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ANALYTIC RANGE FUNCTIONS OF SEVERAL VARIABLES

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ABSTRACT. Let $\mathcal{C}$ be a separable Hilbert Space, and let Λ be the halfplane

$$\{(m,n) \in \mathbf{Z}^2 : m \geq 1\} \cup \{(0,n) \in \mathbf{Z}^2 : n \geq 0\}$$

of the integer lattice. Consider the subspace $\mathcal{M}_{\mathcal{C}}(\Lambda)$ of $L^2_{\mathcal{C}}$ on the torus spanned by the $\mathcal{C}$-valued trigonometric functions $\{ce^{ims+int} : c \in \mathcal{C}, (m,n) \in \Lambda\}$. The notion of a Λ-analytic operator on $\mathcal{M}_{\mathcal{C}}(\Lambda)$ is defined with respect to the family of shift operators $\{S_{mn}\}_\Lambda$ on $\mathcal{M}_{\mathcal{C}}(\Lambda)$ given by

$$(S_{mn}f)(e^{is}, e^{it}) = e^{ims+int} f(e^{is}, e^{it}).$$

The corresponding concepts of inner function, outer function and analytic range function are explored. These ideas are applied to the spectral factorization problem in prediction theory.

This paper is a continuation of the works [1, 2] on the analysis of operator valued functions of several variables. The present purpose is to extend some of the work of Loubaton [7] for matrix valued functions to the infinite rank case; or what is equivalent, to establish a several variables version of some results on range functions from Helson [4]. Thus we shall consider notions of analytic, inner and outer functions in this setting, and develop the related concept of analytic range function. These ideas are applied to the spectral factorization problem from prediction theory. At issue is whether a given nonnegative operator valued function W has a factorization $W = \Phi^*\Phi$, where Φ is outer in a certain sense. The final theorem is a reduction of this issue to the special case that the values of W are full rank. The results obtained here do not displace those of [7]: In passing to the infinite rank picture we need to alter some of the underlying definitions, and this precludes the detailed accounting of dimensionality that was possible in [7]. Moreover, a general logarithmic integrability criterion for the desired factorization is not possible for W having infinite rank values.

We adopt the following notation and definitions. Throughout, σ is normalized Lebesgue measure on the unit circle $\mathbf{T}$. Let $\mathcal{C}$ be a separable Hilbert space, and write $\mathcal{B}(\mathcal{C})$ for the Banach space of (bounded linear) operators on $\mathcal{C}$. By $L^p_{\mathcal{C}}(\sigma^2)$ and $L^p_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ we mean Lebesgue spaces of vector and operator valued functions on the torus $\mathbf{T}^2$; the corresponding Hardy spaces are written $H^p_{\mathcal{C}}(\sigma^2)$ and $H^p_{\mathcal{B}(\mathcal{C})}(\sigma^2)$. Fix a nonempty subset Ω of the integer lattice $\mathbf{Z}^2$. We define the space $\mathcal{M}_{\mathcal{C}}(\Omega)$ to be that subspace of $L^2_{\mathcal{C}}(\sigma^2)$

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spanned by the trigonometric functions $\{ce^{ims+int} : c \in \mathcal{C}, (m,n) \in \Omega\}$. For each (m,n) in $\mathbf{Z}^2$, let S_{mn} be the operation

$$f(e^{is}, e^{it}) \longrightarrow e^{ims+int} f(e^{is}, e^{it}),$$

where f is any $\mathcal{C}$-valued trigonometric polynomial on the torus. Each S_{mn} determines an isometry on $L^2_{\mathcal{C}}(\sigma^2)$. Moreover, if Ω is a semigroup in $\mathbf{Z}^2$, and (m,n) belongs to Ω, then S_{mn} restricts to an isometry on $\mathcal{M}_{\mathcal{C}}(\Omega)$. We may write $S_1 = S_{1,0}$ and $S_2 = S_{0,1}$. If a subspace $\mathcal{K}$ of $L^2_{\mathcal{C}}(\sigma^2)$ is invariant under S_{mn} for every $(m,n) \in \Omega$, then we say that $\mathcal{K}$ is Ω-*invariant.*

In the paper [1], the space $\mathcal{M}_{\mathcal{C}}(\Omega)$ was examined, where Ω was any *sector* of the integer lattice. Here, however, we are primarily concerned with the halfplanes

$$\Lambda = \{(m,n) \in \mathbf{Z}^2 : m \geq 1, n \in \mathbf{Z}\} \cup \{(0,n) \in \mathbf{Z}^2 : n \geq 0\}$$
$$\Pi = \{(m,n) \in \mathbf{Z}^2 : m \geq 0, n \in \mathbf{Z}\}.$$

Loubaton [7] justifies the attention to Λ by viewing it as a halfline with respect to a lexicographical ordering of $\mathbf{Z}^2$. In addition, Λ is an example of a halfplane in the sense of Helson and Lowdenslager [5], which was introduced for algebraic reasons. We are ultimately concerned with a notion of analyticity that is connected to Λ. The halfplane Π is needed here as an auxiliary tool. Later, we also use $(-\Lambda) = \{(-m,-n) : (m,n) \in \Lambda\}$.

