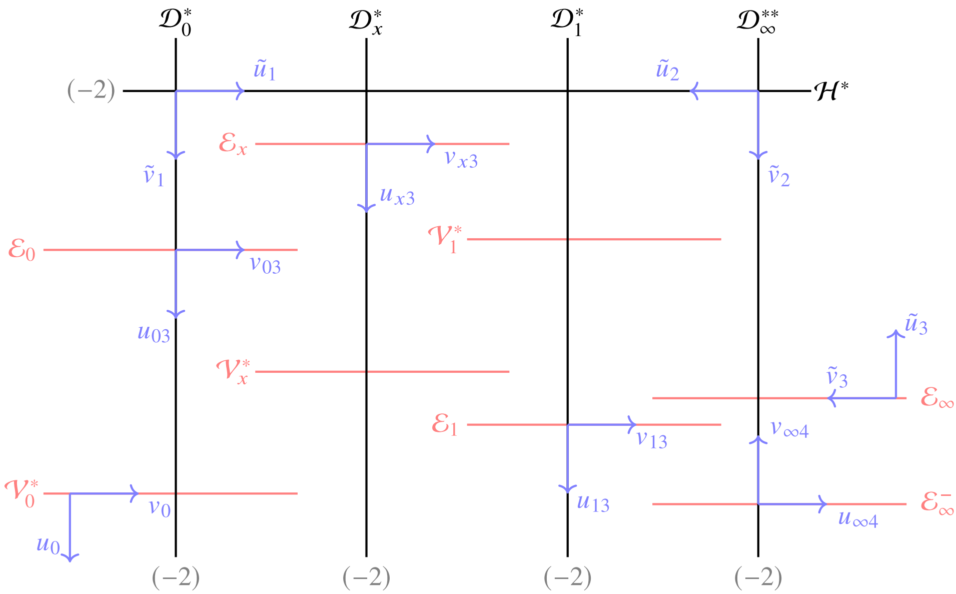

Figure 4. *The space of initial values of the resolved Painlevé VI vector field for $x \neq 0$.*

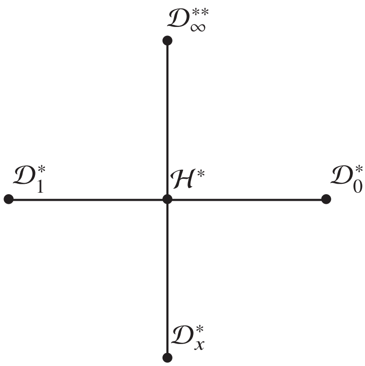

Figure 5. *The Dynkin diagram with nodes representing (-2)-lines in Okamoto's space, for $x \neq 0, 1$, is equivalent to that for $D_4^{(1)}$.*

that the initial value space is resolved or regularised. Moreover, the function E is well-defined there, that is when resolving the base points of the vector field, we also resolved the indeterminacy points of E. For each of the new coordinate charts (u_{mn}, v_{mn}), we also define the Jacobian

$$\omega_{mn} = \frac{\partial u_{mn}}{\partial u}\frac{\partial v_{mn}}{\partial v} - \frac{\partial u_{mn}}{\partial v}\frac{\partial v_{mn}}{\partial u}.$$

Figure 4 illustrates a schematic drawing of the resultant collection of exceptional lines, $\mathcal{H}^*$ and D_∞^{**} and their intersections in the resolved space, as well as the coordinates that will be most important in the following. For each $x = x_0 \neq 0, 1$, this regularised space will be denoted as $\mathcal{S}(x_0)$. Moreover, we define $\mathcal{S}(0)$ to be the result of the blow-up procedure for $x = 0$. Its relation to the vector field is studied in the next section. The union of $S(x_0)$ forms a fibre bundle

$$\mathcal{S} := \bigcup_{x_0 \in \mathbb{C}\setminus\{1\}} \mathcal{S}(x_0).$$

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From the detailed charts in the Appendix Section B.2, one sees that for $x_0 \neq 0$, the Painlevé vector field is 'vertical' or tangent to the lines $\mathcal{H}^*$, $\mathcal{D}_\infty^{**}$, $\mathcal{D}_0^*$, $\mathcal{D}_x^*$, $\mathcal{D}_1^*$, which each have self-intersection -2. For this reason, such curves are often referred to as 'vertical leaves' in Okamoto's construction. For each $x = x_0 \neq 0, 1$, we define

$$\mathcal{I}(x_0) := \mathcal{H}^* \cup \mathcal{D}_\infty^{**} \cup \mathcal{D}_0^* \cup \mathcal{D}_x^* \cup \mathcal{D}_1^*$$

the *infinity set*, corresponding to the black part of the diagram shown in Figure 4. Okamoto's space of initial values for $x_0 \neq 0, 1$ is $\mathrm{Oka}(x_0) := \mathcal{S}(x_0) \setminus \mathcal{I}(x_0)$.

Note that the strict transforms $\mathcal{H}^*$, $\mathcal{D}_\infty^{**}$, $\mathcal{D}_0^*$, $\mathcal{D}_x^*$, $\mathcal{D}_1^*$ each have self-intersection -2. The corresponding Dynkin diagram reflecting their intersections, given in Figure 5, is equivalent to that for $D_4^{(1)}$.

4. The vector field in the limit space

When $x \longrightarrow 0$ (or, more precisely, $\Re(t) \longrightarrow -\infty$), we get the autonomous limiting system

$$\begin{cases} \dot{u} = -u\{(u-1)(2uv - 2\theta + \theta_\infty) - \theta_1\}, \\ \dot{v} = uv((3u-2)v - 4\theta + 2\theta_\infty) + (2\theta - \theta_\infty - \theta_1)v + \theta(\theta - \theta_\infty), \end{cases} \tag{4.1}$$

where $\dot{u} = \partial E_0/\partial v$, $\dot{v} = -\partial E_0/\partial u$, with

$$E_0 := -u\{(u-1)v(uv - 2\theta + \theta_\infty) - \theta_1 v + \theta(\theta - \theta_\infty)\}.$$

We can solve this Hamiltonian system completely: if the values of the θ_i's are generic, that is if they belong to an open dense subset of the set of all possible values of those parameters, we obtain a one-parameter family of solutions that lies on the line $\{u = 0\}$. Again for generic θ_i values, no solutions lie on the line $\{u = 1\}$. Let us assume $u \not\equiv 0, 1$. Then the Hamiltonian system (4.1) yields

$$v = -\frac{\dot{u}}{2u^2(u-1)} + \frac{\theta_1}{2u(u-1)} + \frac{\theta + \overline{\theta}}{2u},$$

leading to

$$\ddot{u} = \frac{3u-2}{2u(u-1)}\dot{u}^2 + \frac{\theta_\infty^2}{2}(u-1)u^2 - \frac{\theta_1^2 u^2}{2(u-1)}.$$

Note that if $(u(t), v(t))$ is a solution of the autonomous Hamiltonian system, then $\eta_0 := E_0(u(t), v(t))$ is constant. Setting $u_3 := 1/u$, the autonomous differential equation for u yields $(\dot{u}_3)^2 = \alpha u_3^2 + \beta u_3 + \gamma$, where $\alpha = 4\eta_0 + (\theta + \overline{\theta} - \theta_1)^2$, $\beta = \theta_1^2 - \theta_\infty^2 - \alpha$ and $\gamma = \theta_\infty^2$. This integrates as

$$u_3(t) = \begin{cases} \frac{\sqrt{4\gamma\alpha - \beta^2}\sinh(\sqrt{\alpha}\, t + \eta_1) - \beta}{2\alpha} & \text{if} \quad \alpha \neq 0 \\ \left(\frac{\sqrt{\beta}}{2}\, t + \eta_1\right)^2 - \frac{\gamma}{\beta} & \text{if} \quad \alpha = 0\,, \end{cases}$$

where η_1 is an arbitrary integration constant. In particular, we find the following list of equilibrium points (trajectories reduced to one point) of the autonomous Hamiltonian system for generic θ_i's:

$$\begin{cases} (u, v) = \left(\frac{\theta_\infty - \theta_1}{\theta_\infty}, \frac{\theta_\infty \overline{\theta}}{\theta_\infty - \theta_1}\right) \text{ for } \eta_0 = \frac{(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x - 1)}{4} \\ (u, v) = \left(\frac{\theta_\infty + \theta_1}{\theta_\infty}, \frac{\theta_\infty \theta}{\theta_\infty + \theta_1}\right) \text{ for } \eta_0 = \frac{(\theta_1 + \theta_\infty)^2 - (\theta_0 + \theta_x - 1)}{4}\,. \end{cases}$$

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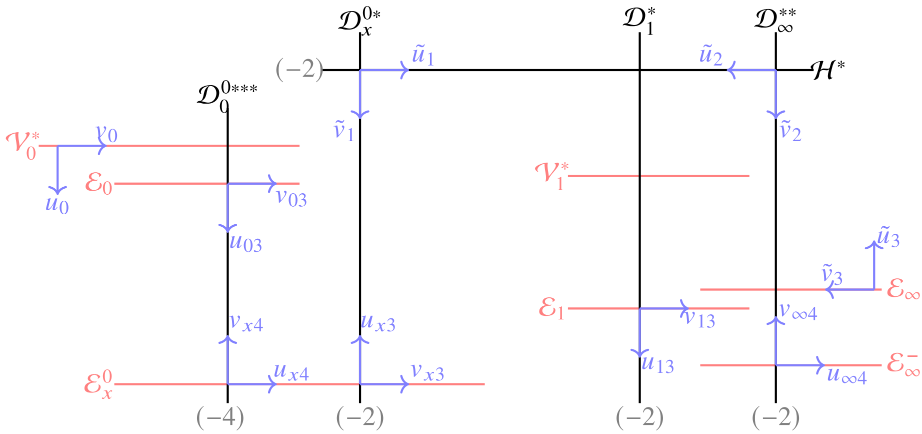

Figure 6. *The limit space for $x = 0$ of the space of initial values for $x \neq 0$.*

We may now compactify the space of initial values $\mathbb{C}^2_{u,v}$ to $\mathcal{S}(0)$. Figure 6 contains a schematic drawing of how the limits of the components of the infinity set and the exceptional lines arrange in this space. Here, as usual, red lines have self-intersection (-1). The notable differences with the configuration in $\mathcal{S}(x)$ with $x = 0$ are the following, where we use the superscript '0' when convenient to indicate particularities for the $x = 0$ case:

- After blow-up of $\beta_0 : (u_1, v_1) = (0,0)$, the point $\beta_x^0 : (u_{01}, v_{01}) = (0,0)$ which has to be blown up lies on the intersection of (the strict transform) of $\mathcal{H}$ and the exceptional line $\mathcal{D}_0 = \mathcal{D}_0^0$.
- As a result, we still have $\mathcal{H}^* = \{\tilde{v}_1 = 0\} \cup \{\tilde{v}_2 = 0\}$, but $\mathcal{D}_0^0 = \mathcal{D}_0^{0*} \cup \mathcal{D}_x^0$.
- Moreover, the point $\gamma_x^0 : (u_{x2}, v_{x2}) = (0,0)$ now corresponds to the intersection $\mathcal{D}_0^{0*} \cap \mathcal{D}_x^0$.
- As a result, we still have $\mathcal{D}_x^{0*} = \{v_{x3} = 0\} \cup \{\tilde{u}_1 = 0\}$, but $\mathcal{D}_0^{0*} = \mathcal{D}_0^{0**} \cup \mathcal{E}_x^0$, where $\mathcal{E}_x^0 : \{u_{x3} = 0\} \cup \{v_{x4} = 0\}$.
- Finally, the blow-up of $\gamma_0^0 : (u_{02}, v_{02}) = (\theta_0, 0)$ yields the strict transform $\mathcal{D}_0^{0***} : \{v_{03} = 0\} \cup \{u_{x4} = 0\}$ of self-intersection (-4).

The resulting autonomous vector field in $\mathcal{S}(0)$ is obtained from the one in the Appendix Section B.2 by systematically setting $x = 0$. For convenience of the reader, the formulae are given in the Appendix Section D.

We use the term *elliptic base points* for a point where the induced autonomous vector field in $\mathcal{S}(0)$ is undetermined. There is one such elliptic base point, given by

$$\mathfrak{u} : (u_{x4}, v_{x4}) = (0,0) \in \mathcal{D}_0^{***} \cap \mathcal{E}_x^0 .$$

This elliptic base point cannot be resolved by blow-ups![1] Note, however, that the autonomous energy function E_0 is well-defined (and infinite) at $\mathfrak{u}$.

Let us denote $\mathcal{I}^0$ the subset of $\mathcal{S}(0)$ where the autonomous vector field is infinite or undefined. We find

$$\mathcal{I}^0 = \mathcal{E}_x^0 \cup \mathcal{D}_x^{0*} \cup \mathcal{H}^* \cup \mathcal{D}_1^* \cup \mathcal{D}_\infty^{**}.$$

This set corresponds precisely to the points where the autonomous energy function E_0 is infinite. As explained above, we have

$$\lim_{x \to 0} \mathcal{I}(x) = \mathcal{D}_0^{0***} \cup \mathcal{I}^0 .$$

[1]Moreover, when following through the process of Okamoto desingularisation, one realises that $\gamma_0^0 : (u_{02}, v_{02}) = (\theta_0, 0)$ was in fact not an elliptic base point.

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In order to complete the description of the autonomous vector field in $\mathcal{S}(0) \setminus \{\mathfrak{u}\}$, it remains to investigate trajectories that might be contained in $\mathcal{S}(0) \setminus \left(\mathcal{I}^0 \cup \mathbb{C}^2_{u_0, v_0}\right)$. We find the following, where, as usual, we assume the values of the θ_i's to be generic:

- There is no trajectory contained in any of the following:
  - $\mathcal{E}_\infty \setminus \mathcal{D}^{**}_\infty : \{\tilde{u}_3 = 0\}$,
  - $\mathcal{E}^-_\infty \setminus \mathcal{D}^{**}_\infty : \{u_{\infty 3} = 0\}$,
  - $\mathcal{E}_1 \setminus \mathcal{D}^*_1 : \{v_{14} = 0\}$.
- The line $\mathcal{E}^0_0 \setminus \mathcal{D}^{0***}_0 : \{v_{04} = 0\}$ is the union of one trajectory and one equilibrium point, given by

$$u_{04} = -\frac{(\theta_0 - \theta)(\theta_0 - \overline{\theta})\theta_0}{2\theta_0 + \theta_1 - (\theta + \overline{\theta})}$$

  with energy $\eta_0 = -(\theta_x - 1)\theta_0$.
- Every point of $\mathcal{D}^{0***}_0 \setminus \mathcal{E}^0_x : \{v_{03} = 0\}$ is an equilibrium point of the autonomous vector field, with energy $\eta_0 = (u_{03} - (\theta_x - 1))(u_{03} + \theta_0)$.

5. Movable singularities in the Okamoto's space

In this section, we will consider neighbourhoods of exceptional lines where the Painlevé vector field becomes unbounded. The construction given in Appendix B shows that these are given by the lines $\mathcal{E}_0$, $\mathcal{E}_x$, $\mathcal{E}_1$, $\mathcal{E}_\infty$, $\mathcal{E}^-_\infty$.

5.1. Points where u has a zero and v a pole.

The set $\mathcal{E}_0 \setminus \mathcal{I}$ is given by $\{v_{04} = 0\}$, in the (u_{04}, v_{04}) chart (see Section B.2.6). Suppose $u_{04}(\tau) = B$, $v_{04}(\tau) = 0$, for some complex numbers τ, B. From the system of differential equations in Section B.2.6, we get:

$$\begin{aligned} v_{04}(t) = {} & \frac{e^\tau}{e^\tau - 1}(t-\tau) - e^\tau \frac{\theta_0(1+e^\tau) - \theta_x - \theta_1 e^\tau + 2}{2(e^\tau - 1)^2}(t-\tau)^2 \\ & + \left(\frac{2Be^{2\tau}(1+e^\tau)}{3(1-e^\tau)^3} + F_1(\tau)\right)(t-\tau)^3 + O((t-\tau)^4), \end{aligned}$$

with

$$\begin{aligned} F_1(\tau) = {} & -\frac{e^\tau}{6(1-e^\tau)^3}\Big[3(\theta_0 - \theta_x + 1) + (\theta_0 - \theta_x)^2 + 3e^\tau - 8\theta_0(\theta_x + \theta_1)e^\tau + 13\theta_0 e^\tau - 2\theta_x e^\tau \\ & + \theta_1(2\theta_x - 5)e^\tau + 2(\theta + \theta_0)(\bar{\theta} + \theta_0)e^\tau + (\theta_0 - \theta_1)^2 e^{2\tau}\Big]. \end{aligned}$$

Since (see Section B.2.6)

$$u = u_{04}v_{04}^2 + \theta_0 v_{04}, \quad v = \frac{1}{v_{04}},$$

we obtain the series expansions for (u, v):

$$\begin{cases} u(t) = \frac{\theta_0 e^\tau}{e^\tau - 1}(t-\tau) + \left(B\frac{e^{2\tau}}{(e^\tau-1)^2} - \theta_0 e^\tau \frac{\theta_0(1+e^\tau) - \theta_x - \theta_1 e^\tau + 2}{2(e^\tau-1)^2}\right)(t-\tau)^2 + O((t-\tau)^3), \\ v(t) = \frac{e^\tau - 1}{e^\tau}\frac{1}{t-\tau} + \frac{\theta_0(1+e^\tau) - \theta_x - \theta_1 e^\tau + 2}{2e^\tau} + \left(\frac{2B(e^\tau+1)}{3(e^\tau-1)} + F_2(t)\right)(t-\tau) + O((t-\tau)^2), \end{cases}$$

with

$$F_2(\tau) = \frac{1}{12e^\tau(e^\tau-1)}\Big[(\theta_0-\theta_x+3)^2-3-4(\theta-\theta_0)(\bar\theta-\theta_0)e^\tau+4\theta_0^2e^\tau-2(2\theta_0+1)^2e^\tau$$
$$+2(\theta_0+\theta_x-1)(\theta_0+\theta_1+2)e^\tau+(\theta_0-\theta_1)^2e^{2\tau}\Big].$$

Note that u has a simple zero at $t=\tau$ and v a simple pole with residue $1-e^{-\tau}$.