We say that an operator A on $\mathcal{M}_{\mathcal{C}}(\Pi)$ is Π-*analytic*, if A commutes with S_{mn} for all $(m,n) \in \Pi$. In particular, a Π-analytic operator is S_1-analytic in the one-variable sense; we may view such an operator as acting upon $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}\big(\sigma(e^{is})\big)$, which is isomorphic to $\mathcal{M}_{\mathcal{C}}(\Pi)$ in an obvious way. Hence we may speak of a Π-analytic operator as being S_1-inner or S_1-outer. In specific, we say that an operator A on $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}\big(\sigma(e^{is})\big)$ is

S_1-*outer*, if A is S_1-analytic and $\big[AH^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}\big(\sigma(e^{is})\big)\big]^- = H^2_M\big(\sigma(e^{is})\big)$ for some subspace $M = M_{\text{out}}(A)$ of $L^2_{\mathcal{C}}\big(\sigma(e^{it})\big)$;

S_1-*inner*, if A is S_1-analytic and partially isometric.

S_1-*constant*, if both A and A^* are S_1-analytic.

These definitions are conceptually consistent with those from the classical one-variable theory; see [10, Chapter 3]. By the *initial space* of a partial isometry we mean the orthocomplement of its kernel. If A is S_1-inner, then its initial space as a partial isometry on $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}\big(\sigma(e^{is})\big)$ is of the form $H^2_M\big(\sigma(e^{is})\big)$, where $M = M_{\text{in}}(A)$ is a subspace of $L^2_{\mathcal{C}}\big(\sigma(e^{it})\big)$. This follows from [10, Theorem A, p. 96].

In an analogous way, we say that an operator A on $\mathcal{M}_{\mathcal{C}}(\Lambda)$ is

Λ-*analytic*, if A commutes with S_{mn} for all $(m,n) \in \Lambda$;

Λ-*outer*, if A is Λ-analytic and $[A\mathcal{M}_{\mathcal{C}}(\Lambda)]^- = \mathcal{M}_N(\Lambda)$ for some subspace $N = N_{\text{out}}(A)$ of $\mathcal{C}$;

Λ-*inner*, if A is Λ-analytic and partially isometric;

Λ-*constant*, if both A and A^* are Λ-analytic.

If A is Λ-inner, then its initial space is of the form $\mathcal{M}_N(\Lambda)$, where $N = N_{\rm in}(A)$ is a subspace of $\mathcal{C}$ [1, Proposition 1.1]. The proof builds upon the argument due to Lax from the one-variable case.

Every Π- or Λ-analytic operator has a functional realization. That is, if A is Π- or Λ-analytic, then there exists a function $F(e^{is}, e^{it})$ in $L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ such that A is given by multiplication by F. The correspondence is linear and respects adjoints. Thus, we may speak of Π- and Λ-analytic *functions*, as they arise from operators in this fashion. Since $\Lambda \subseteq \Pi$, a Λ-analytic function is automatically Π-analytic. If F is some $\mathcal{B}(\mathcal{C})$-valued function, we define $\tilde{F}$ to be that $\mathcal{B}(\mathcal{C})$-valued function given by

$$\tilde{F}(e^{is}, e^{it}) = F(e^{-is}, e^{-it})^*.$$

We now establish some properties of these inner and outer functions. Suppose that $B \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ is S_1-inner. Then its initial space is of the form $H^2_M\big(\sigma(e^{is})\big)$, where $M = M_{\rm in}(B)$ is a subspace of $L^2_{\mathcal{C}}\big(\sigma(e^{it})\big)$. Now B commutes with both S_2 and S_2^*. Thus if $f \in M$, we have

$$\begin{aligned} \|BS_2f\|_2 &= \|S_2Bf\|_2 \\ &= \|Bf\|_2 \\ &= \|f\|_2 \\ &= \|S_2f\|_2, \end{aligned}$$

and similarly with S_2 replaced by S_2^*. Note also that S_2f and S_2^*f lie in $L^2_{\mathcal{C}}\big(\sigma(e^{it})\big)$. Therefore M is a doubly invariant subspace of $L^2_{\mathcal{C}}\big(\sigma(e^{it})\big)$ with respect to S_2; by [4, Theorem 8, p. 59], M must be of the form

$$M = \int \oplus M(e^{it})\, d\sigma(e^{it}), \tag{1}$$

where each $M(e^{it})$ is a subspace of $\mathcal{C}$.

It is easy to check that $B(e^{is}, e^{it})^*B(e^{is}, e^{it})$ agrees with the projection operator $P_{M(e^{it})}$ of $\mathcal{C}$ onto $M(e^{it})$. Consequently, the values of $B(\cdot, e^{it})$ are partial isometries on $\mathcal{C}$ with initial space $M(e^{it})$. We may therefore view each $B(\cdot, e^{it})$ as an S_1-inner function with related space $M(e^{it})$.

PROPOSITION 1. *A function $B \in L^\infty_{\mathcal{B}(\mathcal{C})}(\sigma^2)$ is S_1-inner on $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}\big(\sigma(e^{is})\big)$ if and only if $B(\cdot, e^{it})$ is S_1-inner on $H^2_{\mathcal{C}}\big(\sigma(e^{is})\big)$, a.e. $[\sigma(e^{it})]$. In this case, equation (1) holds with $M = M_{\rm in}(B)$ and $M(e^{it}) = M_{\rm in}\big(B(\cdot, e^{it})\big)$; furthermore, we have $B(e^{is}, e^{it})^*B(e^{is}, e^{it}) = P_{M(e^{it})}$.*

PROOF. If $B(\cdot, e^{it})$ is S_1-inner on $H^2_{\mathcal{C}}\big(\sigma(e^{is})\big)$, a.e. $[\sigma(e^{it})]$, then B is itself S_1-analytic. Hence in this situation, B is in fact Π-analytic. Observe that B^*B is the projection operator of $H^2_{L^2_{\mathcal{C}}(\sigma(e^{it}))}\big(\sigma(e^{is})\big)$ onto M, where

$$M = \int \oplus M_{\rm in}\big(B(\cdot, e^{it})\big)\, d\sigma(e^{it}),$$

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It follows that B is S_1-inner. The rest was proved above. ∎

The analogous statement concerning S_1-outer functions is [2, Proposition 6.2].

The classical canonical factorization theorem states that an S_1-analytic function can be written as a product of an S_1-inner function and an S_1-outer function. The following assertion slightly strengthens this claim when the given function is Π-analytic.

PROPOSITION 2. *If $A \in L^\infty_{\mathcal{B}(C)}(\sigma^2)$ is Π-analytic, then there exist Π-analytic functions B and C such that B is S_1-inner, C is S_1-outer, and $A = BC$.*

PROOF. Let $\mathcal{K}$ be the Π-invariant subspace $\big(A\mathcal{M}_C(\Pi)\big)^-$. By [1, Theorem 3.2], there exists a Π-analytic, S_1-inner function B such that $\mathcal{K} = B\mathcal{M}_C(\Pi)$. Since B is a partial isometry, B^*B is the projection of $\mathcal{M}_C(\Pi)$ onto the initial space $H^2_M\big(\sigma(e^{is})\big)$ of B, where $M = M_{\text{in}}(B)$. Now with $C = B^*A$ we have

$$\big(A\mathcal{M}_C(\Pi)\big)^- = B\mathcal{M}_C(\Pi)$$
$$\big(B^*A\mathcal{M}_C(\Pi)\big)^- = B^*B\mathcal{M}_C(\Pi)$$
$$\big(C\mathcal{M}_C(\Pi)\big)^- = H^2_M\big(\sigma(e^{is})\big).$$

Thus C is Π-analytic and S_1-outer with $M_{\text{out}}(C) = M$. ∎

The corresponding statement is false with Λ in place of Π; see [1, Theorem 2.2]. This fact was already pointed out [5] in the case C is the complex field.

Functions which are Λ-inner enjoy a characterization similar to Proposition 1. The proof is similar.

PROPOSITION 3. *Suppose that $J \in L^\infty_{\mathcal{B}(C)}(\sigma^2)$ is Λ-analytic. Then J is Λ-inner if and only if J^*J takes a constant value P_N, where P_N is the projection operator of C onto some subspace N of C. In this case, $N = N_{\text{in}}(J)$. Furthermore, J is then S_1-inner with related space $M_{\text{in}}(J) = L^2_N\big(\sigma(e^{it})\big)$.*

The following characterization of the Λ-invariant subspaces of $\mathcal{M}_C(\Lambda)$ is useful. Established as [1, Theorem 3.3], it is the analogue for $\mathcal{M}_C(\Lambda)$ of the Beurling-Lax theorem. A version for matrix valued functions is provided by Loubaton [7].

LEMMA 4. *A subspace $\mathcal{N}$ of $\mathcal{M}_C(\Lambda)$ is Λ-invariant if and only if there exist functions $J(e^{is}, e^{it})$ and $B(e^{is}, e^{it})$ in $L^\infty_{\mathcal{B}(C)}(\sigma^2)$ such that*

(i) $J(e^{is}, e^{it})$ is Λ-inner on $\mathcal{M}_C(\Lambda)$;

(ii) $B(\cdot, e^{it})$ is S_1-inner on $H^2_{L^2_C(\sigma(e^{it}))}\big(\sigma(\cdot)\big)$;

(iii) $J^*B = 0$; and

(iv) $\mathcal{N} = J\mathcal{M}_C(\Lambda) \oplus S_1B\mathcal{M}_C(\Pi)$.

The above result yields general structural information about Λ-analytic functions, analogous to the inner-outer factorization from the one-variable picture. We shall call it the *canonical representation* of a Λ-analytic function.

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THEOREM 5. *If F is a Λ-analytic function, then $F(e^{is}, e^{it})$ has the representation*

$$F(e^{is}, e^{it}) = J(e^{is}, e^{it})G(e^{is}, e^{it}) + e^{is}B(e^{is}, e^{it})A(e^{is}, e^{it}), \tag{2}$$

where

(i) J is a Λ-inner function;
(ii) G is a Λ-outer function with $N_{\text{out}}(G) = N_{\text{in}}(J)$;
(iii) B is a Π-analytic, S_1-inner function on $H^2_{L^2_C(\sigma(e^{it}))}\big(\sigma(\cdot)\big)$;
(iv) A is a Π-analytic, S_1-outer function on $H^2_{L^2_C(\sigma(e^{it}))}\big(\sigma(\cdot)\big)$ with $M_{\text{out}}(A) = M_{\text{in}}(B)$;
(v) $J^*B = 0$; and
(vi) $N_{\text{in}}(J) \perp M_{\text{in}}\big(B(\cdot, e^{it})\big)$, a.e. $[\sigma(e^{it})]$.