5.2. *Points where* $u=1$ *and* v *has a pole.*

The set $\mathcal{E}_1\setminus\mathcal{I}$ is given by $\{v_{14}=0\}$, in the (u_{14},v_{14}) chart (see Section B.2.10). Suppose $u_{14}(\tau)=B$, $v_{14}(\tau)=0$, for some complex numbers τ, B. From the system of differential equations in Section B.2.10, we get:

$$v_{14}(t) = -(t-\tau)+\frac{\theta+\bar\theta+\theta_1(e^\tau-3)-\theta_0e^\tau}{2(e^\tau-1)}(t-\tau)^2+\left(\frac{2B(e^\tau-2)}{3(1-e^\tau)}+F_3(\tau)\right)(t-\tau)^3+O((t-\tau)^4),$$

with

$$F_3(\tau) = -\frac{1}{6(e^\tau-1)^2}\Big[(\theta+\bar\theta-5\theta_1)^2+2(\theta\bar\theta-5\theta_1^2)+(\theta_0-\theta_1)^2e^{2\tau}-2(\theta_0(\theta+\bar\theta+2)+\theta\bar\theta)e^\tau$$
$$+2(3\theta_1+1)(\theta+\bar\theta+\theta_0-2\theta_1)e^\tau\Big].$$

Since (see Section B.2.10)

$$u=u_{14}v_{14}^2+\theta_1v_{14}+1,\qquad v=\frac{1}{v_{14}},$$

we obtain the series expansions for (u,v):

$$\begin{cases} u(t) &=& 1-\theta_1(t-\tau)+\left(B+\theta_1\frac{\theta+\bar\theta+\theta_1(e^\tau-3)-\theta_0e^\tau}{2(e^\tau-1)}\right)(t-\tau)^2+O((t-\tau)^3),\\ v(t) &=& -\frac{1}{t-\tau}-\frac{\theta+\bar\theta+\theta_1(e^\tau-3)-\theta_0e^\tau}{2(e^\tau-1)}-\left(\frac{2B(e^\tau-2)}{3(1-e^\tau)}+F_4(\tau)\right)(t-\tau)+O((t-\tau)^2),\end{cases}$$

with

$$F_4(\tau)=F_3(\tau)+\left(\frac{\theta+\bar\theta+\theta_1(e^\tau-3)-\theta_0e^\tau}{2(e^\tau-1)}\right)^2.$$

At $t=\tau$, $u-1$ has a simple zero, while v has a simple pole with residue -1.

5.3. *Points where* $u(\tau)=e^\tau$ *and* v *has a pole.*

The set $\mathcal{E}_x\setminus\mathcal{I}$ is given by $\{v_{x4}=0\}$, in the (u_{x4},v_{x4}) chart (see Section B.2.8). Suppose $u_{x4}(\tau)=B$, $v_{x4}(\tau)=0$, for some complex number B. From the system of differential equations in Section B.2.8, we get:

$$v_{x4}=(t-\tau)+\frac{\theta_0+\theta_x-(\theta+\bar\theta-\theta_x)e^\tau}{2(e^\tau-1)}(t-\tau)^2+\left(\frac{B(e^{2\tau}-1)}{3e^\tau(e^\tau-1)^2}+F_5(\tau)\right)(t-\tau)^3+O((t-\tau)^4),$$

with

$$F_5(\tau)=\frac{\Big[(\theta+\bar\theta-2\theta_x)^2-3\theta_x^2+2\theta\bar\theta\Big]e^{2\tau}+(\theta_0+2\theta_x)^2-\theta_x^2-2\Big[1+\theta_0(\theta+\bar\theta)+\theta_x-\theta_1+\theta\bar\theta\Big]e^{\tau}}{6(e^\tau-1)^2}.$$

Since, as calculated in Section B.2.8:

$$u=(u_{x4}v_{x4}+e^t\theta_x)v_{x4}+e^t,\qquad v=\frac{1}{((u_{x4}v_{x4}+e^t\theta_x)v_{x4}+e^t)v_{x4}},$$

we obtain:

$$\begin{cases} u(t) &= e^\tau+e^\tau(\theta_x+1)(t-\tau)+\Big(B+\frac{e^\tau}{2}+e^\tau\theta_x\frac{\theta_0+\theta_x-2-(\theta_0+\theta_1-3)e^\tau}{2(e^\tau-1)}\Big)(t-\tau)^2+O((t-\tau)^3)\,,\\ v(t) &= \frac{e^{-\tau}}{t-\tau}-e^{-\tau}\frac{\theta_0-\theta_x-2-(\theta+\bar\theta-3\theta_x-2)e^\tau}{2(e^\tau-1)}+e^{-\tau}\Big(B\cdot\frac{2-4e^\tau}{3e^\tau(e^\tau-1)}+F_6(\tau)\Big)(t-\tau)+O((t-\tau)^2)\,,\end{cases}$$

with

$$\begin{aligned}F_6(\tau)=-\frac{1}{12(e^\tau-1)^2}\Big\{&\big[\theta_\infty^2+2(\theta+\bar\theta+2\theta_x)(\theta_x-3)+(3\theta_x+5)^2-19\big]e^{2\tau}\\ &-\big[\theta_\infty^2-(\theta_1-2\theta_x)^2+2(\theta_x-12)+(5\theta_x+1)^2+(\theta_0-6)^2\big]e^\tau\\ &+\big[8\theta_x^2+(\theta_0-\theta_x-3)^2-2\big]\Big\}.\end{aligned}$$

At $t=\tau$, obviously, v has a simple pole with residue $e^{-\tau}$, while $u(t)-e^\tau$ has a simple zero.

5.4. *Points where u has a pole and v a zero.*

Such points belong to $\mathcal{E}_\infty$ and $\mathcal{E}_\infty^-$, which are obtained by blowing up the points β_∞ and β_∞^- on $\mathcal{D}_\infty$. We notice that the initial vector field (see Section B.2.1) does not depend on the sign of θ_∞. Moreover, if we replace θ_∞ by $-\theta_\infty$, the roles of β_∞ and β_∞^- are interchanged. Because of that symmetry, we may consider only the case when the solution intersects $\mathcal{E}_\infty$.

The set $\mathcal{E}_\infty\setminus\mathcal{I}$ is given by $\{\tilde u_3=0\}$ in the $(\tilde u_3,\tilde v_3)$ chart (see Section B.2.4). Suppose $\tilde u_3(\tau)=0$, $\tilde v_3(\tau)=B$. From the differential equations in Section B.2.4, we get:

$$\tilde u_3(t)=\frac{\theta_\infty}{1-e^\tau}(t-\tau)-\theta_\infty\frac{(\theta_\infty+\theta_x-2)e^\tau+\theta_\infty+\theta_1+2B}{2(e^\tau-1)^2}(t-\tau)^2+O((t-\tau)^3).$$

Then, using the relations:

$$u=\frac{1}{\tilde u_3},\qquad v=(\theta-\tilde u_3\tilde v_3)\tilde u_3,$$

we get:

$$\begin{cases} u(t) &= \frac{1-e^\tau}{\theta_\infty(t-\tau)}+\frac{(\theta_\infty+\theta_x-2)e^\tau+\theta_\infty+\theta_1+2B}{2\theta_\infty}+O(t-\tau),\\ v(t) &= \frac{\theta\theta_\infty}{(1-e^\tau)}(t-\tau)-\theta_\infty\frac{\theta(\theta_\infty+\theta_x-2)e^\tau+\theta(\theta_\infty+\theta_1)+2(\theta+\theta_\infty)B}{2(e^\tau-1)^2}(t-\tau)^2+O((t-\tau)^3).\end{cases}$$

Note that u has a simple pole with residue $(1-e^\tau)/\theta_\infty$, while v has a simple zero. In the intersection points with $\mathcal{E}_\infty^-$, u has a simple pole with residue $-(1-e^\tau)/\theta_\infty$ and v a simple zero.

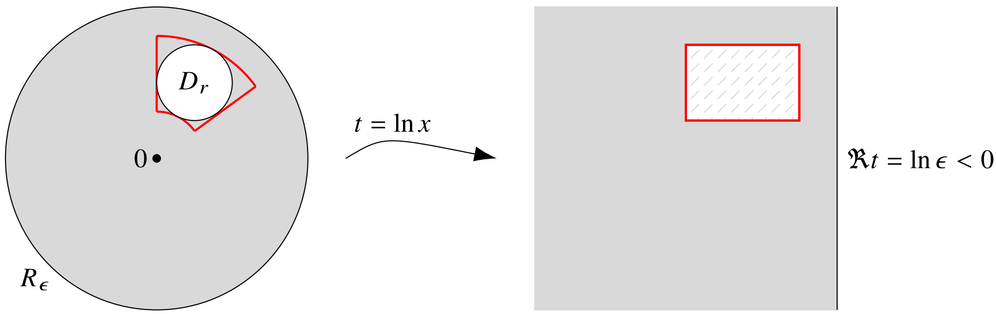

Figure 7. *Domains R_ϵ and D_r. R_ϵ is the disk centred at the origin with radius $0 < \epsilon < 1$. D_r is a disk within R_ϵ and does not contain the origin. The image of R_ϵ by the logarithmic function is the half-plane placed on the left to the boundary $\Re t = \ln \epsilon$. On the left side of the figure, notice a curvilinear 'quadrangle' consisting of two circular arcs centred at the origin and two segments placed on radii of R_ϵ, such that it is circumscribed about D_r. That 'quadrangle' is mapped to the red rectangle on the right side, which thus will contain the image of D_r.*

6. Estimates and the main result

In this section, we estimate the distance of the vector field from each vertical leaf, for sufficiently small x. These estimates allow us to describe the domain of each solution in $\mathcal{S}\backslash\mathcal{I}$, which is Okamoto's space of initial values. The results will be used to prove properties of the limit set of each solution.

Given $0 < \epsilon < 1$, $\epsilon \in \mathbb{R}$, define a disk $R = R_\epsilon = \{x \in \mathbb{C} \mid |x| < \epsilon\}$. Letting $\xi \in R$, $r < |\xi| < \epsilon$, define a disk $D = D_r(\xi) = \{x \in R \bigm| |x - \xi| < r\}$ that lies in the interior of R. Defining a new time coordinate $t = \ln x$, we have corresponding domains R_t and D_t in the t-plane. Note that D_t is no longer a circular disk but lies inside a rectangular region in the left half of the t-plane (see Figure 7).

The reader may find it useful to consult Figure 4 in the proofs of the following results.

Lemma 6.1. *Given $x \in \mathbb{C}\backslash\{0\}$, there exists a continuous complex valued function d in a neighbourhood of the infinity set $\mathcal{I}$ in Okamoto's space, such that*

$$d = \begin{cases} \frac{1}{E} & \textit{in a neighbourhood of } \mathcal{H}^* \cup \mathcal{D}_1^* \cup \mathcal{D}_\infty^{**} \setminus (\mathcal{D}_0^* \cup \mathcal{D}_x^*), \\ -\frac{x-1}{x}\omega_{03} & \textit{in a neighbourhood of } \mathcal{D}_0^*\backslash\mathcal{H}^*, \\ -\omega_{x3} & \textit{in a neighbourhood of } \mathcal{D}_x^*\backslash\mathcal{H}^*. \end{cases}$$

Note that d vanishes on $\mathcal{I}$ and that d is not defined at $x = 0$.

Proof. From Section B.2.5, the set $\mathcal{D}_0^*\backslash\mathcal{H}^*$ is given by $v_{03} = 0$ in the (u_{03}, v_{03}) chart. As we approach $\mathcal{D}_0^*$, we have:

$$E\omega_{03} \sim -\frac{x}{x-1}, \quad \text{as} \quad v_{03} \to 0.$$

From Section B.2.7, the set $\mathcal{D}_x^*\backslash\mathcal{H}^*$ is given by $v_{x3} = 0$ in the (u_{x3}, v_{x3}) chart. As we approach $\mathcal{D}_x^*$, we have:

$$E\omega_{x3} \sim -1 - \frac{x}{u_{x3}} \quad \text{as} \quad v_{x3} \to 0.$$

Thus, as we approach $\mathcal{H}^*$: $u_{x3} \to \infty$, we have that $-\omega_{x3} \sim 1/E$. □

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Lemma 6.2. *For every $\epsilon > 0$, there exists a neighbourhood U of $\mathcal{D}_\infty^{**}$, such that*

$$\left|\frac{\dot{E}}{E} + \frac{e^t}{e^t - 1}\right| < \epsilon.$$

Proof. The proof follows from the expressions for $\dot{E}/E$ in (u_2, v_2) and (u_3, v_3) charts (see Sections A.0.3 and A.0.4), where $\mathcal{D}_\infty$ is given by $u_2 = 0$ and $u_3 = 0$, respectively. □

Lemma 6.3. *For every compact subset K of $\mathcal{D}_\infty^{**} \cup \mathcal{D}_1^* \cup \mathcal{H}^* \backslash \mathcal{D}_x^*$, there exists a neighbourhood V of K and a constant $C > 0$, such that*

$$\left|e^{-t}\,\frac{\dot{E}}{E}\right| < C$$

in V for all t, such that e^t is bounded away from 1.

Proof. Note that $\mathcal{H}^* = \{\tilde{v}_1 = 0\} \cup \{\tilde{v}_2 = 0\}$ is parametrised by $\tilde{u}_1$ and $\tilde{u}_2 = 1/\tilde{u}_1$, respectively. Moreover, $\mathcal{D}_x^*$ is given in these charts by $\{\tilde{u}_1 = x\}$ and $\{x\tilde{u}_2 = 1\}$. In the respective coordinate charts $(\tilde{u}_1, \tilde{v}_1)$, $(\tilde{u}_2, \tilde{v}_2)$ (see Sections B.2.2 and B.2.3), we have

$$e^{-t}\,\frac{\dot{E}}{E} = \begin{cases} \frac{\tilde{u}_1 - 1}{(e^t - 1)(e^t - \tilde{u}_1)} + \frac{(\theta_x - 1)\tilde{u}_1(\tilde{u}_1 - 1)}{e^t - \tilde{u}_1}\tilde{v}_1 + O\big(\tilde{v}_1^2\big), \\ -\frac{\tilde{u}_2 - 1}{(e^t - 1)(e^t\tilde{u}_2 - 1)} - \frac{(\theta_x - 1)\tilde{u}_2(\tilde{u}_2 - 1)}{e^t\tilde{u}_2 - 1}\tilde{v}_2 + O\big(\tilde{v}_2^2\big). \end{cases}$$

So as long as we consider compact subsets of $\mathcal{H}^* \backslash \mathcal{D}_x^*$, the values of $\frac{1}{\tilde{u}_1 - x}$ and $\frac{1}{x\tilde{u}_2 - 1}$ are bounded. We have now proven the desired result in a neighbourhood of any compact subset of $\mathcal{H}^* \setminus \mathcal{D}_x^*$. Since $\mathcal{D}_\infty^{**}$ intersects with $\mathcal{H}^* \setminus \mathcal{D}_x^*$, the result holds in a neighbourhood of $\mathcal{D}_\infty^{**} \cap \mathcal{H}^*$.

On the other hand, near $\mathcal{D}_\infty^{**} \setminus \mathcal{H}^*$, given by $\{u_{\infty 4} = 0\}$, we may consider only bounded values of $v_{\infty 4}$, and so we have (see Section B.2.11)

$$e^{-t}\,\frac{\dot{E}}{E} = -\frac{1}{e^t - 1} + \Big[\theta v_{\infty 4} - \Big(v_{\infty 4} + \overline{\theta}\Big)(v_{\infty 4} + \theta_x - 1)\Big]u_{\infty 4} + O(u_{\infty 4}^2)\,.$$

Hence, the result holds in a neighbourhood of the compact set $\mathcal{D}_\infty^{**} = \{u_{\infty 4} = 0\} \cup \{\tilde{u}_2 = 0\}$. Similarly, near $\mathcal{D}_1^{**} \setminus \mathcal{H}^*$, given by $\{v_{13} = 0\}$, where we may consider only bounded values of u_{13}, we have (see Section B.2.9)

$$e^{-t}\,\frac{\dot{E}}{E} = \frac{(u_{13} + \theta_1)(u_{13} - \theta_x + 1)}{(e^t - 1)^2}v_{13} + O(v_{13}^2)\,.$$

Hence, the result holds for any compact subset K of $\mathcal{D}_\infty^{**} \cup \mathcal{D}_1^* \cup \mathcal{H}^* \backslash \mathcal{D}_x^*$ and any t as long as $\frac{1}{e^t - 1}$ is bounded. □

Remark 6.4. The estimate in the above Lemma 6.3 applies to all compact subsets of $\mathcal{D}_\infty^{**} \cup \mathcal{D}_1^* \cup \mathcal{H}^* \backslash \mathcal{D}_x^*$ and, therefore, in particular to $\mathcal{D}_\infty^{**} \cup \mathcal{D}_1^*$.

Definition 6.5. An *approximate disk* with centre τ and radius R is an open simply connected set which, for some $\varepsilon > 0$, contains the disk centred at τ with radius $R - \varepsilon$ and is contained in the disk centred at τ with radius $R + \varepsilon$.

Lemma 6.6 (Behaviour near $\mathcal{D}_x^* \setminus \mathcal{H}^*$). *If a solution at a complex time t is sufficiently close to $\mathcal{D}_x^* \setminus \mathcal{H}^*$, then there exists a unique $\tau \in \mathbf{C}$, such that $(u(\tau), v(\tau))$ belongs to the exceptional line $\mathcal{E}_x$. In other words, $u(\tau) = e^\tau$ and $v(t)$ has a pole at $t = \tau$. Moreover, for sufficiently small $d(t)$ and bounded u_{x3}, we have $|t - \tau| = O(|e^{-t}d(t)||u_{x3}(t)|)$.*

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For large $R_x > 0$, consider the set $\{t \in \mathbb{C} \mid |u_{x3}(t)| \le R_x\}$. Its connected component containing τ is an approximate disk Δ_x with centre τ and radius $|d(\tau)e^{-\tau}|R_x$, and $t \mapsto u_{x3}(t)$ is a complex analytic diffeomorphism from Δ_x onto $\{u \in \mathbb{C} \mid |u| \le R_x\}$.

Proof. For the study of the solutions near $\mathcal{D}_x^* \setminus \mathcal{H}^*$, we use coordinates (u_{x3}, v_{x3}) (see Section B.2.7). In this chart, the set $\mathcal{D}_x^* \setminus \mathcal{H}^*$ is given by $v_{x3} = 0$ and parametrised by $u_{x3} \in \mathbb{C}$. Moreover, $\mathcal{E}_x$ is given by $\{u_{x3} = 0\}$ and parametrised by the variable v_{x3}. From Lemma 6.1, we recall that $d = -\omega_{x3}$ in this chart.

Asymptotically, for $v_{x3} \to 0$, bounded u_{x3} and $x = e^t$ bounded away from 0 and 1, we have:

$$\dot{u}_{x3} \sim \frac{1}{v_{x3}}, \tag{6.1a}$$

$$\dot{v}_{x3} \sim \left(\frac{2u_{x3}}{x(x-1)} - (\theta_0 + \theta_x) - \frac{x\theta_1 - \theta_x - 1}{x-1}\right) v_{x3}, \tag{6.1b}$$

$$\omega_{x3} \sim -x v_{x3}, \tag{6.1c}$$

$$\frac{\dot{\omega}_{x3}}{\omega_{x3}} \sim (1 - \theta_0 - \theta_1)\frac{x}{x-1} + \frac{2u_{x3} + \theta_0 + \theta_x}{x-1}, \tag{6.1d}$$

$$E\omega_{x3} \sim -1 - \frac{x}{u_{x3}}. \tag{6.1e}$$

Note that integrating Equation (6.1d) from t_0 to t_1, where $t_0, t_1 \in D_t$ (see Figure 7) leads to

$$\log\left(\frac{\omega_{x3}(t_1)}{\omega_{x3}(t_0)}\right) \sim (1 - \theta_0 - \theta_1)\log\left(\frac{1 - e^{t_1}}{1 - e^{t_0}}\right) + \int_{t_0}^{t_1} \frac{2u_{x3} + \theta_0 + \theta_x}{e^t - 1}\, dt.$$

Therefore, if for all t on the line segment from t_0 to t_1, we have $|e^t - e^{t_0}| \ll |e^{t_0}|$ and $|u_{x3}(t)|$ is bounded, then $\omega_{x3}(t)/\omega_{x3}(t_0) \sim ((1 - e^t)/(1 - e^{t_0}))^{1-\theta_0-\theta_1}$, where the right side is upperbounded by e^{t_0}. In view of this situation, Equation (6.1c) shows that v_{x3} is approximately given by a small constant. We take $t_0 = \tau$ in the following analysis. From (6.1a), it follows that:

$$u_{x3} \sim u_{x3}(\tau) + \frac{t - \tau}{v_{x3}(\tau)}.$$

Thus, if t runs over an approximate disk Δ centred at τ with radius $|v_{x3}|R$, then u_{x3} fills an approximate disk centred at $u_{x3}(\tau)$ with radius R. Therefore, if $|v_{x3}| \ll 1/|\tau|$, the solution has the following properties for $t \in \Delta$:

$$\frac{v_{x3}(t)}{v_{x3}(\tau)} \sim 1,$$

and u_{x3} is a complex analytic diffeomorphism from Δ onto an approximate disk with centre $u_{x3}(\tau)$ and radius R. If R is sufficiently large, we will have $0 \in u_{x3}(\Delta)$, that is the solution of the Painlevé equation will have a pole at a unique point in Δ. Now, it is possible to take τ to be the pole point. We have:

$$u_{x3}(t) \sim \frac{t - \tau}{v_{x3}(\tau)} \sim -\frac{(t - \tau)e^{\tau}}{d(\tau)}.$$

Let R_x be a large positive real number. Then the equation $|u_{x3}(t)| = R_x$ corresponds to $|t - \tau| \sim |e^{-\tau}d(\tau)|R_x$, which is still small compared to $|\tau|$ if $|d(\tau)|$ is sufficiently small. It follows that the connected component Δ_x of the set of all $t \in \mathbb{C}$, such that $\{t \mid |u_{x3}(t)| \le R_x\}$ is an approximate disk with centre τ and radius $|d(\tau)e^{-\tau}|R_x$. More precisely, u_{x3} is a complex analytic diffeomorphism from Δ_x onto $\{u \in \mathbb{C} \mid |u| \le R_x\}$, and $\frac{d(t)}{d(\tau)} \sim 1$ for all $t \in \Delta_x$. □

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Remark 6.7. Similar arguments show that if a solution comes sufficiently close to $\mathcal{D}_1^*$ or $\mathcal{D}_\infty^{**}$, then it will cross the corresponding exceptional lines $\mathcal{E}_1$, respectively $\mathcal{E}_\infty$ and $\mathcal{E}_\infty^-$ transversally at a unique nearby value of time. We prove this in Appendix C. This is, however, not needed for our main result.

Lemma 6.8 (Behaviour near $\mathcal{D}_0^* \setminus \mathcal{H}^*$). *If a solution at a complex time t is sufficiently close to $\mathcal{D}_0^* \setminus \mathcal{H}^*$, then there exists unique $\tau \in \mathbf{C}$, such that $(u(\tau), v(\tau))$ belongs to the line $\mathcal{E}_0$. In other words, u vanishes and v has a pole at $t = \tau$. Moreover, $|t - \tau| = O(|d(t)||u_{03}(t)|)$ for sufficiently small $d(t)$ and bounded u_{03}.*

For large $R_0 > 0$, consider the set $\{t \in \mathbb{C} \mid |u_{03}(t)| \leq R_0\}$. Its connected component containing τ is an approximate disk Δ_0 with centre τ and radius $|d(\tau)(e^\tau + e^{-\tau})|R_0$, and $t \mapsto u_{03}(t)$ is a complex analytic diffeomorphism from that Δ_0 onto $\{u \in \mathbb{C} \mid |u| \leq R_0\}$.

Proof. For the study of the solutions near $\mathcal{D}_0^* \setminus \mathcal{H}^*$, we use coordinates (u_{03}, v_{03}) (see Section B.2.5). In this chart, the set $\mathcal{D}_0^* \setminus \mathcal{H}^*$ is given by $v_{03} = 0$ and parametrised by $u_{03} \in \mathbb{C}$. Moreover, $\mathcal{E}_0$ is given by $u_{03} = 0$ and parametrised by v_{03}.

Asymptotically, for $v_{03} \longrightarrow 0$, bounded u_{03} and $x = e^t$ bounded away from 0 and 1, we have:

$$\dot{u}_{03} \sim -\frac{x}{(1-x)v_{03}}, \tag{6.2a}$$

$$\dot{v}_{03} \sim -\frac{(x+1)(2u_{03}+\theta_0) - \theta_x + 1 - x\theta_1}{x-1} v_{03}, \tag{6.2b}$$

$$\omega_{03} = -v_{03}, \tag{6.2c}$$

$$\frac{\dot{\omega}_{03}}{\omega_{03}} \sim 2u_{03} + \theta_0 - \theta_1 + \frac{4u_{03} + \theta_0 - \theta_1 + 1}{x-1}, \tag{6.2d}$$

$$E\omega_{03} \sim -\frac{x}{x-1}. \tag{6.2e}$$

Arguments similar to those in the proof of Lemma 6.6 show that v_{03} is approximately equal to a small constant, and from (6.2a), it follows that:

$$u_{03} \sim u_{03}(\tau) - \frac{e^t - e^\tau}{v_{03}(\tau)}.$$

Thus, if t runs over an approximate disk Δ centred at τ with radius $|v_{03}| \log R$, then u_{03} fills an approximate disk centred at $u_{03}(\tau)$ with radius R. Therefore, if $|v_{03}| \ll e^{-|\tau|}$, the solution has the following properties for $t \in \Delta$:

$$\frac{v_{03}(t)}{v_{03}(\tau)} \sim 1,$$

and u_{03} is a complex analytic diffeomorphism from Δ onto an approximate disk with centre $u_{03}(\tau)$ and radius R. If R is sufficiently large, we will have $0 \in u_{03}(\Delta)$, that is the solution of the Painlevé equation will vanish at a unique point in Δ. Now, it is possible to take τ to be that point. We have:

$$u_{03}(t) \sim -\frac{e^t - e^\tau}{v_{03}(\tau)} \sim -\frac{(e^t - e^\tau)e^\tau}{(e^\tau - 1)d(\tau)}.$$

Let R_0 be a large positive real number. Then the equation $|u_{03}(t)| = R_0$ corresponds to $|1 - e^{t-\tau}| \sim |e^{-2\tau}(e^\tau - 1)d(\tau)|R_0$, which is still small compared to $|e^\tau|$ if $|d(\tau)|$ is sufficiently small. It follows that the connected component Δ_0 of the set of all $t \in \mathbb{C}$, such that $\{t \mid |u_{03}(t)| \leq R_0\}$ is an approximate disk with centre τ and radius $|d(\tau)(e^{-\tau} + e^\tau)|R_0$. More precisely, u_{03} is a complex analytic diffeomorphism from Δ_0 onto $\{u \in \mathbb{C} \mid |u| \leq R_0\}$, and $\frac{d(t)}{d(\tau)} \sim 1$ for all $t \in \Delta_0$. □

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Theorem 6.9. *Let ϵ_1, ϵ_2, ϵ_3 be given, such that $0 < \epsilon_1 < 1$, $0 < \epsilon_2 < 1$, $0 < \epsilon_3 < 1$. Then there exists $\delta > 0$, such that if $|e^{t_0}| < \epsilon_1$ and $|d(t_0)| < \delta$, it follows that*

$$\rho = \inf\{r < |e^{t_0}|\,,\ \textit{such that}\ |d(t)| < \delta\ \textit{whenever}\ |e^{t_0}| \geq |e^t| \geq r\}$$

satisfies:

(i) *$\rho > 0$ and is bounded below by the relation:*

$$\delta \geq |d(t_0)|\big((1-\rho)/|1-e^{t_0}|\big)^{1-\epsilon_2}(1-\epsilon_3);$$

(ii) *if $|e^{t_0}| \geq |e^t| \geq \rho$, then*

$$d(t) = d(t_0)\left(\frac{1-e^t}{1-e^{t_0}}\right)^{1+\varepsilon_2(t)}(1+\varepsilon_3(t)),$$

where $|\varepsilon_2(t)| \leq \epsilon_2$ and $|\varepsilon_3(t)| \leq \epsilon_3$; and,

(iii) *if $|e^t|$ is less than ρ, but still sufficiently close to ρ, then $|d(t)| \geq \delta(1-\epsilon_3)$.*

Proof. Suppose a solution of the system (3.1) is close to the infinity set at times t_0 and t_1. If follows from Lemmas 6.6 and 6.8 that for every solution close to $\mathcal{I}$, the set of complex times t, such that the solution is not close to $\mathcal{I} \setminus (\mathcal{D}_0^* \cup \mathcal{D}_x^*)$ is the union of approximate disks of radius $\sim |d|$. Hence, if the solution is near $\mathcal{I}$ for all complex times t, such that $|e^{t_0}| \geq |e^t| \geq |e^{t_1}|$, then there exists a path $\mathcal{P}$ from t_0 to t_1, such that the solution is close to $\mathcal{I} \setminus (\mathcal{D}_0^* \cup \mathcal{D}_x^*)$ for all $t \in \mathcal{P}$ and $\mathcal{P}$ is C^1-close to the path: $s \mapsto t_1^s t_0^{1-s}$, $s \in [0,1]$.

Then Lemma 6.2 implies that near $\mathcal{D}_\infty^{**}$:

$$\frac{E(t)}{E(t_0)} \sim \frac{1-e^{t_0}}{1-e^t},$$

and by using Lemma 6.1, we find:

$$d(t) \sim d(t_0)\frac{1-e^t}{1-e^{t_0}}. \tag{6.3}$$

For the first statement of the theorem, we have:

$$\delta > |d(t)| \geq |d(t_0)|\left(\frac{1-|e^t|}{|1-e^{t_0}|}\right)^{1-\epsilon_2}(1-\epsilon_3),$$

and the desired result follows from $\rho \leq e^t$. For $|e^t| \leq |e^{t_0}|$, the second statement follows from (6.3) and the third one from the definition of ρ. The symmetries of the sixth Painlevé equation show that the same statements follow near other lines of the infinity set $\mathcal{I}$. □

As a consequence of Theorem 6.9, we can prove the repelling property of the set $\mathcal{I}$.

Corollary 6.10. *No solution with the initial conditions in the space of the initial values intersects $\mathcal{I}$. A solution that is close to $\mathcal{I}$ for a certain value of the independent variable t will stay in the vicinity of $\mathcal{I}$ only for a limited range of t. Moreover, if a solution is sufficiently close to $\mathcal{I}$ at a point t, then it will have a pole in a neighbourhood of t.*

Proof. The statement follows from Theorem 6.9 and Lemmas 6.6, 6.8. □

Remark 6.11. Parts (i) and (ii) of Theorem 6.9 give estimates on the behaviour of the solutions near the infinity set. Part (iii) implies that a solution does not stay indefinitely near the infinity set as $e^t \to 0$.

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7. The limit set

Our definition of the limit set is the extension of the standard concept of limit sets in dynamical systems to complex-valued solutions.

Definition 7.1. Let $(u(t), v(t))$ be a solution of (3.1). *The limit set* $\Omega_{u,v}$ of $(u(t), v(t))$ is the set of all $s \in \mathcal{S}(0) \setminus \mathcal{I}(0)$, such that there exists a sequence $t_n \in \mathbf{C}$ satisfying:

$$\lim_{n\to\infty} \Re(t_n) = -\infty \quad \text{and} \quad \lim_{n\to\infty} (u(t_n), z(v_n)) = s.$$

Theorem 7.2. *There exists a compact set $K \subset \mathcal{S}(0) \setminus \mathcal{I}(0)$, such that the limit set $\Omega_{u,v}$ of any solution (u, v) of (3.1) is contained in K. Moreover, $\Omega_{u,v}$ is a nonempty, compact and connected set, which is invariant under the flow of the autonomous system given in Section 4.*

Proof. For any positive numbers η, r, let $K_{\eta,r}$ denote the set of all $s \in \mathcal{S}(x)$, such that $|x| \le r$ and $|d(s)| \ge \eta$. Since $\mathcal{S}(x)$ is a complex analytic family over $\mathbb{P}^1 \setminus \{0, 1\}$ of compact surfaces $\mathcal{S}(x)$, $K_{\eta,r}$ is also compact. Furthermore, $K_{\eta,r}$ is a compact subset of the Okamoto's space $\mathcal{S} \setminus \mathcal{I}(0)$. When r approaches 0, the sets $K_{\eta,r}$ shrink to the compact set:

$$K_{\eta,0} = \{s \in \mathcal{S}(0) \mid |d(s) \ge \eta\} \subset \mathcal{S}(0) \setminus \mathcal{I}(0).$$

If follows from Theorem 6.9 that there is $\eta > 0$, such that for every solution (u, v), there exists $r_0 > 0$ with the following property:

$$(u(t), v(t)) \in K_{\eta,r_0} \text{ for every } t, \text{ such that } |e^t| \le r_0.$$

Hereafter, we take $r \le r_0$, when it follows that $(u(t), v(t)) \in K_{\eta,r}$ whenever $|e^t| \le r$.

Let $T_r = \{t \in \mathbb{C} \mid |e^t| \le r\}$, and let $\Omega_{(u,v),r}$ denote the closure of the image set $\big(u(T_r), v(T_r)\big)$ in $\mathcal{S}$. Since T_r is connected and (u, v) is continuous, $\Omega_{(u,v),r}$ is also connected. Since $\big(u(T_r), v(T_r)\big)$ is contained in the compact set $K_{\eta,r}$, its closure $\Omega_{(u,v),r}$ is also contained in $K_{\eta,r}$, and, therefore, $\Omega_{(u,v),r}$ is a nonempty compact and connected subset of $\mathcal{S} \setminus \mathcal{S}(0)$. The intersection of a decreasing sequence of nonempty, compact and connected sets is a nonempty, compact and connected. Therefore, as $\Omega_{(u,v),r}$ decreases to $\Omega_{(u,v)}$ as r approaches zero, it follows that $\Omega_{(u,v)}$ is a nonempty, compact and connected subset of $\mathcal{S}$. Since $\Omega_{(u,v),r} \subset K_{\eta,r}$, for all $r \le r_0$, and the sets $K_{\eta,r}$ shrink to the compact subset $K_{\eta,0}$ of $\mathcal{S}(0) \setminus \mathcal{I}(0)$ as r decreases to zero, it follows that $\Omega_{(u,v)} \subset K_{\eta,0}$. This proves the first statement of the theorem with $K = K_{\eta,0}$.