PROOF. Apply Lemma 4 to the Λ-invariant subspace $\mathcal{N} = [F\mathcal{M}_C(\Lambda)]^-$ to get J and B satisfying (i), (iii) and (v). Now note that

$$[J^*F\mathcal{M}_C(\Lambda)]^- = \mathcal{M}_M(\Lambda),$$

and

$$[S_1^*B^*F\mathcal{M}_C(\Pi)]^- = H^2_N\big(\sigma(e^{is})\big).$$

Thus, the functions $G = J^*F$ and $A = B^*F$ satisfy (ii) and (iv), respectively, and the representation (2) holds.

Observe that the ranges of J and B are orthogonal pointwise, hence

$$\dim[J(e^{is}, e^{it})C] + \dim[B(e^{is}, e^{it})C] \leq \dim C.$$

Since the values of J and B are partial isometries, this gives

$$\dim N_{\text{in}}(J) + \dim M_{\text{in}}\big(B(\cdot, e^{it})\big) \leq \dim C.$$

It follows that in the construction of B in the proof of [1, Theorem 3.2] we may take the vectors $\{c_k\}$ to be an orthonormal set in $C \ominus N_{\text{in}}(J)$ rather than merely in C. With that change, we may insist that condition (vi) hold. ■

We now follow [4], and define a *range function* to be a function $\mathcal{F}$ on $\mathbf{T}^2$ with values in the subspaces of C. A range function $\mathcal{F}$ is *measurable* if the corresponding projection valued function $P_{\mathcal{F}}$ is weakly measurable. As argued in [4, p. 58], we may view $P_{\mathcal{F}}$ itself as a projection of $L^2_C(\sigma^2)$ onto the subspace $\mathcal{M}_{\mathcal{F}}$ defined by

$$\mathcal{M}_{\mathcal{F}} = \{f \in L^2_C(\sigma^2) : f(e^{is}, e^{it}) \in \mathcal{F}(e^{is}, e^{it}), \text{ a.e. } [\sigma^2]\}.$$

If F is a $\mathcal{B}(C)$-valued function such that

$$[F(e^{is}, e^{it})C]^- = \mathcal{F}(e^{is}, e^{it}), \text{ a.e. } [\sigma^2],$$

then we speak of $\mathcal{F}$ as being the range function of F.

For a range function $\mathcal{F}$ and for a semigroup Ω of $\mathbf{Z}^2$ define $\mathcal{A}_{\mathcal{F}}(\Omega)$ to be the subspace

$$\mathcal{A}_{\mathcal{F}}(\Omega) = \{f \in \mathcal{M}_C(\Omega) : f(e^{is}, e^{it}) \in \mathcal{F}(e^{is}, e^{it}), \text{ a.e. } [\sigma^2]\}.$$

A range function $\mathcal{F}$ is said to be Π-*analytic* if there exists a sequence $\{f_k\}$ in $\mathcal{M}_C(\Pi)$ such that

$$\mathcal{F}(e^{is}, e^{it}) = \bigvee_k \{f_k(e^{is}, e^{it})\}, \text{ a.e. } [\sigma^2]. \tag{3}$$

A range function $\mathcal{F}$ is said to be Λ-*analytic* if there exists a sequence $\{f_k\}$ in $\mathcal{M}_C(\Lambda)$ such that equation (3) holds, and in addition we have

$$\mathcal{A}_{\mathcal{F}}(\Lambda) = \sum \oplus \{S_{mn}[\mathcal{A}_{\mathcal{F}}(\Lambda) \ominus S_2\mathcal{A}_{\mathcal{F}}(\Lambda)] : (m,n) \in \Lambda\}. \tag{4}$$

The necessity of the condition (4) was noted in [7]; it arises because a complication in the general structure of Λ-invariant subspaces of $\mathcal{M}_C(\Lambda)$.

If $\mathcal{F}$ is Π-analytic, then $\mathcal{A}_{\mathcal{F}}(\Pi)$ is a Π-invariant subspace of $\mathcal{M}_C(\Pi)$. By [1, Theorem 3.2], there exists a Π-analytic, S_1-inner function B such that $\mathcal{A}_{\mathcal{F}}(\Pi) = B\mathcal{M}_C(\Pi)$. Likewise, if $\mathcal{F}$ is Λ-analytic, then by equation (4) and [1, Theorem 3.1] there exists a Λ-inner function J such that $\mathcal{A}_{\mathcal{F}}(\Lambda) = J\mathcal{M}_C(\Lambda)$. Following [7] we speak of B and J as the inner functions *associated* with $\mathcal{A}_{\mathcal{F}}(\Pi)$ and $\mathcal{A}_{\mathcal{F}}(\Lambda)$, respectively. They are unique up to partially isometric factors which are constant in the appropriate sense. Since $\Lambda \subseteq \Pi$, the following assertion is not surprising.