Since $\Omega_{(u,v)}$ is the intersection of the decreasing family of compact sets $\Omega_{(u,v),r}$, there exists for every neighbourhood A of $\Omega_{(u,v)}$ in $\mathcal{S}$, an $r > 0$, such that $\Omega_{(u,v),r} \subset A$. Hence, $(u(t), v(t)) \in A$ for every $t \in \mathbb{C}$, such that $|e^t| \le r$. If $\{t_j\}$ is any sequence in $\mathbb{C} \setminus \{0\}$, such that $|t_j| \to 0$, then the compactness of $K_{\eta,r}$, in combination with $\big(u(T_r), v(T_r)\big) \subset K_{\eta,r}$, implies that there is a subsequence $j = j(k) \to \infty$ as $k \to \infty$ and an $s \in K_{\eta,r}$, such that:

$$(u(t_{j(k)}), v(t_{j(k)})) \to s \text{ as } k \to \infty.$$

It follows, therefore, that $s \in \Omega_{(u,v)}$. Next, we prove that $\Omega_{(u,v)}$ is invariant under the flow Φ^τ of the autonomous Hamiltonian system. Let $s \in \Omega_{(u,v)}$ and t_j be a sequence in $\mathbb{C} \setminus \{0\}$, such that $e^{t_j} \to 0$ and $(u(t_j), v(t_j)) \to s$. Since the t-dependent vector field of the Painlevé system converges in C^1 to the vector field of the autonomous Hamiltonian system as $e^t \to 0$, it follows from the continuous dependence on initial data and parameters that the distance between $(u(t_j + \tau), v(t_j + \tau))$ and $\Phi^\tau(u(t_j), v(t_j))$ converges to zero as $j \to \infty$. Since $\Phi^\tau(u(t_j), v(t_j)) \to \Phi^\tau(s)$ and $|e^t_j| \to 0$ as $j \to \infty$, it follows that $(u(t_j + \tau), v(t_j + \tau)) \to \Phi^\tau(s)$ and $e^{t_j+\tau} \to 0$ as $j \to \infty$, hence, $\Phi^\tau(s) \in \Omega_{(u,v)}$. □

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Proposition 7.3. *Every solution $(u(t), v(t))$ with the essential singularity at $x = 0$ intersects each of the exceptional lines $\mathcal{E}_0$, $\mathcal{E}_x$, $\mathcal{E}_1$, $\mathcal{E}_\infty$, $\mathcal{E}_\infty^-$ infinitely many times in any neighbourhood of that singular point.*

Proof. For conciseness, we refer to the solution $(u(t), v(t))$ of the system as the Painlevé vector field and denote the vector field near each of the five exceptional lines $\mathcal{E}_0$, $\mathcal{E}_x$, $\mathcal{E}_1$, $\mathcal{E}_\infty$, $\mathcal{E}_\infty^-$ by $(U(t), V(t))$. Furthermore, let

$$\mathscr{E} = \mathcal{E}_0 \cup \mathcal{E}_x \cup \mathcal{E}_1 \cup \mathcal{E}_\infty \cup \mathcal{E}_\infty^-.$$

Now suppose that $(U(t), V(t))$ intersects $\mathscr{E}$ only finitely many times. According to Theorem 7.2, the limit set $\Omega_{(u,v)}$ is a compact set in $\mathcal{S}(0) \setminus \mathcal{I}(0)$. If $\Omega_{(u,v)}$ intersects one the five exceptional lines $\mathcal{E}_0$, $\mathcal{E}_x$, $\mathcal{E}_1$, $\mathcal{E}_\infty$, $\mathcal{E}_\infty^-$ at a point p, then there exists a t, such that e^t is arbitrarily close to zero and the Painlevé vector field is arbitrarily close to p, when the transversality of the vector field to the exceptional line implies that $(U(\tau), V(\tau)) \in \mathscr{E}$ for a unique τ ($\neq t$) near t. This is a contradiction to our assumption, as it follows that $(U(t), V(t))$ intersects $\mathscr{E}$ infinitely many times. Therefore, we must have that $\Omega_{(u,v)}$ is a compact subset of $\mathcal{S}(0) \setminus (\mathcal{I}(0) \cup \mathscr{E})$.

However, $\mathscr{E}$ is equal to the set of all points in $\mathcal{S}(0) \setminus \mathcal{I}(0)$, which project (blow-down) to the line $\mathcal{D}_\infty \cup \mathcal{H}^*$, and, therefore, $\mathcal{S}(0) \setminus (\mathcal{I}(0) \cup \mathscr{E})$ is the affine (u, v)-coordinate chart, of which $\Omega_{(u,v)}$ is a compact subset, which implies that $u(t)$ and $v(t)$ remain bounded for small $|e^t|$. $|e^t| \to 0$. From there, $x = 0$ is not an essential singularity. □

Theorem 7.4. *Every solution of the sixth Painlevé equation has infinitely many poles, infinitely many zeroes and infinitely many times takes value 1 in any neighbourhood of its essential singularity.*

Proof. At the intersection points with $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_\infty$, $\mathcal{E}_\infty^-$, the solution will have zeroes, 1s and poles, as explained in detail in Section 5. Thus, the statement for an essential singularity at $x = 0$ follows from Proposition 7.3. If $y(x)$ is a solution of (1.1), observe that the following Bäcklund transformations:

$$\mathcal{S}_1 : y_1(x_1) = \frac{y}{x}, \quad x_1 = \frac{1}{x}, \quad (\theta_{\infty,1}, \theta_{0,1}, \theta_{1,1}, \theta_{x,1}) = \left(\theta_\infty, \theta_0, \sqrt{\theta_x^2 + \frac{1}{2}}, \sqrt{\theta_1^2 + \frac{1}{2}}\right)$$

$$\mathcal{S}_2 : y_2(x_2) = 1 - y, \quad x_2 = 1 - x, \quad (\theta_{\infty,2}, \theta_{0,2}, \theta_{1,2}, \theta_{x,2}) = (\theta_\infty, i\theta_1, i\theta_0, \theta_x)$$

give the solutions $y_1(x_1)$ and $y_2(x_2)$ of the sixth Painlevé equation with respective parameters $(\theta_{\infty,1}, \theta_{0,1}, \theta_{1,1}, \theta_{x,1})$ and $(\theta_{\infty,2}, \theta_{0,2}, \theta_{1,2}, \theta_{x,2})$, [24, Section 32.7(vii)]. Transformation $\mathcal{S}_1$ maps point $x = \infty$ to $x_1 = 0$, while $\mathcal{S}_2$ maps point $x = 1$ to $x_2 = 0$, thus, the statement will also hold for essential singularities at $x = 1$ and $x = \infty$. □

8. Conclusion

The Painlevé equations have been playing an increasingly important role in mathematical physics, especially in the applications to classical and quantum integrable systems and random matrix theory. The sixth Painlevé equation, which is the focus of this work, is very prominent in these areas, in particular, in conformal field theory in recent times [7]. For further relations with conformal block expansions and supersymmetric gauge theories, see the references in [7].

Although the initial values space for the Painlevé equations was described by Okamoto [25], our aim in this work was to describe the dynamics of the solutions by analysing that construction.

Many questions beyond the limit behaviour remain open about particular families of transcendental solutions, from the dynamical systems point of view. For example, the existence of limit cycles of transcendental solutions with particular symmetry properties and whether there are periodic cycles in the combined space of parameters and initial values remain open.

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A. Charts of the initial surface $\mathbb{F}_1$

A.0.1. Initial chart $(u_0, v_0) = (u, v)$

$$\begin{cases} (u,v) &= (u_0, v_0) \\ \omega_0 &= 1 \\ E &= \frac{1}{x-1}\Big[u_0(u_0-x)(u_0-1)v_0^2 - (\theta+\overline{\theta})u_0^2 v_0 + \theta\overline{\theta}(u_0-x) - x\theta_0 v_0 + \\ & \qquad +\Big((x+1)\theta_0 + x\theta_1 + (\theta_x - 1)\Big)u_0 v_0\Big] \\ \dot{E} &= -\frac{x(u_0-1)}{(x-1)^2}\Big[(u_0v_0-\theta)(u_0v_0-\overline{\theta}) - (u_0v_0-\theta_0)v_0\Big] \end{cases}$$

$$\begin{cases} \dot{u}_0 &= -2\frac{u_0(u_0-x)(u_0-1)v_0}{1-x} + \frac{\theta+\overline{\theta}}{1-x}u_0(u_0-1) + \theta_1 u_0 - \frac{x}{1-x}\theta_0(u_0-1) \\ \dot{v}_0 &= \frac{(3u_0-2)u_0}{1-x}v_0^2 - \Big[\frac{\theta+\overline{\theta}}{1-x}(2u_0-1) + \theta_1\Big]v_0 + \frac{\theta\overline{\theta}}{1-x} - \frac{x}{1-x}[(2u_0-1)v_0 - \theta_0]v_0 \\ \frac{\dot{\omega}_0}{\omega_0} &= 0. \end{cases}$$

No base points.
No elliptic base points.
No visible components of the infinity set.

A.0.2. First chart $(u_1, v_1) = \left(u, \frac{1}{v}\right)$

$$\begin{cases} (u,v) &= \left(u_1, \frac{1}{v_1}\right) \\ \omega_1 &= -v_1^2 \\ E &= \frac{1}{(x-1)v_1^2}\Big[u_1(u_1-x)(u_1-1) - (\theta+\overline{\theta})u_1^2 v_1 + \theta\overline{\theta}(u_1-x)v_1^2 - x\theta_0 v_1 + \\ & \qquad +\Big((x+1)\theta_0 + x\theta_1 + (\theta_x - 1)\Big)u_1 v_1\Big] \\ \dot{E} &= -\frac{x(u_1-1)}{(x-1)^2 v_1^2}\Big[(u_1-\theta v_1)(u_1-\overline{\theta}v_1) - (u_1-\theta_0 v_1)\Big] \end{cases}$$

$$\begin{cases} \dot{u}_1 &= -2\frac{u_1(u_1-x)(u_1-1)}{(1-x)v_1} + \frac{\theta+\overline{\theta}}{1-x}u_1(u_1-1) + \frac{\theta_1}{1-x}u_1 - \frac{x}{1-x}[\theta_0(u_1-1) + \theta_1 u_1] \\ \dot{v}_1 &= -\frac{\theta\overline{\theta}}{1-x}v_1^2 + \frac{1}{1-x}\Big[(\theta+\overline{\theta})(2u_1-1) + \theta_1\Big]v_1 - \frac{(3u_1-2)u_1}{1-x} - \frac{x}{1-x}[(\theta_0+\theta_1)v_1 - (2u_1-1)] \\ \frac{\dot{\omega}_1}{\omega_1} &= -2\frac{\theta\overline{\theta}}{1-x}v_1 + 2\frac{1}{1-x}\Big[(\theta+\overline{\theta})(2u_1-1) + \theta_1\Big] - 2\frac{(3u_1-2)u_1}{(1-x)v_1} - 2\frac{x}{1-x}\Big[(\theta_0+\theta_1) - \frac{2u_1-1}{v_1})\Big]. \end{cases}$$

Base points of the vector field:

$$b_0 : (u_1, v_1) = (0,0), \quad b_x : (u_1, v_1) = (x, 0), \quad b_1 : (u_1, v_1) = (1, 0).$$

Elliptic base points are b_0 and b_1.
Visible components of the infinity set: $H : \{v_1 = 0\}$

Estimates near $\boldsymbol{H}$, that is $\boldsymbol{v_1 \longrightarrow 0}$:

$$\omega = -v_1^2 \tag{A.1}$$

$$\omega E \sim \frac{u_1(u_1-1)(u_1-x)}{1-x} \tag{A.2}$$

$$\frac{\dot{E}}{E} \sim \frac{x}{1-x} - \frac{x}{u_1-x} \tag{A.3}$$

$$\frac{\dot{\omega}}{\omega} \sim \left(\frac{1}{u_1} + \frac{1}{u_1-x} + \frac{1}{u_1-1}\right)\dot{u}_1 \tag{A.4}$$

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$$\dot{u}_1 \sim 2\frac{u_1(u_1-1)(u_1-x)}{(x-1)v_1} \tag{A.5}$$

$$\dot{v}_1 \sim \frac{3u_1^2-2(1+x)u_1+x}{x-1} \tag{A.6}$$

$$\omega E_0 \sim u_1^2(u_1-1) \tag{A.7}$$

$$\frac{\dot{E}_0}{E_0} \sim \left(\frac{1}{u}-\frac{1}{u-x}\right)\dot{u}_1. \tag{A.8}$$

A.0.3. Second chart $(u_2, v_2) = \left(\frac{1}{u}, \frac{1}{uv}\right)$

$$\begin{cases} (u,v) &= \left(\frac{1}{u_2}, \frac{u_2}{v_2}\right) \\ \omega_2 &= u_2 v_2^2 \\ E &= \frac{1}{(x-1)v_2^2}\left[\frac{(xu_2-1)(u_2-1)-(\theta+\overline{\theta})v_2}{u_2} - \theta\overline{\theta}\frac{xu_2-1}{u_2}v_2^2 - x\theta_0 u_2 v_2 + \Big((x+1)\theta_0 + x\theta_1 + (\theta_x - 1)\Big)v_2\right] \\ \dot{E} &= \frac{x(u_2-1)}{(x-1)^2 v_2^2}\left[\frac{(\theta v_2-1)(\overline{\theta}v_2-1)}{u_2} + \theta_0 v_2 - 1)\right] \end{cases}$$

$$\begin{cases} \dot{u}_2 &= \frac{(u_2-1)((\theta+\overline{\theta})v_2-2)}{(1-x)v_2} - \theta_1 u_2 - \frac{x}{(1-x)v_2}u_2(u_2-1)(\theta_0 v_2 - 2) \\ \dot{v}_2 &= -\frac{1}{(1-x)u_2}(\theta v_2 - 1)(\overline{\theta}v_2 - 1) - \frac{x}{1-x}(\theta_0 v_2 - 1)u_2 \\ \frac{\dot{\omega}_2}{\omega_2} &= \frac{(\theta+\overline{\theta})(u_2+1)-2\theta\overline{\theta}v_2}{(1-x)u_2} - \theta_1 + 2\frac{-1+x(2u_2-1)}{(1-x)v_2} - \frac{x\theta_0(3u_2-1)}{1-x}. \end{cases}$$

No new base points.

Other visible base points:

$$b_x : (u_2, v_2) = \left(\frac{1}{x}, 0\right), \quad b_1 : (u_2, v_2) = (1, 0), \quad b_\infty : (u_2, v_2) = \left(0, \frac{1}{\theta}\right), \quad b_\infty^- : (u_2, v_2) = \left(0, \frac{1}{\overline{\theta}}\right).$$

No new elliptic base points.
Visible components of the infinity set: $H : \{v_2 = 0\}$, $D_\infty : \{u_2 = 0\}$.

Estimates near $\boldsymbol{D_\infty}$, that is $\boldsymbol{u_2 \longrightarrow 0}$:

$$\omega = u_2 v_2^2 \tag{A.9}$$

$$\omega E \sim \frac{(v_3\theta - 1)(v_3\overline{\theta} - 1)}{x-1} \tag{A.10}$$

$$\frac{\dot{E}}{E} \sim \frac{x}{1-x} \tag{A.11}$$

$$\frac{\dot{\omega}}{\omega} \sim \left(\frac{1}{v_2 - \frac{1}{\theta}} + \frac{1}{v_2 - \frac{1}{\overline{\theta}}}\right)\dot{v}_2 \tag{A.12}$$

$$\dot{u}_2 \sim \frac{\theta + \overline{\theta}}{x-1} - \frac{2}{(x-1)v_2} \tag{A.13}$$

$$\dot{v}_2 \sim \frac{(v_2\theta - 1)(v_2\overline{\theta} - 1)}{(x-1)u_2} \tag{A.14}$$

$$\omega E_0 \sim -(v_2\theta - 1)(v_2\overline{\theta} - 1) \tag{A.15}$$

$$\frac{\dot{E}_0}{E_0} \sim x\frac{\theta_0 + \theta_1}{x-1} - \frac{2x}{(x-1)v_2}. \tag{A.16}$$

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Estimates near $\boldsymbol{H}$, that is $\boldsymbol{v_2} \longrightarrow \mathbf{0}$:

$$\omega = u_2 v_2^2 \tag{A.17}$$

$$\omega E \sim \frac{(u_2-1)(xu_2-1)}{x-1} \tag{A.18}$$

$$\frac{\dot{E}}{E} \sim \frac{1}{1-x} + \frac{1}{xu_2-1} \tag{A.19}$$

$$\frac{\dot{\omega}}{\omega} \sim \left(\frac{1}{u_2-1} + \frac{x}{xu_2-1}\right)\dot{u}_2 \tag{A.20}$$

$$\dot{u}_2 \sim \frac{2(u_2-1)(xu_2-1)}{(1-x)v_2} \tag{A.21}$$

$$\dot{v}_2 \sim -\frac{xu_2}{x-1} + \frac{1}{(x-1)u_2} \tag{A.22}$$

$$\omega E_0 \sim u_2 - 1 \tag{A.23}$$

$$\frac{\dot{E}_0}{E_0} \sim -\frac{x\dot{u}_2}{xu_2-1}. \tag{A.24}$$

A.0.4. Third chart $(u_3, v_3) = \left(\frac{1}{u}, uv\right)$

$$\begin{cases} (u,v) &= \left(\frac{1}{u_3}, u_3 v_3\right) \\ \omega_3 &= -u_3 \\ E &= \frac{1}{x-1}\left[\frac{(xu_3-1)(u_3-1)v_3-(\theta+\overline{\theta})}{u_3}v_3 - \theta\overline{\theta}\frac{xu_3-1}{u_3} - x\theta_0 u_3 v_3 + ((x+1)\theta_0 + x\theta_1 + (\theta_x - 1))v_3\right] \\ \dot{E} &= \frac{x(u_3-1)}{(x-1)^2}\left[\frac{(v_3-\theta)(v_3-\overline{\theta})}{u_3} - (v_3-\theta_0)v_3\right] \end{cases}$$

$$\begin{cases} \dot{u}_3 &= -\frac{(u_3-1)(2v_3-(\theta+\overline{\theta}))}{1-x} - \theta_1 u_3 + \frac{x}{1-x}u_3(u_3-1)(2v_3-\theta_0) \\ \dot{v}_3 &= \frac{1}{(1-x)u_3}(v_3-\theta)(v_3-\overline{\theta}) - x\frac{(v_3-\theta_0)u_3 v_3}{1-x} \\ \frac{\dot{\omega}_3}{\omega_3} &= -\frac{(u_3-1)(2v_3-(\theta+\overline{\theta}))}{(1-x)u_3} - \theta_1 + \frac{x}{1-x}(u_3-1)(2v_3-\theta_0) \end{cases}.$$

New base points:

$$b_\infty : (u_3, v_3) = (0, \theta), \quad b_\infty^- : (u_3, v_3) = \left(0, \overline{\theta}\right).$$

No other visible base points.

New elliptic base points are b_∞ and b_∞^-.

Visible components of the infinity set: $D_\infty : \{u_3 = 0\}$.