PROPOSITION 6. *Suppose that $\mathcal{F}$ is Λ-analytic with associated Λ-inner function J. Then $\mathcal{F}$ is Π-analytic with associated S_1-inner function J.*

PROOF. By hypothesis, $\mathcal{A}_{\mathcal{F}}(\Lambda) = J\mathcal{M}_C(\Lambda)$, and hence $JC = \mathcal{A}_{\mathcal{F}}(\Lambda) \ominus S_2\mathcal{A}_{\mathcal{F}}(\Lambda)$. By Proposition 3, J is S_1-inner. Now observe that

$$\begin{aligned}
S_1\mathcal{A}_{\mathcal{F}}(\Pi) &= \mathcal{A}_{\mathcal{F}}\big(\Pi + (1,0)\big) \\
&\subseteq \bigcap_{n=0}^{\infty} S_2^n \mathcal{A}_{\mathcal{F}}(\Lambda) \\
&= \bigcap_{n=0}^{\infty} S_2^n \sum \oplus \{S_{jk}[\mathcal{A}_{\mathcal{F}}(\Lambda) \ominus S_2\mathcal{A}_{\mathcal{F}}(\Lambda)] : (j,k) \in \Lambda\} \\
&= \sum \oplus \{S_{jk}[\mathcal{A}_{\mathcal{F}}(\Lambda) \ominus S_2\mathcal{A}_{\mathcal{F}}(\Lambda)] : (j,k) \in \Pi + (1,0)\} \\
&= \sum \oplus \{S_{jk}JC : (j,k) \in \Pi + (1,0)\} \\
&= S_1 J\mathcal{M}_C(\Pi).
\end{aligned}$$

So $\mathcal{A}_{\mathcal{F}}(\Pi) \subseteq J\mathcal{M}_C(\Pi)$, and the reverse inclusion is immediate. Since $\mathcal{F}$ is Λ-analytic, there are functions $\{f_k\}$ in $\mathcal{M}_C(\Lambda)$ such that equation (3) holds. Since each f_k already lies in $\mathcal{M}_C(\Pi)$, it follows that $\mathcal{F}$ is Π-analytic. By the above argument, J is its associated S_1-inner function. ■

Associated inner functions have the expected and desirable property described below.

PROPOSITION 7. *(i) If B is the S_1-inner, Π-analytic function associated with a Π-analytic range function $\mathcal{F}$, then $\mathcal{F}$ is the range function of B. (ii) If J is the Λ-inner function associated with a Λ-analytic range function $\mathcal{F}$, then $\mathcal{F}$ is the range function of J.*

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PROOF. (i) By hypothesis, the subspace $\mathcal{A}_{\mathcal{F}}(\Pi)$ is of the form $B\mathcal{M}_C(\Pi)$. Clearly, for any $c \in C$, we have $Bc \in \mathcal{A}_{\mathcal{F}}(\Pi)$, a.e. $[\sigma^2]$. Hence

$$B(e^{is}, e^{it})C \subseteq \mathcal{F}(e^{is}, e^{it}).$$

On the other hand, there exist $\{f_k\}$ in $\mathcal{M}_C(\Pi)$ such that condition (3) holds. Since each f_k belongs to $\mathcal{A}_{\mathcal{F}}(\Pi)$, there are functions g_k in the initial space of B such that $f_k = Bg_k$. Thus

$$\begin{aligned}\mathcal{F}(e^{is}, e^{it}) &= B(e^{is}, e^{it}) \vee \{g_k(e^{is}, e^{it})\} \\ &\subseteq B(e^{is}, e^{it})C.\end{aligned}$$

This shows that $\mathcal{F}$ is the range function of B.

The proof of part (ii) is similar. ∎

We may impose some useful structure on a Π-analytic function giving rise to a certain range function. Let V be Π-analytic, and let $\mathcal{F}$ be its range function. Then $W = VV^*$ is nonnegative valued with range function $\mathcal{F}$. Since

$$W(e^{-is}, e^{-it}) = V(e^{-is}, e^{-it})V(e^{-is}, e^{-it})^*$$

is an S_1-analytic factorization, [2, Lemma 6.1] gives

$$W(e^{-is}, e^{-it}) = F(e^{is}, e^{it})^* F(e^{is}, e^{it})$$

for some S_1-outer, Π-analytic function F. And now $\mathcal{F}$ is the range of $\tilde{F}$. Similarly there is an S_1-outer, Π-analytic function G such that

$$F(e^{-is}, e^{-it})F(e^{-is}, e^{-it})^* = G(e^{is}, e^{it})^* G(e^{is}, e^{it}).$$