Estimates near $\boldsymbol{D_\infty}$, that is $\boldsymbol{u_3} \longrightarrow \mathbf{0}$:

$$\omega = -u_3 \tag{A.25}$$

$$\omega E \sim \frac{(v_3-\theta)(v_3-\overline{\theta})}{1-x} \tag{A.26}$$

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$$\frac{\dot{E}}{E} \sim \frac{x}{1-x} \tag{A.27}$$

$$\frac{\dot{\omega}}{\omega} \sim \frac{1}{u_3}\dot{u}_3 \tag{A.28}$$

$$\dot{u}_3 \sim \frac{2v_3 - (\theta + \overline{\theta})}{1-x} \tag{A.29}$$

$$\dot{v}_3 \sim \frac{(v_3-\theta)(v_3-\overline{\theta})}{(1-x)u_3} \tag{A.30}$$

$$\omega E_0 \sim (v_3-\theta)(v_3-\overline{\theta}) \tag{A.31}$$

$$\frac{\dot{E}_0}{E_0} \sim -\frac{x(\theta_x-1)}{x-1} + x\dot{u}_3. \tag{A.32}$$

B. Okamoto desingularisation

B.1. Details for the blow-up procedure

B.1.1. Blow up of $\beta_0, \beta_1, \beta_x, \beta_\infty$

Let us first blow up the points $\beta_0, \beta_1, \beta_x$.

★ Recall that for $i \in \{0, 1, x\}$, we have $\mathcal{V}_i : \{u_0 = i\} \cup \{u_1 = i\} \cup \{u_2 = 1/i\} \cup \{u_3 = 1/i\}$ and that $\beta_i \in \mathcal{V}_i$. Note further that $\mathcal{V}_i \setminus \{\beta_i\} \subset \mathbb{C}^2_{u,v}$. So whenever we remove one of the points β_i from a chart other than $\mathbb{C}^2_{u,v}$, we may just as well remove the visible part of the whole line $\mathcal{V}_i$, without changing the global picture.

★ Replace the chart $\mathbb{C}^2_{u_1,v_1}$ by the following six $\mathbb{C}^2$-charts:

$$\begin{aligned}
(u_{01}, v_{01}) &:= \left(u_1, \tfrac{v_1}{u_1}\right) & (u_{02}, v_{02}) &:= \left(\tfrac{u_1}{v_1}, v_1\right)\\
(u_{11}, v_{11}) &:= \left(u_1 - 1, \tfrac{v_1}{u_1-1}\right) & (u_{12}, v_{12}) &:= \left(\tfrac{u_1-1}{v_1}, v_1\right)\\
(u_{x1}, v_{x1}) &:= \left(u_{01} - x, \tfrac{v_{01}}{u_{01}-x}\right) = \left(u_1 - x, \tfrac{v_1}{u_1(u_1-x)}\right) & (u_{x2}, v_{x2}) &:= \left(\tfrac{u_{01}-x}{v_{01}}, v_{01}\right) = \left(\tfrac{u_1(u_1-x)}{v_1}, \tfrac{v_1}{u_1}\right).
\end{aligned}$$

★ In each pair of charts $\mathbb{C}^2_{u_{i1},v_{i1}}, \mathbb{C}^2_{u_{i2},v_{i2}}$ (which effectively replaces β_i by the exceptional line $\mathcal{D}_i := \{u_{i1} = 0\} \cup \{v_{i2} = 0\}$), we have to remove the points β_j for $j \in \{0, 1, x\} \setminus \{i\}$ if visible. Yet these points are visible only in the $\mathbb{C}^2_{u_{i1},v_{i1}}$ charts. By the remark above, we may remove

the following visible parts of $\mathcal{V}_0 : \{u_{11} = -1\}$,

the following visible parts of $\mathcal{V}_1 : \{u_{01} = 1\} \cup \{u_{x1} = 1 - x\}$,

the following visible parts of $\mathcal{V}_x : \{u_{01} = x\} \cup \{u_{11} = x - 1\}$.

★ The three charts $\mathbb{C}^2_{u_{i1},v_{i1}}$ with the removed lines are equivalent to a single $\mathbb{C}^2$-chart, namely

$$(\tilde{u}_1, \tilde{v}_1) := \left(u_{11} + 1, \frac{v_{11}}{(u_{11}+1)(u_{11}+1-x)}\right) = \left(u_1, \frac{v_1}{u_1(u_1-1)(u_1-x)}\right).$$

For $x = 0$, the lines $\mathcal{V}_0$ and $\mathcal{V}_x$ cannot be distinguished. Yet then the pair of charts $\mathbb{C}^2_{u_{x1},v_{x1}}$,$\mathbb{C}^2_{u_{x2},v_{x2}}$ replaces the chart $\mathbb{C}^2_{u_{01},v_{01}}$. Therefore, this coordinate change is still valid.

★ Similarly, we need to remove the visible points β_x, β_1 from the chart $\mathbb{C}^2_{u_2,v_2}$, which can effectively be done by setting

$$(\tilde{u}_2, \tilde{v}_2) := \left(u_2, \frac{v_2}{(1-u_2)(1-xu_2)}\right) = \left(\frac{1}{u_1}, \frac{v_1 u_1}{(u_1-1)(u_1-x)}\right) = \left(\frac{1}{\tilde{u}_1}, \tilde{v}_1\tilde{u}_1^2\right).$$

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Indeed, removing $\{\tilde{u}_2 \in \{1, 1/t\}$, this chart is isomorphic to $\mathbb{C}^2_{u_2,v_2} \setminus \mathcal{V}_x \cup \mathcal{V}_1$, removing $\{\tilde{u}_2 = 0\}$, this chart is isomorphic to $\mathbb{C}^2_{\tilde{u}_1,\tilde{v}_1} \setminus \{\tilde{u}_1 = 0\}$. Note that this holds also for $x = 0$.

For the blow-up of β_∞, we may stick to the standard procedure:

★ Remove the point β_∞ from the chart $\mathbb{C}^2_{\tilde{u}_2,\tilde{v}_2}$. Note that then it remains visible only in the chart $\mathbb{C}^2_{u_3,v_3}$, as $\beta_\infty : (0, \theta)$.

★ Replace the chart $\mathbb{C}^2_{u_3,v_3}$ by the following pair of $\mathbb{C}^2$-charts:

$$(u_{\infty 1}, v_{\infty 1}) := \left(u_3, \frac{v_3 - \theta}{u_3}\right), \quad (u_{\infty 2}, v_{\infty 2}) := \left(\frac{u_3}{v_3 - \theta}, v_3 - \theta\right).$$

★ Note that $\mathbb{C}^2_{u_{\infty 1},-v_{\infty 1}}$ (with a minus sign) corresponds to the classical chart of a certain surface, Σ_θ much used in publications concerning the Okamoto desingularisation of the sixth Painlevé equation (see, for example [29]). Hence, for traditional reasons, we denote this chart by $\mathbb{C}^2_{\tilde{u}_3,\tilde{v}_3} := \mathbb{C}^2_{u_{\infty 1},-v_{\infty 1}}$.

B.1.2. The vector field in the resulting new charts

In our seven new charts, that we have to add to the chart $\mathbb{C}^2_{u_0,v_0}$ to obtain the global picture after blow-up of $\beta_0, \beta_1, \beta_x, \beta_\infty$, the vector field, respectively, reads as follows.

$$\begin{cases} \dot{\tilde{u}}_1 = \frac{2}{(x-1)\tilde{v}_1} - \frac{\tilde{u}_1(\tilde{u}_1-1)(\tilde{u}_1-x)}{(x-1)}\left(\frac{\theta_0}{\tilde{u}_1} + \frac{\theta_1}{\tilde{u}_1-1} + \frac{\theta_x-1}{\tilde{u}_1-x}\right) \\ \dot{\tilde{v}}_1 = \frac{x\theta_0\tilde{v}_1-1}{(x-1)\tilde{u}_1} - \frac{(x-1)\theta_1\tilde{v}_1+1}{(x-1)(\tilde{u}_1-1)} + \frac{x(x-1)\theta_x\tilde{v}_1-1}{(x-1)(\tilde{u}_1-x)} + \left(\frac{\theta+\overline{\theta}}{x-1}(\tilde{u}_1 + x - 1) - 2\frac{x\theta_0+(x-1)\theta_1}{x-1}\right)\tilde{v}_1 + \\ \qquad + \frac{\theta\overline{\theta}}{x-1}\tilde{u}_1(\tilde{u}_1 - 1)(\tilde{u}_1 - x)\tilde{v}_1^2, \end{cases}$$

$$\begin{cases} \dot{\tilde{u}}_2 = -\frac{2}{(x-1)\tilde{v}_2} + \frac{(1-\tilde{u}_2)(1-x\tilde{u}_2)}{x-1}\left(\theta_0 + \frac{\theta_1}{1-\tilde{u}_2} + \frac{\theta_x-1}{1-x\tilde{u}_2}\right) \\ \dot{\tilde{v}}_2 = \frac{(\theta\tilde{v}_2-1)(\overline{\theta}\tilde{v}_2-1)}{(x-1)\tilde{u}_2} + \frac{(x-1)\theta_x\tilde{v}_2-x}{(x-1)(1-x\tilde{u}_2)} - \frac{(x-1)\theta_1\tilde{v}_2+1}{(x-1)(1-\tilde{u}_2)} + \left(2\frac{x\theta_1+\theta_x-1}{x-1} - 1\right)\tilde{v}_2 + \frac{\theta\overline{\theta}\tilde{v}_2-\theta_0}{x-1}(x\tilde{u}_2 - x - 1)\tilde{v}_2, \end{cases}$$

$$\begin{cases} \dot{\tilde{u}}_3 = \frac{\tilde{u}_3(1-\tilde{u}_3)(1-x\tilde{u}_3)}{(x-1)}\left(2\tilde{v}_3 - \frac{\theta_\infty}{\tilde{u}_3} + \frac{x(\theta_x-1)}{1-x\tilde{u}_3} + \frac{\theta_1}{1-\tilde{u}_3}\right) \\ \dot{\tilde{v}}_3 = -\frac{3x\tilde{u}_3^2-2(x+1)\tilde{u}_3+1}{x-1}\tilde{v}_3^2 + \frac{x(2\theta-\theta_0)(2\tilde{u}_3-1)+(x-1)\theta_1-\theta_\infty}{x-1}\tilde{v}_3 - x\frac{\theta(\theta-\theta_0)}{x-1}, \end{cases}$$

$$\begin{cases} \dot{u}_{02} = x\frac{u_{02}-\theta_0}{(x-1)v_{02}} - \frac{u_{02}^3 v_{02}}{x-1} + \frac{\theta+\overline{\theta}}{x-1}u_{02}^2 v_{02} - \frac{\theta\overline{\theta}}{x-1}u_{02}v_{02} \\ \dot{v}_{02} = \frac{3(u_{02}v_{02})^2-2(x+1)u_{02}v_{02}+x}{x-1} - 2\frac{\theta+\overline{\theta}}{x-1}u_{02}v_{02}^2 + \left(\frac{x\theta_0}{x-1} + \theta_1 + \frac{\theta+\overline{\theta}}{x-1}\right)v_{02} + \frac{\theta\overline{\theta}}{x-1}v_{02}^2, \end{cases}$$

$$\begin{cases} \dot{u}_{x2} = -\frac{(\theta_x-1)(u_{x2}v_{x2}+x)-u_{x2}+x}{v_{x2}} - \frac{u_{x2}(u_{x2}+x\theta_0)}{(x-1)(u_{x2}v_{x2}+x)} + \frac{u_{x2}(u_{x2}+\theta_0)}{x-1} - \frac{\theta\overline{\theta}u_{x2}v_{x2}(u_{x2}v_{x2}+x)}{x-1} \\ \dot{v}_{x2} = \frac{(u_{x2}v_{x2}+x)(\theta v_{x2}-1)(\overline{\theta}v_{x2}-1)}{x-1} + x\frac{\theta_0 v_{x2}-1}{(x-1)(u_{x2}v_{x2}+x)}, \end{cases}$$

$$\begin{cases} \dot{u}_{12} = -\frac{u_{12}-\theta_1}{v_{12}} - \frac{u_{12}^3 v_{12}}{x-1} + \frac{\theta+\overline{\theta}}{x-1}u_{12}^2 v_{12} - \frac{\theta\overline{\theta}}{x-1}u_{12}v_{12} \\ \dot{v}_{12} = \frac{3u_{12}^2 v_{12}^2-2(x-2)(u_{12}v_{12})-x+1}{x-1} - 2\frac{\theta+\overline{\theta}}{x-1}u_{12}v_{12}^2 + \left(\frac{x\theta_0}{x-1} + \theta_1 - \frac{\theta+\overline{\theta}}{x-1}\right)v_{12} + \frac{\theta\overline{\theta}}{x-1}v_{12}^2, \end{cases}$$

$$\begin{cases} \dot{u}_{\infty 2} = -\frac{3x(u_{\infty 2}v_{\infty 2})^2-2(x+1)u_{\infty 2}v_{\infty 2}+1}{x-1} - 2x\frac{2\theta-\theta_0}{x-1}u_{\infty 2}^2 v_{\infty 2} + \left(\frac{\theta_\infty}{x-1} - \theta_1 + \frac{x(2\theta-\theta_0)}{x-1}\right)u_{\infty 2} - x\frac{\theta(\theta-\theta_0)}{x-1}u_{\infty 2}^2 \\ \dot{v}_{\infty 2} = \frac{v_{\infty 2}+\theta_\infty}{(1-x)u_{\infty 2}} - x\frac{u_{\infty 2}v_{\infty 2}(v_{\infty 2}+\theta-\theta_0)(v_{\infty 2}+\theta)}{1-x}. \end{cases}$$

B.2. Detailed charts of Okamoto's space

B.2.1. The chart $(u_0, v_0) = (u, v)$

Domain of definition: $\mathbb{C}^2$.
Visible components of the infinity set: $\emptyset$.
Visible exceptional lines: $\emptyset$

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$$\begin{cases} (u_0, v_0) &= (u, v) \\ \omega_0 &= 1 \\ E &= \frac{u_0(u_0-1)(u_0-x)}{x-1}\left\{v_0^2 - v_0\left(\frac{\theta_0}{u_0} + \frac{\theta_1}{u_0-1} + \frac{\theta_x-1}{u_0-x}\right) + \frac{\theta\overline{\theta}}{u_0(u_0-1)}\right\}. \end{cases}$$

$$\begin{cases} \dot{u}_0 = \frac{u_0(u_0-1)(u_0-x)}{x-1}\left(2v_0 - \frac{\theta_0}{u_0} - \frac{\theta_1}{u_0-1} - \frac{\theta_x-1}{u_0-x}\right) \\ \dot{v}_0 = -\frac{3u_0^2-2(x+1)u_0+x}{x-1}v_0^2 + 2\frac{\theta+\overline{\theta}}{x-1}u_0v_0 - \left(\frac{x\theta_0}{x-1} + \theta_1 + \frac{\theta+\overline{\theta}}{x-1}\right)v_0 - \frac{\theta\overline{\theta}}{x-1}. \end{cases}$$

***B.2.2.* The chart** $(\tilde{u}_1, \tilde{v}_1) = \left(u, \frac{1}{u(u-x)(u-1)v}\right)$

Domain of definition: $\mathbb{C}^2 \setminus \{\gamma_0, \gamma_x, \gamma_1\}$, where

$$\gamma_0 : \left(0, \frac{1}{x\theta_0}\right), \quad \gamma_x : \left(x, \frac{1}{x(x-1)\theta_x}\right), \quad \gamma_1 : \left(1, \frac{1}{(1-x)\theta_1}\right).$$

Visible components of the infinity set:

$$\mathcal{H}^* : \{\tilde{v}_1 = 0\}, \quad \mathcal{D}_0^* : \{\tilde{u}_1 = 0\}, \quad \mathcal{D}_x^* : \{\tilde{u}_1 = x\}, \quad \mathcal{D}_1^* : \{\tilde{u}_1 = 1\}.$$

Visible exceptional lines: $\emptyset$

$$\begin{cases} (\tilde{u}_1, \tilde{v}_1) &= \left(u, \frac{1}{u(u-x)(u-1)v}\right) \\ (u, v) &= \left(\tilde{u}_1, \frac{1}{\tilde{u}_1(\tilde{u}_1-x)(\tilde{u}_1-1)\tilde{v}_1}\right) \\ \tilde{\omega}_1 &= -\tilde{u}_1(\tilde{u}_1 - x)(\tilde{u}_1 - 1)\tilde{v}_1^2 \\ \dot{E} &= -\frac{x}{(x-1)^2}\left\{\frac{1}{\tilde{u}_1(\tilde{u}_1-x)^2\tilde{v}_1^2} - \frac{(\theta+\overline{\theta})\tilde{u}_1-\theta_0}{\tilde{v}_1\tilde{u}_1(\tilde{u}_1-x)} + \theta\overline{\theta}(\tilde{u}_1 - 1)\right\} \\ E &= \frac{1}{x-1}\left\{\frac{1}{\tilde{u}_1(\tilde{u}_1-x)(\tilde{u}_1-1)\tilde{v}_1^2} - \frac{1}{\tilde{v}_1}\left(\frac{\theta_0}{\tilde{u}_1} + \frac{\theta_1}{\tilde{u}_1-1} + \frac{\theta_x-1}{\tilde{u}_1-x}\right) + \theta\overline{\theta}(\tilde{u}_1 - x)\right\}. \end{cases}$$

$$\begin{cases} \dot{\tilde{u}}_1 = \frac{2}{(x-1)\tilde{v}_1} - \frac{\tilde{u}_1(\tilde{u}_1-1)(\tilde{u}_1-x)}{x-1}\left(\frac{\theta_0}{\tilde{u}_1} + \frac{\theta_1}{\tilde{u}_1-1} + \frac{\theta_x-1}{\tilde{u}_1-x}\right) \\ \dot{\tilde{v}}_1 = \frac{x\theta_0\tilde{v}_1-1}{(x-1)\tilde{u}_1} - \frac{(x-1)\theta_1\tilde{v}_1+1}{(x-1)(\tilde{u}_1-1)} + \frac{x(x-1)\theta_x\tilde{v}_1-1}{(x-1)(\tilde{u}_1-x)} + \left(\frac{\theta+\overline{\theta}}{x-1}(\tilde{u}_1 + x - 1) - 2\frac{x\theta_0+(x-1)\theta_1}{x-1}\right)\tilde{v}_1 + \\ \quad + \frac{\theta\overline{\theta}}{x-1}\tilde{u}_1(\tilde{u}_1 - 1)(\tilde{u}_1 - x)\tilde{v}_1^2. \end{cases}$$

***B.2.3.* The chart** $(\tilde{u}_2, \tilde{v}_2) = \left(\frac{1}{u}, \frac{u}{(u-x)(u-1)v}\right)$

Domain of definition: $\mathbb{C}^2 \setminus \{\gamma_x, \gamma_1, \beta_\infty, \beta_\infty^-\}$, where

$$\gamma_x : \left(\frac{1}{x}, \frac{x}{(x-1)\theta_x}\right), \quad \gamma_1 : \left(1, \frac{1}{(1-x)\theta_1}\right), \quad \beta_\infty : \left(0, \frac{1}{\theta}\right), \quad \beta_\infty^- : \left(0, \frac{1}{\overline{\theta}}\right).$$

Visible components of the infinity set:

$$\mathcal{H}^* : \{\tilde{v}_2 = 0\}, \quad \mathcal{D}_\infty^{**} : \{\tilde{u}_2 = 0\}, \quad \mathcal{D}_x^* : \{\tilde{u}_2 = 1/x\}, \quad \mathcal{D}_1^* : \{\tilde{u}_2 = 1\}.$$

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Visible exceptional lines: $\emptyset$

$$
\begin{cases}
(\tilde{u}_2, \tilde{v}_2) = \left(\frac{1}{u}, \frac{u}{(u-x)(u-1)v}\right) \\
(u, v) = \left(\frac{1}{\tilde{u}_2}, \frac{\tilde{u}_2}{(1-x\tilde{u}_2)(1-\tilde{u}_2)\tilde{v}_2}\right) \\
\tilde{\omega}_2 = \tilde{u}_2(1 - x\tilde{u}_2)(1 - \tilde{u}_2)\tilde{v}_2^2 \\
\dot{E} = -\frac{x}{(x-1)^2}\left\{\frac{1}{\tilde{u}_2(1-x\tilde{u}_2)^2\tilde{v}_2^2} - \frac{(\theta+\overline{\theta})-\theta_0\tilde{u}_2}{\tilde{v}_2\tilde{u}_2(1-x\tilde{u}_2)} + \frac{\theta\overline{\theta}}{\tilde{u}_2}(1 - \tilde{u}_2)\right\} \\
E = \frac{(\theta\tilde{v}_2-1)(\overline{\theta}\tilde{v}_2-1)}{(x-1)\tilde{u}_2\tilde{v}_2^2} - \frac{(x-1)\theta_1\tilde{v}_2+1}{(x-1)^2(1-\tilde{u}_2)\tilde{v}_2^2} - x\frac{(x-1)(\theta_x-1)\tilde{v}_2-x}{(x-1)^2(1-x\tilde{u}_2)\tilde{v}_2^2} - \frac{x\theta\overline{\theta}}{x-1}
\end{cases}
$$

$$
\begin{cases}
\dot{\tilde{u}}_2 = -\frac{2}{(x-1)\tilde{v}_2} + \frac{(1-\tilde{u}_2)(1-x\tilde{u}_2)}{x-1}\left(\theta_0 + \frac{\theta_1}{1-\tilde{u}_2} + \frac{\theta_x-1}{1-x\tilde{u}_2}\right) \\
\dot{\tilde{v}}_2 = \frac{(\theta\tilde{v}_2-1)(\overline{\theta}\tilde{v}_2-1)}{(x-1)\tilde{u}_2} + \frac{(x-1)\theta_x\tilde{v}_2-x}{(x-1)(1-x\tilde{u}_2)} - \frac{(x-1)\theta_1\tilde{v}_2+1}{(x-1)(1-\tilde{u}_2)} + \left(2\frac{x\theta_1+\theta_x-1}{x-1} - 1\right)\tilde{v}_2 + \frac{\theta\overline{\theta}\tilde{v}_2-\theta_0}{x-1}(x\tilde{u}_2 - x - 1)\tilde{v}_2\,.
\end{cases}
$$

***B.2.4.* The chart** $(\tilde{u}_3, \tilde{v}_3) = \left(u_3, -\frac{v_3-\theta}{u_3}\right)$

Domain of definition: $\mathbb{C}^2$.

Visible components of the infinity set: $\emptyset$.

Visible exceptional lines:$\mathcal{E}_\infty : \{\tilde{u}_3 = 0\}$

$$
\begin{cases}
(\tilde{u}_3, \tilde{v}_3) = \left(\frac{1}{u}, -u(uv - \theta)\right) \\
(u, v) = \left(\frac{1}{\tilde{u}_3}, -\tilde{u}_3^2\tilde{v}_3 + \theta\tilde{u}_3\right) \\
\tilde{\omega}_3 = 1 \\
E = (1 - \tilde{u}_3)(1 - x\tilde{u}_3)\left(\frac{\tilde{v}_3(\tilde{u}_3\tilde{v}_3-2\theta+\theta_0)}{x-1} + \frac{\theta\overline{\theta}+\theta_1(\tilde{v}_3-\theta)}{(x-1)(1-\tilde{u}_3)} + \frac{(\theta_x-1)(\tilde{v}_3-x\theta)}{(x-1)(1-x\tilde{u}_3)}\right)
\end{cases}
$$

$$
\begin{cases}
\dot{\tilde{u}}_3 = \frac{\tilde{u}_3(1-\tilde{u}_3)(1-x\tilde{u}_3)}{(x-1)}\left(2\tilde{v}_3 - \frac{\theta_\infty}{\tilde{u}_3} + \frac{x(\theta_x-1)}{1-x\tilde{u}_3} + \frac{\theta_1}{1-\tilde{u}_3}\right) \\
\dot{\tilde{v}}_3 = -\frac{3x\tilde{u}_3^2-2(x+1)\tilde{u}_3+1}{x-1}\tilde{v}_3^2 + \frac{x(2\theta-\theta_0)(2\tilde{u}_3-1)+(x-1)\theta_1-\theta_\infty}{x-1}\tilde{v}_3 - x\frac{\theta(\theta-\theta_0)}{x-1}.
\end{cases}
$$

***B.2.5.* The chart** $(u_{03}, v_{03}) = \left(u_{02} - \theta_0, \frac{v_{02}}{u_{02}-\theta_0}\right)$

Domain of definition: $\mathbb{C}^2$.

Visible components of the infinity set: $\mathcal{D}_0^* : \{v_{03} = 0\}$.

Visible exceptional lines: $\mathcal{E}_0 : \{u_{03} = 0\}$

$$
\begin{cases}
(u_{03}, v_{03}) = \left(uv - \theta_0, \frac{1}{uv^2-\theta_0 v}\right) \\
(u, v) = \left(u_{03}^2 v_{03} + \theta_0 u_{03} v_{03}, \frac{1}{u_{03}v_{03}}\right) \\
\omega_{03} = -v_{03} \\
E = \frac{1}{x-1}\Big[\frac{((u_{03}+\theta_0)u_{03}v_{03}-x)((u_{03}+\theta_0)u_{03}v_{03}-1)}{v_{03}} - (\theta + \overline{\theta} - \theta_0)(u_{03} + \theta_0)^2 u_{03}v_{03} + \\
\qquad + \theta\overline{\theta}((u_{03} + \theta_0)u_{03}v_{03} - x) + \left(x\theta_1 + (\theta_x - 1)\right)(u_{03} + \theta_0)\Big]
\end{cases}
$$

$$
\begin{cases}
\dot{u}_{03} = \frac{1}{1-x}(u_{03} + \theta_0 - \theta)(u_{03} + \theta_0 - \overline{\theta})(u_{03} + \theta_0)u_{03}v_{03} - \frac{x}{(1-x)v_{03}} \\
\dot{v}_{03} = -\frac{1}{1-x}(u_{03} + \theta_0 - \theta)(u_{03} + \theta_0 - \overline{\theta})(2u_{03} + \theta_0)v_{03}^2 - \frac{2(u_{03}+\theta_0)-(\theta+\overline{\theta})}{1-x}(u_{03} + \theta_0)u_{03}v_{03}^2 + \, . \\
\qquad + \frac{2u_{03}+\theta_0-\theta_x+1}{1-x}v_{03} + \frac{x(2u_{03}+\theta_0-\theta_1)}{1-x}v_{03}
\end{cases}
$$

We have:

$$\begin{aligned}
&\omega_{x3} = -\frac{1}{((u-x)uv - x\theta_x)v} = -\frac{u_{03}v_{03}}{u_{03}v_{03}(u_{03}+\theta_0)^2 - x(u_{03}+\theta_0+\theta_x)},\\
&\omega_{x3} \sim \frac{u_{03}v_{03}}{x(u_{03}+\theta_0+\theta_x)},\\
&E \sim \frac{1}{x-1}\frac{x}{v_{03}},\\
&E\omega_{x3} \sim \frac{1}{x-1}\frac{u_{03}}{(u_{03}+\theta_0+\theta_x)} \longrightarrow \frac{1}{x-1},\\
&(x-1)\omega_{x3} \longrightarrow \frac{1}{E}.
\end{aligned}$$

Further formulae:

$$\begin{cases}
-\frac{(x-1)\omega_{03}}{x}E = 1 - \frac{v_{03}}{x}\Big[(\theta+\overline{\theta}-\theta_0-u_{03})(u_{03}+\theta_0)^2u_{03}v_{03} + (u_{03}+\theta_0)u_{03}(\theta\overline{\theta}v_{03}+x+1)- \\
\qquad -\theta\overline{\theta}x + \Big(x\theta_1+\theta_x-1\Big)(u_{03}+\theta_0)\Big] \\
\frac{\dot{\omega}_{03}}{\omega_{03}} = -\frac{1}{1-x}(u_{03}+\theta_0-\theta)(u_{03}+\theta_0-\overline{\theta})(2u_{03}+\theta_0)v_{03} - \frac{2(u_{03}+\theta_0)-(\theta+\overline{\theta})}{1-x}(u_{03}+\theta_0)u_{03}v_{03}+ \\
\qquad +\frac{2u_{03}+\theta_0-\theta_x+1}{1-x} + \frac{x(2u_{03}+\theta_0-\theta_1)}{1-x}
\end{cases}.$$

_B.2.6._ The chart $(u_{04}, v_{04}) = \left(\frac{1}{v_{03}}, u_{03}v_{03}\right)$

Domain of definition: $\mathbb{C}^2$.

Visible components of the infinity set: $\emptyset$.

Visible exceptional lines:$\mathcal{E}_0 : \{v_{04} = 0\}$

$$\begin{cases}
(u_{04}, v_{04}) = \left(uv^2 - \theta_0 v, \frac{1}{v}\right) \\
(u, v) = \left((u_{04}v_{04}+\theta_0)v_{04}, \frac{1}{v_{04}}\right) \\
\omega_{04} = -1 \\
E = \frac{1}{x-1}\Big[((u_{04}v_{04}+\theta_0)v_{04}-x)((u_{04}v_{04}+\theta_0)v_{04}-1)u_{04} - (\theta+\overline{\theta}-\theta_0)(u_{04}v_{04}+\theta_0)^2v_{04}+ \\
\qquad +\theta\overline{\theta}((u_{04}v_{04}+\theta_0)v_{04}-x) + (x\theta_1+(\theta_x-1))(u_{04}v_{04}+\theta_0)\Big]
\end{cases}$$

$$\begin{cases}
\dot{u}_{04} = \frac{1}{1-x}(u_{04}v_{04}+\theta_0-\theta)(u_{04}v_{04}+\theta_0-\overline{\theta})(2u_{04}v_{04}+\theta_0) - \theta_1 u_{04}- \\
\qquad -\frac{(\theta+\overline{\theta})-2(u_{04}v_{04}+\theta_0)}{1-x}[(u_{04}v_{04}+\theta_0)v_{04}-1]u_{04} - \frac{x(2u_{04}v_{04}+\theta_0)}{1-x}u_{04} \\
\dot{v}_{04} = -\frac{1}{1-x}\Big[[2(u_{04}v_{04}+\theta_0)-\theta]v_{04}-1\Big]\Big[\Big[2(u_{04}v_{04}+\theta_0)-\overline{\theta}\Big]v_{04}-1\Big]+ \\
\qquad +\frac{1}{1-x}[(u_{04}v_{04}+\theta_0)v_{04}-1]^2 + \theta_1 v_{04} + \frac{x}{1-x}[(2u_{04}v_{04}+\theta_0)v_{04}-1].
\end{cases}$$

_B.2.7._ The chart $(u_{x3}, v_{x3}) = \left(u_{x2} - x\theta_x, \frac{v_{x2}}{u_{x2}-x\theta_x}\right)$

Domain of definition: $\mathbb{C}^2 \setminus \{\gamma_0, \alpha\}$, where

$$\gamma_0 : \left(-x(\theta_0+\theta_x), -\frac{1}{x\theta_0(\theta_0+\theta_x)}\right), \quad \alpha : \left(-x\frac{\theta_x}{2}, \frac{4}{x\theta_x^2}\right).$$

Here, α is an apparent base point on $\mathcal{D}_0 \setminus \mathcal{H}$, not a base point in the charts $\mathbb{C}^2_{u_{03},v_{03}}$ and $\mathbb{C}^2_{u_{04},v_{04}}$.

Visible components of the infinity set: $\mathcal{D}_0 \setminus \gamma_0 : \{u_{x3}v_{x3}(u_{x3}+x\theta_x) = -x\}$, $\mathcal{D}_x^* : \{v_{x3} = 0\}$.

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Visible exceptional lines: $\mathcal{E}_x : \{u_{x3} = 0\}$

$$\left\{\begin{array}{rcl}(u_{x3}, v_{x3}) &=& \left((u-x)uv - x\theta_x, \tfrac{1}{((u-x)uv-x\theta_x)uv}\right)\\ (u,v) &=& \left((u_{x3}+x\theta_x)u_{x3}v_{x3}+x, \tfrac{1}{((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)u_{x3}v_{x3}}\right)\\ \omega_{x3} &=& -((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)v_{x3}\\ E &=& \tfrac{1}{(x-1)u_{x3}v_{x3}}\Big[-\tfrac{u_{x3}+x\theta_x+x\theta_0}{(u_{x3}+x\theta_x)u_{x3}v_{x3}+x} - (\theta u_{x3}v_{x3}-1)(\overline{\theta}u_{x3}v_{x3}-1)(u_{x3}+x\theta_x)+\\ && \hfill +\theta_0 - (x-1)(\theta_x-1)\Big]\end{array}\right.$$

$$\left\{\begin{array}{rcl}\dot{u}_{x3} &=& -u_{x3}(\theta_x-1) - x\theta_x^2 - \tfrac{(u_{x3}+x\theta_x+\theta_0)(u_{x3}+x\theta_x)}{1-x} + \tfrac{1}{v_{x3}} + \tfrac{(u_{x3}+x(\theta_0+\theta_x))(u_{x3}+x\theta_x)}{(1-x)((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)}+\\ && +\tfrac{\theta\overline{\theta}}{1-x}((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)(u_{x3}+x\theta_x)u_{x3}v_{x3}\\ \dot{v}_{x3} &=& \tfrac{((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)v_{x3}}{1-x}\Big[-\theta\overline{\theta}(2u_{x3}+x\theta_x)v_{x3} + (\theta+\overline{\theta})\Big]-\\ && -\tfrac{(2u_{x3}+x\theta_x)v_{x3}}{(1-x)((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)} + \tfrac{(\theta_0+\theta_x)v_{x3}}{1-x}\left(1 + \tfrac{x\theta_x(u_{x3}+x\theta_x)v_{x3}-2x}{(u_{x3}+x\theta_x)u_{x3}v_{x3}+x}\right) - v_{x3}.\end{array}\right.$$

Further formulae:

$$\left\{\begin{array}{rcl}-\omega_{x3}E &=& -\Big[\tfrac{x(\theta\overline{\theta}u_{x3}v_{x3}-(\theta+\overline{\theta}+\theta_x-1))+(\theta+\overline{\theta}-\theta_1)+(\theta u_{x3}v_{x3}-1)(\overline{\theta}u_{x3}v_{x3}-1)(u_{x3}+x\theta_x)}{1-x}\Big](u_{x3}+x\theta_x)v_{x3}+\\ && +\left(1+\tfrac{x}{u_{x3}}\right)\\ \tfrac{\dot{\omega}_{x3}}{\omega_{x3}} &=& -\tfrac{((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)(2u_{x3}+x\theta_x)v_{x3}}{1-x}\cdot\theta\overline{\theta} + \tfrac{2u_{x3}+x\theta_x}{x-1}+\\ && +\tfrac{\theta+\overline{\theta}}{1-x}(2(u_{x3}+x\theta_x)u_{x3}v_{x3}+x) - x(\theta_0+\theta_x)\tfrac{\theta_x(u_{x3}+x\theta_x)v_{x3}-1}{(x-1)((u_{x3}+x\theta_x)u_{x3}v_{x3}+x)}\\ &=& \tfrac{2u_{x3}+x\theta_x}{1-x}\left(\theta\overline{\theta}\omega_{x3} - (\theta_0+\theta_x)\theta_x\tfrac{xv_{x3}}{\omega_{x3}}v_{x3}+1\right) - x\tfrac{\theta+\overline{\theta}}{1-x}(2\tfrac{\omega_{x3}}{xv_{x3}}+1) - \tfrac{(\theta_0+\theta_x)}{x-1}\tfrac{xv_{x3}}{\omega_{x3}}.\end{array}\right.$$

B.2.8. The chart $(u_{x4}, v_{x4}) = \left(\frac{u_{x2}-x\theta_x}{v_{x2}}, v_{x2}\right)$

Domain of definition: $\mathbb{C}^2 \setminus \{\gamma_0, \alpha\}$, where

$$\gamma_0 : \left(-x(\theta_0+\theta_x)\theta_0\,,\, \frac{1}{\theta_0}\right), \quad \alpha : \left(x\frac{\theta_x^2}{4}\,,\, -\frac{2}{\theta_x}\right).$$