Let K be the partial isometry valued function satisfying

$$F(e^{-is}, e^{-it})^* = K(e^{is}, e^{it})G(e^{is}, e^{it}).$$

Note that the operator $K(e^{is}, e^{it})$ has initial space $[G(e^{is}, e^{it})C]^-$ and final space $[F(e^{-is}, e^{-it})^* C]^-$. The function $G(\cdot, e^{it})$ is S_1-outer on $H^2_C\big(\sigma(\cdot)\big)$, $\sigma(e^{it})$-almost everywhere [2, Proposition 6.2]. Thus by [10, Section 5.4], $[G(e^{is}, e^{it})C]^- = M_{\mathrm{out}}\big(G(\cdot, e^{it})\big)$ independently of e^{is}. Note also that

$$\begin{aligned}K\mathcal{M}_C(\Pi) &= KH^2_M\big(\sigma(e^{is})\big) \\ &= K[G\mathcal{M}_C(\Pi)]^- \\ &= \tilde{F}\mathcal{M}_C(\Pi),\end{aligned}$$

where $M = M_{\mathrm{out}}(G)$. Thus K is S_1-inner with range $\mathcal{F}$. Finally, there exists a Π-analytic function U such that $\tilde{U}$ is S_1-outer and

$$\tilde{K}^*\tilde{K} = \tilde{U}^*\tilde{U}. \tag{5}$$

Check that U has range function $\mathcal{F}$ as well, and note that the left side of equation (5) is projection valued. Furthermore since $\tilde{U}$ is S_1-outer, the space $U(e^{-is}, e^{-it})^* \mathcal{C}$ is some subspace $M(e^{it})$ of $\mathcal{C}$ not depending on e^{is}. Hence

$$U(e^{-is}, e^{-it})^* \mathcal{M}_C(\Pi) = H^2_M\big(\sigma(e^{is})\big),$$

where

$$M = \int \oplus M(e^{it})\, d\sigma(e^{it}).$$

Evidently $U(\cdot, e^{it})$ is S_1-inner with initial space $M(e^{-it})$. It follows from Proposition 1 that U is S_1-inner. This proves the following claim.

PROPOSITION 8. *If $\mathcal{F}$ is the range function of a Π-analytic function, then $\mathcal{F}$ is the range function of an S_1-inner, Π-analytic function U such that $\tilde{U}$ is S_1-outer.*

The above proof fails with Π replaced with Λ, since the existence of a Λ-analytic factorization does not imply that of a Λ-outer factorization.

We next see that range inclusion gives rise to a factorization.

LEMMA 9. *(i) Suppose that $\mathcal{F}$ is a Π-analytic range function, and let B be an associated S_1-inner, Π-analytic function. If A is a Π-analytic, partial isometry valued function with range included pointwise in $\mathcal{F}$, then*

$$A = BC$$

for some Π-analytic, partial isometry valued function C. (ii) Suppose that $\mathcal{F}$ is a Λ-analytic range function, and let J be the associated Λ-inner function. If A is a Λ-analytic, partial isometry valued function with range included pointwise in $\mathcal{F}$, then

$$A = JC$$

for some Λ-analytic, partial isometry valued function C.

PROOF.

$$\begin{aligned} A\mathcal{M}_C(\Pi) &\subseteq \mathcal{A}_{\mathcal{F}}(\Pi) \\ &= B\mathcal{M}_C(\Pi). \end{aligned}$$

Therefore

$$\begin{aligned} B^*A\mathcal{M}_C(\Pi) &\subseteq B^*B\mathcal{M}_C(\Pi) \\ &= H^2_M\big(\sigma(e^{is})\big), \end{aligned}$$

where $M = M_{\text{in}}(B)$. In particular, $C = B^*A$ is Π-analytic and partial isometry valued. Now

$$\begin{aligned} C &= B^*A \\ BC &= BB^*A \\ BC &= P_{\mathcal{F}}A \\ BC &= A. \end{aligned} \tag{6}$$

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This verifies the assertion (i). Again, the proof of (ii) is similar. ∎

An inner function need not be associated with its range function. Indeed, this occurs only under a rather stringent condition.

PROPOSITION 10. *(i) A Π-analytic, S_1-inner function B is associated with its range function if and only if $\tilde{B}$ is Π-outer. (ii) A Λ-inner function J is associated with its range function if and only if $\tilde{J}$ is Λ-outer.*

PROOF. (i) Let B be associated with its Π-analytic range function $\mathcal{F}$. Suppose that U is any Π-analytic function with the property $U^*U = \tilde{B}^*\tilde{B}$. Then $\tilde{U}$ has the same range function as B, namely $\mathcal{F}$. Lemma 9 asserts that there exists a Π-analytic function D such that $U = D\tilde{B}$. It must be that $\tilde{B}$ is Π-outer.

For the converse, let V be a Π-analytic, S_1-inner function associated with the range function of B. Then Lemma 9 provides that $\tilde{B} = \tilde{C}\tilde{V}$ for some Π-analytic, partial isometry valued function C. If $\tilde{B}$ is S_1-outer, then $\tilde{V}$ is S_1-outer and $\tilde{C}$ is S_1-constant. Thus B is associated with its range function as well.

(ii) Suppose that J is associated with its Λ-analytic range function $\mathcal{F}$. Then by Proposition 6, the range function is also Π-analytic with associated S_1-inner function J. By part (i), $\tilde{J}$ is S_1-outer. Thus the canonical representation of $\tilde{J}$ (see Theorem 5) reduces to $\tilde{J} = F\Gamma_1 + S_1 0$. If U is any Λ-analytic function such that $U^*U = \tilde{J}^*\tilde{J}$, then Lemma 9 provides for a Λ-analytic function D such that $U = D\tilde{J}$. Since this is true for $U = \Gamma_1$, the Λ-inner factor F must have been Λ-constant. Thus, $\tilde{J}$ is Λ-outer.

Conversely, assume that $\tilde{J}$ is Λ-outer, and let V be a Λ-inner function associated with the range function $\mathcal{F}$ of J. Then there exists a Λ-analytic function C such that $\tilde{J} = \tilde{C}\tilde{V}$. From part (i), $\tilde{V}$ is already Π-outer, so that its canonical representation must be reduced to the form $\tilde{V} = F\Gamma_1 + S_1 0$. With that, F and C must be Λ-constant partial isometries. It follows that $J = VC$ was already associated with $\mathcal{F}$. ∎

We are concerned with applying these ideas toward the the spectral factorization problem. To this end, it will be necessary to relate an outer function to a range function through the appropriate inner function. These connections are made with the help of the following two lemmata.

LEMMA 11. *Suppose that Φ is a Π-analytic function with the factorization $\Phi = B\Psi$, where B is Π-analytic and S_1-inner, and Ψ is Π-analytic and S_1-outer. If $\tilde{\Phi}$ is S_1-outer, then $\tilde{B}$ is S_1-outer.*

PROOF. By hypothesis, $\tilde{\Phi} = \tilde{\Psi}\tilde{B}$, and so

$$\begin{aligned} \tilde{\Psi}\tilde{B}\mathcal{M}_C(\Pi) &= \tilde{\Phi}\mathcal{M}_C(\Pi) \\ &= H^2_M\big(\sigma(e^{is})\big), \end{aligned} \tag{7}$$

where $M = M_{\text{out}}(\tilde{\Phi})$. Since Ψ is S_1-outer, we may define the related spaces $m(e^{it}) = M_{\text{out}}\big(\Psi(\cdot, e^{it})\big)$. Then each value of $\tilde{B}(e^{is}, e^{it})$ is a partial isometry with final space $m(e^{-it})$.

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So the co-kernel of $\tilde{\Psi}$ is $H^2_{M'}\big(\sigma(e^{is})\big)$, where

$$M' = \int \oplus m(e^{-it})\, d\sigma(e^{it}),$$

and the image of $\mathcal{M}_C(\Pi)$ under $\tilde{B}$ is included in $H^2_{M'}\big(\sigma(e^{is})\big)$. In fact, $\tilde{B}\mathcal{M}_C(\Pi)$ must be dense in $H^2_{M'}\big(\sigma(e^{is})\big)$, in order for equation (7) to hold. This shows that $\tilde{B}$ is S_1-outer. ∎

LEMMA 12. *Suppose that Φ is Λ-analytic and $\tilde{\Phi}$ is Λ-outer. Then the range function $\mathcal{F}$ of Φ is Λ-analytic.*

PROOF. By Lemma 4, the Λ-invariant subspace $\mathcal{A}_{\mathcal{F}}(\Lambda)$ has the structure

$$\mathcal{A}_{\mathcal{F}}(\Lambda) = J\mathcal{M}_C(\Lambda) \oplus S_1 B\mathcal{M}_C(\Pi), \tag{8}$$

where J is Λ-inner, B is Π-analytic and S_1-inner, $J^*B = 0$, and (from the proof of Theorem 5)

$$N_{\text{in}}(J) \perp M_{\text{in}}\big(B(\cdot, e^{it})\big), \quad \text{a.e. } [\sigma(e^{it})].$$

Next, check that from equation (8)

$$\begin{aligned}
[\Phi\mathcal{M}_C(\Lambda)]^- &\subseteq \mathcal{A}_{\mathcal{F}}(\Lambda)\\
J^*[\Phi\mathcal{M}_C(\Lambda)]^- &\subseteq \mathcal{M}_{N(J)}(\Lambda)\\
B^*[\Phi\mathcal{M}_C(\Lambda)]^- &\subseteq S_1 H^2_{M(B)}\big(\sigma(e^{is})\big).
\end{aligned}$$

Thus $\Gamma_1 = J^*\Phi$ is Λ-analytic, and $\Gamma_2 = S_1^* B^*\Phi$ is Π-analytic. Furthermore, we have $[\Gamma_1(e^{is}, e^{it})C]^- \perp [\Gamma_2(e^{is}, e^{it})C]^-$. And now

$$\begin{aligned}
\Phi &= (JJ^* + BS_1S_1^*B^*)\Phi\\
&= J\Gamma_1 + S_1 B\Gamma_2\\
&= (J+B)(\Gamma_1 + S_1\Gamma_2).
\end{aligned}$$

Note that

$$\begin{aligned}
(J+B)^*(J+B) &= J^*J + B^*B\\
&= P_{N(J)} + P_{M(B)}\\
&= P_{N(J)\oplus M(B)}.
\end{aligned}$$

Hence the Π-analytic function $J+B$ is S_1-inner.