Visible components of the infinity set: $\mathcal{D}_0 \setminus \gamma_0 : \{v_{x4}(u_{x4}v_{x4}+x\theta_x) = -x\}$.
Visible exceptional lines: $\mathcal{E}_x : \{v_{x4} = 0\}$

$$\left\{\begin{array}{rcl}(u_{x4}, v_{x4}) &=& \left(((u-x)uv - x\theta_x)uv, \tfrac{1}{uv}\right)\\ (u,v) &=& \left((u_{x4}v_{x4}+x\theta_x)v_{x4}+x, \tfrac{1}{((u_{x4}v_{x4}+x\theta_x)v_{x4}+x)v_{x4}}\right)\\ \omega_{x4} &=& -((u_{x4}v_{x4}+x\theta_x)v_{x4}+x)\\ E &=& \tfrac{1}{(x-1)v_{x4}}\Big[-\tfrac{u_{x4}v_{x4}+x\theta_x+x\theta_0}{(u_{x4}v_{x4}+x\theta_x)v_{x4}+x} - (\theta v_{x4}-1)(\overline{\theta}v_{x4}-1)(u_{x4}v_{x4}+x\theta_x)+\\ && \hfill +\theta_0 - (x-1)(\theta_x-1)\Big]\end{array}\right.$$

$$\left\{\begin{array}{rcl}\dot{u}_{x4} &=& \tfrac{((u_{x4}v_{x4}+x\theta_x)v_{x4}+x)}{1-x}\Big[\theta\overline{\theta}(2u_{x4}v_{x4}+x\theta_x) - (\theta+\overline{\theta})u_{x4}\Big]+\\ && +\tfrac{(2u_{x4}v_{x4}+x\theta_x)u_{x4}}{(1-x)((u_{x4}v_{x4}+x\theta_x)v_{x4}+x)} - \tfrac{\theta_0+\theta_x}{1-x}\left(u_{x4} + \tfrac{x\theta_x(u_{x4}v_{x4}+x\theta_x)-2xu_{x4}}{(u_{x4}v_{x4}+x\theta_x)v_{x4}+x}\right) + u_{x4}\ \cdot\\ \dot{v}_{x4} &=& -\tfrac{(u_{x4}v_{x4}+x\theta_x)v_{x4}+x}{1-x}(\theta v_{x4}-1)(\overline{\theta}v_{x4}-1) + x\tfrac{\theta_0 v_{x4}-1}{(x-1)((u_{x4}v_{x4}+x\theta_x)v_{x4}+x)}\end{array}\right.$$

We also have:

$$\omega_{x3} = -((u_{x4}v_{x4}+x\theta_x)v_{x4}+x)\frac{1}{u_{x4}}.$$

B.2.9. **The chart** $(u_{13}, v_{13}) = \left(u_{12} - \theta_1, \frac{v_{12}}{u_{12}-\theta_1}\right)$

Domain of definition: $\mathbb{C}^2$.
Visible components of the infinity set: $\mathcal{D}_1^* : \{v_{13} = 0\}$.
Visible exceptional lines: $\mathcal{E}_1 : \{u_{13} = 0\}$

$$\begin{cases} (u_{13}, v_{13}) = \left((u-1)v - \theta_1, \frac{1}{(u-1)v^2 - \theta_1 v}\right) \\ (u, v) = \left(u_{13}^2 v_{13} + \theta_1 u_{13} v_{13} + 1, \frac{1}{u_{13} v_{13}}\right) \\ \omega_{13} = -v_{13} \\ \dot{E} = -x\frac{u_{13}+\theta_1}{(x-1)^2}\left[(u_{13} + \theta_1 - \theta)(u_{13} + \theta_1 - \overline{\theta})u_{13}v_{13} + u_{13} - (\theta_x - 1)\right] \\ E = \frac{(u_{13}+\theta_1-\theta)(u_{13}+\theta_1-\overline{\theta})}{x-1}((u_{13} + \theta_1)u_{13}v_{13} + 1) - x\frac{\theta\overline{\theta}-\theta_0(u_{13}+\theta_1)}{x-1} - \frac{(u_{13}+\theta_1)u_{13}v_{13}+1}{v_{13}} \end{cases}$$

$$\begin{cases} \dot{u}_{13} = -\frac{1}{v_{13}} + \frac{1}{1-x}(u_{13} + \theta_1 - \theta)\left(u_{13} + \theta_1 - \overline{\theta}\right)(u_{13} + \theta_1)u_{13}v_{13} \\ \dot{v}_{13} = -(2u_{13} + \theta_1)v_{13} - \frac{1}{1-x}(u_{13} + \theta_1 - \theta)\left(u_{13} + \theta_1 - \overline{\theta}\right)(2u_{13} + \theta_1)v_{13}^2 - \\ \qquad -\frac{1}{1-x}\left(2u_{13} + 2\theta_1 - \theta - \overline{\theta}\right)(u_{13}^2 v_{13} + \theta_1 u_{13} v_{13} + 1)v_{13} - \frac{x\theta_0}{1-x}v_{13} \end{cases}.$$

B.2.10. **The chart** $(u_{14}, v_{14}) = \left(\frac{u_{12}-\theta_1}{v_{12}}, v_{12}\right)$

Domain of definition: $\mathbb{C}^2$.
Visible components of the infinity set: $\emptyset$.
Visible exceptional lines:$\mathcal{E}_1 : \{v_{14} = 0\}$

$$\begin{cases} (u_{14}, v_{14}) = \left((u-1)v^2 - \theta_1 v, \frac{1}{v}\right) \\ (u, v) = \left(u_{14}v_{14}^2 + \theta_1 v_{14} + 1, \frac{1}{v_{14}}\right) \\ \omega_{14} = -1 \\ E = \frac{(u_{14}v_{14}+\theta_1-\theta)(u_{14}v_{14}+\theta_1-\overline{\theta})}{x-1}((u_{14}v_{14} + \theta_1)v_{14} + 1) - x\frac{\theta\overline{\theta}-\theta_0(u_{14}v_{14}+\theta_1)}{x-1} - \\ \qquad -((u_{14}v_{14} + \theta_1)v_{14} + 1)u_{14} \end{cases}$$

$$\begin{cases} \dot{u}_{14} = (2u_{14}v_{14} + \theta_1)u_{14} + \frac{(2u_{14}v_{14}+\theta_1)}{1-x}(u_{14}v_{14} + \theta_1 - \theta)\left(u_{14}v_{14} + \theta_1 - \overline{\theta}\right) + \\ \qquad +2\frac{(u_{14}v_{14}^2+\theta_1 v_{14}+1)u_{14}}{1-x}\left(u_{14}v_{14} + \theta_1 - \frac{\theta+\overline{\theta}}{2}\right) + \frac{x\theta_0 u_{14}}{1-x} \\ \dot{v}_{14} = -\frac{v_{14}^2}{1-x}(u_{14}v_{14} + \theta_1 - \theta)\left(u_{14}v_{14} + \theta_1 - \overline{\theta}\right) - 1 - 2\left(u_{14}v_{14} - \frac{\theta_0-\theta_1}{2}\right)v_{14} - \\ \qquad -\frac{\theta_0 v_{14}}{1-x} - 2\frac{(u_{14}v_{14}^2+\theta_1 v_{14}+1)v_{14}}{1-x}\left(u_{14}v_{14} + \theta_1 - \frac{\theta+\overline{\theta}}{2}\right) \end{cases}.$$

B.2.11. **The chart** $(u_{\infty3}, v_{\infty3}) = \left(u_{\infty2}, \frac{v_{\infty2}+\theta_\infty}{u_{\infty2}}\right)$

Domain of definition: $\mathbb{C}^2$.
Visible components of the infinity set: $\emptyset$.
Visible exceptional lines: $\mathcal{E}_\infty^- : \{u_{\infty3} = 0\}$, $\mathcal{E}_\infty : \{u_{\infty3}v_{\infty3} = \theta_\infty\}$

$$\begin{cases} (u_{\infty3}, v_{\infty3}) = \left(\frac{1}{u^2 v - \theta u}, (uv - \theta)(uv - \overline{\theta})u\right) \\ (u, v) = \left(\frac{1}{(u_{\infty3}v_{\infty3}-\theta_\infty)u_{\infty3}}, \left(u_{\infty3}v_{\infty3} + \overline{\theta}\right)(u_{\infty3}v_{\infty3} - \theta_\infty)u_{\infty3}\right) \\ \omega_{\infty3} = -1 \\ E = \theta_1(u_{\infty3}v_{\infty3} + \overline{\theta}) + \frac{x}{x-1}(u_{\infty3}v_{\infty3} + \overline{\theta} - \theta_0)(u_{\infty3}v_{\infty3} + \overline{\theta})[u_{\infty3}(u_{\infty3}v_{\infty3} - \theta + \overline{\theta}) - 1] - \\ \qquad -\theta\overline{\theta} - \frac{v_{\infty3}}{x-1}[u_{\infty3}(u_{\infty3}v_{\infty3} - \theta + \overline{\theta}) - 1] \end{cases}$$

$$\begin{cases} \dot{u}_{\infty 3} = -\left(\theta_1 + \frac{\theta_\infty}{1-x}\right)u_{\infty 3} - \frac{2(u_{\infty 3}v_{\infty 3}-\theta_\infty)u_{\infty 3}-1}{1-x} - x\frac{(u_{\infty 3}^2 v_{\infty 3}-\theta_\infty u_{\infty 3}-1)^2}{1-x} + \\ \quad + \frac{x}{1-x}\Big((2u_{\infty 3}v_{\infty 3} - \theta - \theta_0 + 2\overline{\theta})u_{\infty 3} - 1\Big)\Big((2u_{\infty 3}v_{\infty 3} - \theta + 2\overline{\theta})u_{\infty 3} - 1\Big) \\ \dot{v}_{\infty 3} = v_{\infty 3}(2u_{\infty 3}v_{\infty 3} + \theta_1 - \theta_\infty) - x\frac{\overline{\theta}(\overline{\theta}-\theta_0)}{1-x}(2u_{\infty 3}v_{\infty 3} - \theta_\infty) - \\ \quad - \frac{x}{1-x}\Big[2(2u_{\infty 3}v_{\infty 3} - \theta_\infty)(u_{\infty 3}^2 v_{\infty 3} - \theta_\infty u_{\infty 3} - 1) - \\ \quad -(\theta + \overline{\theta} - \theta_0)\big(1 - 3u_{\infty 3}^2 v_{\infty 3} + 2\theta_\infty u_{\infty 3}\big)\Big]v_{\infty 3} \end{cases}.$$

***B.2.12.* The chart** $(u_{\infty 4}, v_{\infty 4}) = \left(\frac{u_{\infty 2}}{v_{\infty 2}+\theta_\infty}, v_{\infty 2} + \theta_\infty\right)$

Domain of definition: $\mathbb{C}^2$.
Visible components of the infinity set: $\mathcal{D}_\infty^{**} : \{u_{\infty 4} = 0\}$.
Visible exceptional lines: $\mathcal{E}_\infty^- : \{v_{\infty 4} = 0\}$, $\mathcal{E}_\infty : \{v_{\infty 4} = \theta_\infty\}$

$$\begin{cases} (u_{\infty 4}, v_{\infty 4}) = \left(\frac{1}{(uv-\theta)(uv-\overline{\theta})u}, uv - \overline{\theta}\right) \\ (u, v) = \left(\frac{1}{(v_{\infty 4}-\theta_\infty)u_{\infty 4}v_{\infty 4}}, \left(v_{\infty 4} + \overline{\theta}\right)(v_{\infty 4} - \theta_\infty)u_{\infty 4}v_{\infty 4}\right) \\ \omega_{\infty 4} = -u_{\infty 4} \\ \dot{E} = x\frac{(v_{\infty 4}-\theta_\infty)u_{\infty 4}v_{\infty 4}-1}{(x-1)^2}\left[\frac{1}{u_{\infty 4}} - (v_{\infty 4} + \overline{\theta} - \theta_0)(v_{\infty 4} + \overline{\theta})\right] \\ E = \theta_1(v_{\infty 4} + \overline{\theta}) + \frac{x}{x-1}(v_{\infty 4} + \overline{\theta} - \theta_0)(v_{\infty 4} + \overline{\theta})[u_{\infty 4}v_{\infty 4}(v_{\infty 4} - \theta + \overline{\theta}) - 1] - \\ \quad -\theta\overline{\theta} - \frac{1}{(x-1)u_{\infty 4}}[u_{\infty 4}v_{\infty 4}(v_{\infty 4} - \theta + \overline{\theta}) - 1] \end{cases}$$

$$\begin{cases} \dot{u}_{\infty 4} = -u_{\infty 4}(2v_{\infty 4} + \theta_1 - \theta_\infty) + x\frac{u_{\infty 4}}{1-x}\Big[\overline{\theta}(\overline{\theta} - \theta_0)(2v_{\infty 4} - \theta_\infty)u_{\infty 4} + \\ \quad +(\theta_x + \theta_1 - 1)(3u_{\infty 4}v_{\infty 4}^2 - 2\theta_\infty u_{\infty 4}v_{\infty 4} - 1) + \\ \quad +2(2v_{\infty 4} - \theta_\infty)(u_{\infty 4}v_{\infty 4}^2 - \theta_\infty u_{\infty 4}v_{\infty 4} - 1)\Big] \\ \dot{v}_{\infty 4} = \frac{1}{(1-x)u_{\infty 4}} - \frac{x}{1-x}(v_{\infty 4} + \overline{\theta})(v_{\infty 4} + \overline{\theta} - \theta_0)(v_{\infty 4} - \theta_\infty)u_{\infty 4}v_{\infty 4} \end{cases}.$$

C. Estimates near $\mathcal{D}_\infty^{**} \setminus \mathcal{H}^*$ and $\mathcal{D}_1^* \setminus \mathcal{H}^*$

Lemma C.1 (Behaviour near $\mathcal{D}_1^* \setminus \mathcal{H}^*$). *If a solution at a complex time t is sufficiently close to $\mathcal{D}_1^* \setminus \mathcal{H}^*$, then there exists unique $\tau \in \mathbf{C}$, such that $(u(\tau), v(\tau))$ belongs to line $\mathcal{E}_1$. In other words, the pair $(u(t), v(t))$ has a pole at $t = \tau$.*

Moreover, $|t - \tau| = O(|d(t)||u_{13}(t)|)$ for sufficiently small $d(t)$ and bounded u_{13}.

For large $R_1 > 0$, consider the set $\{t \in \mathbb{C} \mid |u_{13}(t)| \le R_1\}$. Its connected component containing τ is an approximate disk Δ_1 with centre τ and radius $|d(\tau)|R_1$, and $t \mapsto u_{13}(t)$ is a complex analytic diffeomorphism from that approximate disk onto $\{u \in \mathbb{C} \mid |u| \le R_1\}$.

Proof. For the study of the solutions near $\mathcal{D}_1^* \setminus \mathcal{H}^*$, we use coordinates (u_{13}, v_{13}) (see Section B.2.9). In this chart, the set $\mathcal{D}_1^* \setminus \mathcal{H}^*$ is given by $\{v_{13} = 0\}$ and parametrised by $u_{13} \in \mathbb{C}$. Moreover, $\mathcal{E}_1$ is given by $u_{13} = 0$ and parametrised by v_{13}.

Asymptotically, for $v_{13} \to 0$, bounded u_{13} and $x = e^t$ bounded away from 0 and 1, we have:

$$\dot{u}_{13} \sim -\frac{1}{v_{13}} \tag{C.1a}$$

$$\dot{v}_{13} \sim -2u_{13}v_{13}\frac{2-x}{1-x} - (3\theta_1 - \theta - \bar{\theta} - \theta_0)v_{13} - \frac{\theta_0}{1-x}v_{13} \tag{C.1b}$$

$$\omega_{13} = -v_{13} \tag{C.1c}$$

$$\frac{\dot{\omega}_{13}}{\omega_{13}} = -(2u_{13} + \theta_1) - \frac{1}{1-x}\Big(2u_{13} + 2\theta_1 - \theta - \overline{\theta}\Big) - \frac{x\theta_0}{1-x} + O(\omega_{13}) \tag{C.1d}$$

$$E\omega_{13} \sim 1. \tag{C.1e}$$

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Integrating (C.1d) from τ to t, we get:

$$\omega_{13}(t) = \omega_{13}(\tau)e^{-\theta_1(t-\tau)}e^{K(t-\tau)}(1+o(1)),$$

with

$$K = -2u_{13}(\tilde{\tau}) - \frac{1}{1-e^{\tilde{\tau}}}\left(2u_{13}(\tilde{\tau}) + 2\theta_1 - \theta - \overline{\theta}\right) - \frac{e^{\tilde{\tau}}\theta_0}{1-e^{\tilde{\tau}}}$$

and $\tilde{\tau}$ being on the integration path.

Arguments similar to those in the proof of Lemma 6.6 show that v_{13} is approximately equal to a small constant, from (C.1a) follows that:

$$u_{13} \sim u_{13}(\tau) - \frac{t-\tau}{v_{13}}.$$

Thus, if t runs over an approximate disk Δ centred at τ with radius $|v_{13}|R$, then u_{13} fills an approximate disk centred at $u_{13}(\tau)$ with radius R. Therefore, if $|v_{13}| \ll |\tau|$, the solution has the following properties for $t \in \Delta$:

$$\frac{v_{13}(t)}{v_{13}(\tau)} \sim 1,$$

and u_{13} is a complex analytic diffeomorphism from Δ onto an approximate disk with centre $u_{13}(\tau)$ and radius R. If R is sufficiently large, we will have $0 \in u_{13}(\Delta)$, which means that the solution of the Painlevé equation will have a pole at a unique point in Δ. Now, it is possible to take τ to be the pole point. For $|t-\tau| \ll |\tau|$, we have:

$$\frac{d(t)}{d(\tau)} \sim 1, \quad \text{that is} \quad \frac{v_{13}(t)}{d(\tau)} \sim -\frac{\omega_{13}(t)}{d(\tau)} \sim -1, \quad u_{13}(t) \sim -\frac{t-\tau}{v_{13}} \sim \frac{t-\tau}{d(\tau)}.$$

Let R_1 be a large positive real number. Then the equation $|u_{13}(t)| = R_1$ corresponds to $|t-\tau| \sim |d(\tau)|R_1$, which is still small compared to $|\tau|$ if $|d(\tau)|$ is sufficiently small. Denote by Δ_1 the connected component of the set of all $t \in \mathbb{C}$, such that $\{t \mid |u_{13}(t)| \le R_1\}$ is an approximate disk with centre τ and radius $2|d(\tau)|R_1$. More precisely, u_{13} is a complex analytic diffeomorphism from Δ_1 onto $\{u \in \mathbb{C} \mid |u| \le R_1\}$, and $\frac{d(t)}{d(\tau)} \sim 1$ for all $t \in \Delta_1$.

From (C.1e), $E(t)\omega_{13}(t) \sim 1$ for the annular disk $\Delta_1 \setminus \Delta_1'$, where Δ_1' is a disk centred at τ with small radius compared to radius of Δ_1. □

Lemma C.2 (Behaviour near $\mathcal{D}_\infty^{**} \setminus \mathcal{H}^*$). *If a solution at a complex time t is sufficiently close to $\mathcal{D}_\infty^{**} \setminus \mathcal{H}^*$, then there exists unique $\tau \in$ **C**, such that $(u(t), v(t))$ has a pole at $t = \tau$. Moreover, $|t-\tau| = O(|d(t)||v_{\infty 4}(t)|)$ for sufficiently small $d(t)$ and bounded $v_{\infty 4}$.*

For large $R_\infty > 0$, consider the set $\{t \in \mathbb{C} \mid |v_{\infty 4}| \le R_\infty\}$. Its connected component containing τ is an approximate disk Δ_∞ with centre τ and radius $|d(\tau)|R_\infty$, and $t \mapsto v_{\infty 4}(t)$ is a complex analytic diffeomorphism from that approximate disk onto $\{u \in \mathbb{C} \mid |u| \le R_\infty\}$.

Proof. For the study of the solutions near $\mathcal{D}_\infty^{**} \setminus \mathcal{H}^*$, we use coordinates $(u_{\infty 4}, v_{\infty 4})$ (see Section B.2.12). In this chart, the set $\mathcal{D}_\infty^{**} \setminus \mathcal{H}^*$ is given by $\{u_{\infty 4} = 0\}$ and parametrised by $v_{\infty 4} \in \mathbb{C}$. Moreover, $\mathcal{E}_\infty$ is given by $\{v_{\infty 4} = \theta_\infty\}$ and parametrised by $u_{\infty 4}$, while $\mathcal{E}_\infty^-$ is given by $\{v_{\infty 4} = 0\}$ and also parametrised by $u_{\infty 4}$.

Asymptotically, for $u_{\infty 4} \to 0$, $v_{\infty 4}$ bounded and $x = e^t$ bounded away from 0 and 1, we have:

$$\dot{u}_{\infty 4} \sim \frac{(2v_{\infty 4} - \theta_\infty)(x+1) + \theta_1 + x(\theta_x - 1)}{x-1} \cdot u_{\infty 4}, \tag{C.2a}$$

$$\dot{v}_{\infty 4} \sim -\frac{1}{(x-1)u_{\infty 4}}, \tag{C.2b}$$

$$\omega_{\infty 4} = -u_{\infty 4} \tag{C.2c}$$

$$\frac{\dot{\omega}_{\infty 4}}{\omega_{\infty 4}} \sim 2v_{\infty 4} - \theta_\infty + \theta_x - 1 + \frac{4v_{\infty 4} - 2\theta_\infty + \theta_1 + \theta_x - 1}{x-1} \tag{C.2d}$$

$$E\omega_{\infty 4} \sim -\frac{1}{x-1}. \tag{C.2e}$$

Integrating (C.2d) from τ to t, we get:

$$\omega_{\infty 4}(t) = \omega_{\infty 4}(\tau) e^{(\theta_x - \theta_\infty - 1)(t-\tau)} e^{K(t-\tau)}(1 + o(1)),$$

with

$$K = 2v_{\infty_4}(\tilde{\tau}) + \frac{4v_{\infty 4}(\tilde{\tau}) - 2\theta_\infty + \theta_1 + \theta_x - 1}{e^{\tilde{\tau}} - 1}$$

and $\tilde{\tau}$ being on the integration path.

Arguments similar to those in the proof of Lemma 6.6 show that $u_{\infty 4}$ is approximately equal to a small constant, and from (C.2b) follows that:

$$v_{\infty 4} \sim v_{\infty 4}(\tau) + \frac{t - \tau - \log \frac{1-e^t}{1-e^\tau}}{u_{\infty 4}}.$$

Thus, for large R, if t runs over an approximate disk Δ centred at τ with radius $|u_{\infty 4}|R$, then $v_{\infty 4}$ fills an approximate disk centred at $v_{\infty 4}(\tau)$ with radius R. Therefore, if $|u_{\infty 4}| \ll |\tau|$, the solution has the following properties for $t \in \Delta$:

$$\frac{u_{\infty 4}(t)}{u_{\infty 4}(\tau)} \sim 1$$

and $v_{\infty 4}$ is a complex analytic diffeomorphism from Δ onto an approximate disk with centre $v_{\infty 4}(\tau)$ and radius R. If R is sufficiently large, we will have $0 \in v_{\infty 4}(\Delta)$, that is the solution the Painlevé equation will have a pole at a unique point in Δ. Now, it is possible to take τ to be the pole point. For $|t - \tau| \ll |\tau|$, we have that $d(t) \sim d(\tau)$ implies:

$$1 \sim \frac{(1-x)\omega_{\infty 4}(t)}{d(\tau)} \sim \frac{(e^t - 1)u_{\infty 4}(t)}{d(\tau)}, \qquad v_{\infty 4}(t) \sim \frac{t-\tau}{u_{\infty 4}} \sim \frac{(t-\tau)(e^t - 1)}{d(\tau)}.$$

Let R_∞ be a large positive real number. Then the equation $|v_{\infty 4}(t)| = R_\infty$ corresponds to $|(e^t - 1)(t - \tau)| \sim |d(\tau)|R_\infty$, which is still small compared to $|\tau|$ if $|d(\tau)|$ is sufficiently small. Denote by Δ_∞ the connected component of the set of all $t \in \mathbb{C}$, such that $\{t \mid |v_{\infty 4}(t)| \le R_\infty\}$ is an approximate disk with centre τ and radius $2|d(\tau)|R_\infty$. More precisely, $v_{\infty 4}$ is a complex analytic diffeomorphism from Δ_∞ onto $\{v \in \mathbb{C} \mid |v| \le R_\infty\}$, and $\frac{d(t)}{d(\tau)} \sim 1$ for all $t \in \Delta_\infty$.

From (C.2e), $E(t)\omega_{\infty 4}(t) \sim 1/(1 - e^t)$ for the annular disk $\Delta_\infty \setminus \Delta'_\infty$, where Δ'_∞ is a disk centred at τ with small radius compared to radius of Δ_∞. □

D. The vector field in the limit space

$$\begin{cases} E_0 &= -u\{(u-1)v(uv-2\theta+\theta_\infty)-\theta_1 v+\theta(\theta-\theta_\infty)\} \\ \dot{u} &= -2u^2(u-1)v+(\theta+\overline{\theta})u(u-1)+\theta_1 u \\ \dot{v} &= (3u-2)uv^2-2(\theta+\overline{\theta})uv+\left(\theta+\overline{\theta}-\theta_1\right)v+\theta\overline{\theta}\,. \end{cases}$$

$$\begin{cases} (\tilde{u}_1,\tilde{v}_1) &= \left(u, \frac{1}{u^2(u-1)v}\right) \\ E_0 &= -\frac{1}{\tilde{u}_1^2(\tilde{u}_1-1)\tilde{v}_1^2}+\frac{1}{\tilde{v}_1}\left(\frac{\theta+\overline{\theta}-\theta_1}{\tilde{u}_1}+\frac{\theta_1}{\tilde{u}_1-1}\right)-\theta\overline{\theta}\tilde{u}_1 \\ \dot{\tilde{u}}_1 &= -\frac{2}{\tilde{v}_1}+\tilde{u}_1^2(\tilde{u}_1-1)\left(\frac{\theta+\overline{\theta}-\theta_1}{\tilde{u}_1}+\frac{\theta_1}{\tilde{u}_1-1}\right) \\ \dot{\tilde{v}}_1 &= \frac{1}{\tilde{u}_1}-\frac{\theta_1\tilde{v}_1-1}{\tilde{u}_1-1}-\left((\theta+\overline{\theta})(\tilde{u}_1-1)+2\theta_1\right)\tilde{v}_1-\theta\overline{\theta}\tilde{u}_1^2(\tilde{u}_1-1)\tilde{v}_1^2\,. \end{cases}$$

$$\begin{cases} (\tilde{u}_2,\tilde{v}_2) &= \left(\frac{1}{u}, \frac{1}{(u-1)v}\right) \\ E_0 &= -\frac{(\theta\tilde{v}_2-1)(\overline{\theta}\tilde{v}_2-1)}{\tilde{u}_2\tilde{v}_2^2}+\frac{\theta_1\tilde{v}_2-1}{(1-\tilde{u}_2)\tilde{v}_2^2} \\ \dot{\tilde{u}}_2 &= \frac{2}{\tilde{v}_2}+\theta_1-(1-\tilde{u}_2)\left(\theta+\overline{\theta}-\theta_1\right) \\ \dot{\tilde{v}}_2 &= -\frac{(\theta\tilde{v}_2-1)(\overline{\theta}\tilde{v}_2-1)}{\tilde{u}_2}-\frac{\theta_1\tilde{v}_2-1}{(1-\tilde{u}_2)}-(\theta+\overline{\theta}-\theta_1)\tilde{v}_2+\theta\overline{\theta}\tilde{v}_2^2\,. \end{cases}$$

$$\begin{cases} (\tilde{u}_3,\tilde{v}_3) &= \left(\frac{1}{u}, -u(uv-\theta)\right) \\ E_0 &= -(1-\tilde{u}_3)\tilde{u}_3^2\tilde{v}_3-(\theta_\infty+\theta_1)\tilde{u}_3\tilde{v}_3+\theta_\infty\tilde{v}_3-\theta(\overline{\theta}-\theta_1) \\ \dot{\tilde{u}}_3 &= -2\tilde{u}_3(1-\tilde{u}_3)\tilde{v}_3+\theta_\infty(1-\tilde{u}_3)-\theta_1\tilde{u}_3 \\ \dot{\tilde{v}}_3 &= (-2\tilde{u}_3+1)\tilde{v}_3^2-(\theta_1+\theta_\infty)\tilde{v}_3\,. \end{cases}$$

$$\begin{cases} (u_{03},v_{03}) &= \left(uv-\theta_0, \frac{1}{uv^2-\theta_0 v}\right) \\ (u,v) &= \left(u_{03}^2 v_{03}+\theta_0 u_{03}v_{03}, \frac{1}{u_{03}v_{03}}\right) \\ E_0 &= \left[-(u_{03}+\theta_0)u_{03}v_{03}\left(u_{03}+\theta_0-(\theta+\overline{\theta}-\theta_0)\right)-u_{03}(\theta\overline{\theta}v_{03}-1)-(\theta_x-1)\right](u_{03}+\theta_0) \\ \dot{u}_{03} &= (u_{03}+\theta_0-\theta)(u_{03}+\theta_0-\overline{\theta})(u_{03}+\theta_0)u_{03}v_{03} \\ \dot{v}_{03} &= -(u_{03}+\theta_0-\theta)(u_{03}+\theta_0-\overline{\theta})(2u_{03}+\theta_0)v_{03}^2-(2(u_{03}+\theta_0)-(\theta+\overline{\theta}))(u_{03}+\theta_0)u_{03}v_{03}^2+ \\ &\quad +(2u_{03}+\theta_0-\theta_x+1)v_{03} \end{cases}$$

$$\begin{cases} (u_{04},v_{04}) &= \left(uv^2-\theta_0 v, \frac{1}{v}\right) \\ E_0 &= \left[u_{04}v_{04}-(u_{04}v_{04}+\theta_0-(\theta+\overline{\theta}))(u_{04}v_{04}+\theta_0)v_{04}-\theta\overline{\theta}v_{04}-(\theta_x-1)\right](u_{04}v_{04}+\theta_0) \\ \dot{u}_{04} &= (u_{04}v_{04}+\theta_0-\theta)(u_{04}v_{04}+\theta_0-\overline{\theta})(2u_{04}v_{04}+\theta_0)-\theta_1 u_{04}- \\ &\quad -[(\theta+\overline{\theta})-2(u_{04}v_{04}+\theta_0)][(u_{04}v_{04}+\theta_0)v_{04}-1]u_{04} \\ \dot{v}_{04} &= -\left[(2(u_{04}v_{04}+\theta_0)-\theta)v_{04}-1\right]\left[\left(2(u_{04}v_{04}+\theta_0)-\overline{\theta}\right)v_{04}-1\right]+ \\ &\quad +[(u_{04}v_{04}+\theta_0)v_{04}-1]^2+\theta_1 v_{04} \end{cases}$$

$$\begin{cases} (u_{x3},v_{x3}) &= (u_{x2},v_{x2}/u_{x2}) = \left(u^2 v, \frac{1}{u^3v^2}\right) \\ E_0 &= \frac{1-(\theta+\overline{\theta}-\theta_1)u_{x3}v_{x3}+u_{x3}^2v_{x3}}{(u_{x3}v_{x3})^2}+\theta\overline{\theta}u_{x3}^2v_{x3}-(\theta+\overline{\theta})u_{x3} \\ \dot{u}_{x3} &= -(u_{x3}+\theta+\overline{\theta}-\theta_1)u_{x3}+\frac{2}{v_{x3}}+\theta\overline{\theta}u_{x3}^4v_{x3}^2 \\ \dot{v}_{x3} &= -2\theta\overline{\theta}u_{x3}^3v_{x3}^3+(\theta+\overline{\theta})u_{x3}^2v_{x3}^2-\frac{2}{u_{x3}}+(\theta+\overline{\theta}-\theta_1)v_{x3} \end{cases}$$

$$\begin{cases} (u_{x4},v_{x4}) &= (u_{x2}/v_{x2},v_{x2}) = \left(u^3v^2, \frac{1}{uv}\right) \\ E_0 &= \frac{1-(\theta+\overline{\theta}-\theta_1)v_{x4}}{v_{x4}^2}+(\theta v_{x4}-1)(\overline{\theta}v_{x4}-1)u_{x4} \\ \dot{u}_{x4} &= \theta\overline{\theta}u_{x4}^2v_{x4}^3-(\theta+\overline{\theta})u_{x4}^2v_{x4}^2+\frac{2u_{x4}}{v_{x4}}-(\theta+\overline{\theta}-\theta_1)u_{x4} \\ \dot{v}_{x4} &= -u_{x4}v_{x4}^2(\theta v_{x4}-1)(\overline{\theta}v_{x4}-1) \end{cases}$$

$$\begin{cases} (u_{13}, v_{13}) = \left((u-1)v - \theta_1, \frac{1}{(u-1)v^2 - \theta_1 v}\right) \\ E_0 = -(u_{13} + \theta_1 - \theta)(u_{13} + \theta_1 - \overline{\theta})((u_{13} + \theta_1)u_{13}v_{13} + 1) - (u_{13} + \theta_1)u_{13} - \frac{1}{v_{13}} \\ \dot{u}_{13} = -\frac{1}{v_{13}} + (u_{13} + \theta_1 - \theta)\left(u_{13} + \theta_1 - \overline{\theta}\right)(u_{13} + \theta_1)u_{13}v_{13} \\ \dot{v}_{13} = -(2u_{13} + \theta_1)v_{13} - (u_{13} + \theta_1 - \theta)\left(u_{13} + \theta_1 - \overline{\theta}\right)(2u_{13} + \theta_1)v_{13}^2 - \\ \quad -\left(2u_{13} + 2\theta_1 - \theta - \overline{\theta}\right)(u_{13}^2 v_{13} + \theta_1 u_{13} v_{13} + 1)v_{13} \end{cases}$$

$$\begin{cases} (u_{14}, v_{14}) = \left((u-1)v^2 - \theta_1 v, \frac{1}{v}\right) \\ E_0 = -\left[(u_{14}v_{14} + \theta_1 - \theta)(u_{14}v_{14} + \theta_1 - \overline{\theta}) + u_{14}\right]((u_{14}v_{14} + \theta_1)v_{14} + 1) \\ \dot{u}_{14} = (2u_{14}v_{14} + \theta_1)\left[(u_{14}v_{14} + \theta_1 - \theta)\left(u_{14}v_{14} + \theta_1 - \overline{\theta}\right) + u_{14}\right] + \\ \quad +2(u_{14}v_{14}^2 + \theta_1 v_{14} + 1)u_{14}\left(u_{14}v_{14} + \theta_1 - \frac{\theta + \overline{\theta}}{2}\right) \\ \dot{v}_{14} = -v_{14}^2(u_{14}v_{14} + \theta_1 - \theta)\left(u_{14}v_{14} + \theta_1 - \overline{\theta}\right) - 1 - 2\left(u_{14}v_{14} - \frac{\theta_0 - \theta_1}{2}\right)v_{14} - \\ \quad -\theta_0 v_{14} - 2(u_{14}v_{14}^2 + \theta_1 v_{14} + 1)v_{14}\left(u_{14}v_{14} + \theta_1 - \frac{\theta + \overline{\theta}}{2}\right) \end{cases}$$

$$\begin{cases} (u_{\infty 3}, v_{\infty 3}) = \left(\frac{1}{u^2 v - \theta u}, (uv - \theta)(uv - \overline{\theta})u\right) \\ E_0 = (\theta_1 - \theta)\overline{\theta} + u_{\infty 3}v_{\infty 3}(\theta_1 - \theta_\infty) + (u_{\infty 3}v_{\infty 3})^2 - v_{\infty 3} \\ \dot{u}_{\infty 3} = (\theta_\infty - \theta_1)u_{\infty 3} - 2u_{\infty 3}^2 v_{\infty 3} + 1 \\ \dot{v}_{\infty 3} = v_{\infty 3}(2u_{\infty 3}v_{\infty 3} + \theta_1 - \theta_\infty) \end{cases}$$

$$\begin{cases} (u_{\infty 4}, v_{\infty 4}) = \left(\frac{1}{(uv - \theta)(uv - \overline{\theta})u}, uv - \overline{\theta}\right) \\ E_0 = (\theta_1 - \theta)\overline{\theta} + v_{\infty 4}(v_{\infty 4} - \theta_\infty + \theta_1) - \frac{1}{u_{\infty 4}} \\ \dot{u}_{\infty 4} = -u_{\infty 4}(2v_{\infty 4} + \theta_1 - \theta_\infty) \\ \dot{v}_{\infty 4} = \frac{1}{u_{\infty 4}} \end{cases}.$$

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