Let $\Phi = W\Psi$ be a factorization of Φ where W is Π-analytic and S_1-inner, and Ψ is Π-analytic and S_1-outer. By Lemma 11 and Proposition 10, W is the associated S_1-inner function to the range function $\mathcal{F}$. Since the range of $J+B$ is pointwise included in $\mathcal{F}$, we have $(J+B) = WU$ for some Π-analytic, partial isometry valued function U. But then

$$\Phi = W\Psi = WU(\Gamma_1 + S_1\Gamma_2)$$

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implies that

$$\Psi = U(\Gamma_1 + S_1\Gamma_2).$$

Because Ψ is S_1-outer, Γ_2 must be identically zero and U must be S_1-constant. Thus

$$\begin{aligned}\Phi &= J\Gamma_1 + S_1 B\Gamma_2 \\ &= J\Gamma_1,\end{aligned}$$

and J has range function $\mathcal{F}$ as well. By Lemma 11, $\tilde{J}$ is S_1-outer; by Proposition 10, the S_1-inner function J is associated with its Π-analytic range function $\mathcal{F}$.

Finally, from

$$\begin{aligned}\mathcal{A}_{\mathcal{F}}(\Lambda) &= J\mathcal{M}_C(\Lambda) \oplus S_1 B\mathcal{M}_C(\Pi) \\ &\subseteq \mathcal{A}_{\mathcal{F}}(\Pi) \\ &= J\mathcal{M}_C(\Pi)\end{aligned}$$

and the fact that $J\mathcal{M}_C(\Pi) \perp S_1 B\mathcal{M}_C(\Pi)$, we see that $B = 0$. Therefore $\mathcal{A}_{\mathcal{F}}(\Lambda) = J\mathcal{M}_C(\Lambda)$, and $\mathcal{F}$ is Λ-analytic. ∎

Now suppose that W belongs to $L^\infty_{\mathcal{B}(C)}(\sigma^2)$, and that its values are nonnegative operators. A fundamental issue is whether there exists a Λ-outer function Φ such that $W = \Phi^*\Phi$. A "regularity" criterion on W is given in [2], and that result is extended in [1] to allow Λ to be a more general type of semigroup. Loubaton [7] derived a logarithmic integral test for the finite rank case. To accomplish this, it was necessary to reduce the problem to the full rank case. We shall pursue the corresponding reduction below.

THEOREM 13. *In order that the nonnegative weight function $W \in L^\infty_{\mathcal{B}(C)}(\sigma^2)$ have a Λ-outer factorization, it is necessary for the range function $\mathcal{F}$ of W to be $(-\Lambda)$-analytic. Suppose that this necessary condition holds, and let J be the $(-\Lambda)$-inner function associated with $\mathcal{F}$. Then W has a Λ-outer factorization if and only if the weight function $W_1 = J^*WJ$ does.*

PROOF. Suppose that the nonnegative operator valued function W belonging to $L^\infty_{\mathcal{B}(C)}(\sigma^2)$ has a Λ-outer factorization $W = \Phi^*\Phi$. Then the range function $\mathcal{F}$ of W coincides with that of Φ^*. Since Φ is Λ-outer, the function $\tilde{\Phi}^*$ is $(-\Lambda)$-outer. By Lemma 12, with $(-\Lambda)$ in place of Λ, we deduce that $\mathcal{F}$ is then $(-\Lambda)$-analytic. Hence there is a $(-\Lambda)$-inner function J such that

$$\mathcal{A}_{\mathcal{F}}(-\Lambda) = J\mathcal{M}_C\big((-\Lambda)\big).$$

Set $W_1 = J^*WJ$. Now $W_1 = (\Phi J)^*(\Phi J)$. For any $c \in N(\Phi)$, the function Φ^*c lies in $\mathcal{A}_{\mathcal{F}}(-\Lambda)$. Thus $\Phi^*c = Jf$ for some $f \in \mathcal{M}_C(-\Lambda)$. Thus $J^*\Phi^*$ preserves $\mathcal{M}_C(-\Lambda)$ and is therefore $(-\Lambda)$-analytic. The factor ΦJ is Λ-outer, for

$$\begin{aligned}\mathcal{M}_{N(\Phi)}(\Lambda) &= \big(\Phi\mathcal{M}_C(\Lambda)\big)^- \\ &\supseteq \big(\Phi J\mathcal{M}_C(\Lambda)\big)^- \\ &\supseteq \big(\Phi JJ^*\mathcal{M}_C(\Lambda)\big)^- \\ &= \mathcal{M}_{N(\Phi)}(\Lambda).\end{aligned}$$

On the other hand, assume that W has a $(-\Lambda)$-analytic range function $\mathcal{J}$ with associated $(-\Lambda)$-inner function J, and consider again $W_1 = J^* W J$. If W_1 has a Λ-outer factorization, $W_1 = \Psi^*\Psi$, then W has the Λ-analytic factorization $W = (\Psi J^*)^*(\Psi J^*)$. In fact, by Proposition 10, J^* is Λ-outer, hence (ΨJ^*) is Λ-outer. ∎

The point is that we may view the values of W_1 as full rank operators on $N_{\text{in}}(J)$. With that, we may be able to apply a factorization criterion such as [2, Theorem 7.4].

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