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Algebraic cycles and topology of real Enriques surfaces

FRÉDÉRIC MANGOLTE[1]* and JOOST VAN HAMEL[2]**
[1]*Laboratoire de Mathématiques, Université de Savoie, 73376 Le Bourget du Lac Cedex, France; e-mail: mangolte@math.univ-savoie.fr*
[2]*Institute of Mathematics, University of Utrecht, Budapestlaan 6, 3584 CD Utrecht, The Netherlands; e-mail: vanhamel@math.ruu.nl*

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Abstract. For a real Enriques surface Y we prove that every homology class in $H_1(Y(\mathbb{R}), \mathbb{Z}/2)$ can be represented by a real algebraic curve if and only if all connected components of $Y(\mathbb{R})$ are orientable. Furthermore, we give a characterization of real Enriques surfaces which are Galois-Maximal and/or $\mathbb{Z}$-Galois-Maximal and we determine the Brauer group of any real Enriques surface Y.

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1. Introduction

Let Y be a complex algebraic surface. Let us denote by $Y(\mathbb{C})$ the set of closed points of Y endowed with the Euclidean topology and let $H_2^{\mathrm{alg}}(Y(\mathbb{C}), \mathbb{Z})$ be the subgroup of the homology group $H_2(Y(\mathbb{C}), \mathbb{Z})$ generated by the fundamental classes of algebraic curves on Y. If Y is an Enriques surface, we have

$$H_2^{\mathrm{alg}}(Y(\mathbb{C}), \mathbb{Z}) = H_2(Y(\mathbb{C}), \mathbb{Z}).$$

One of the goals of the present paper is to prove a similar property for real Enriques surfaces with orientable real part. See Theorem 1.1 below.

By an *algebraic variety Y over $\mathbb{R}$* we mean a geometrically integral, separated scheme of finite type over the real numbers. The Galois group $G = \{1, \sigma\}$ of $\mathbb{C}/\mathbb{R}$ acts on $Y(\mathbb{C})$, the set of complex points of Y, via an antiholomorphic involution and the real part $Y(\mathbb{R})$ is precisely the set of fixed points under this action. An algebraic variety Y over $\mathbb{R}$ will be called a *real Enriques surface*, a real K3-surface, etc., if the complexification $Y_{\mathbb{C}} = Y \otimes \mathbb{C}$ is a complex Enriques surface, resp. a complex K3-surface, etc. Consider the following two classification problems:

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- classification of topological types of algebraic varieties Y over $\mathbb{R}$ (the manifolds $Y(\mathbb{C})$ up to equivariant diffeomorphism),
- classification of topological types of the real parts $Y(\mathbb{R})$.

For real Enriques surfaces the two classifications have been investigated recently by Nikulin in [Ni2]. The topological classification of the real parts was completed by Degtyarev and Kharlamov, who give in [DKh1] a description of all 87 topological types. Let us mention here that the real part of a real Enriques surface Y need not be connected and that a connected component V of $Y(\mathbb{R})$ is either a nonorientable surface of genus $\leqslant 11$ or it is homeomorphic to a sphere or to a torus.

The problem of classifying $Y(\mathbb{C})$ up to equivariant diffeomorphism still lacks a satisfactory solution. In the attempts to solve this problem, equivariant (co)homology plays an important role (see [Ni2], [NS], [DKh2]). It establishes for any algebraic variety Y over $\mathbb{R}$ a link between the action of G on the (co)homology of $Y(\mathbb{C})$ and the topology of $Y(\mathbb{R})$. For example, if Y is of dimension d, the famous inequalities

$$\dim H_*(Y(\mathbb{R}), \mathbb{Z}/2) \leqslant \sum_{r=0}^{2n} \dim H^1(G, H_r(Y(\mathbb{C}), \mathbb{Z}/2)), \tag{1}$$

$$\dim H_{\text{even}}(Y(\mathbb{R}), \mathbb{Z}/2) \leqslant \sum_{r=0}^{2n} \dim H^2(G, H_r(Y(\mathbb{C}), \mathbb{Z})), \tag{2}$$

$$\dim H_{\text{odd}}(Y(\mathbb{R}), \mathbb{Z}/2) \leqslant \sum_{r=0}^{2n} \dim H^1(G, H_r(Y(\mathbb{C}), \mathbb{Z})), \tag{3}$$

(cf. [Kr1] or [Si]) can be proven using equivariant homology.

We will say that Y is *Galois-Maximal* or a *GM-variety* if the first inequality turns into equality, and Y will be called $\mathbb{Z}$*-Galois-Maximal*, or a $\mathbb{Z}$*-GM-variety* if inequalities (2) and (3) are equalities. When the homology of $Y(\mathbb{C})$ is torsion free, the two notions coincide (see [Kr1, Prop. 3.6]).

A nonsingular projective surface Y over $\mathbb{R}$ with $Y(\mathbb{R}) \neq \emptyset$ is both *GM* and $\mathbb{Z}$-*GM* if it is simply connected (see [Kr1, Sect. 5.3]). If $H_1(Y(\mathbb{C}), \mathbb{Z}) \neq 0$, as in the case of an Enriques surface, the situation can be much more complicated. The necessary and sufficient conditions for a real Enriques surface Y to be a *GM*-variety were found in [DKh2]; in the present paper we will give necessary and sufficient conditions for Y to be $\mathbb{Z}$-*GM*. See Theorem 1.2.

As far as we know, this is the first paper on real Enriques surfaces in which equivariant (co)homology with integral coefficients is studied instead of coefficients in $\mathbb{Z}/2$. We expect that the extra information that can be obtained this way (compare for example equations (1)–(3)) will be useful in the topological classification of real Enriques surfaces.

In Section 6 we demonstrate the usefulness of integral coefficients by computing the Brauer group $\mathrm{Br}(Y)$ of any real Enriques surface Y. This completes the partial results on the 2-torsion of $\mathrm{Br}(Y)$ obtained in [NS] and [Ni1]. See Theorem 1.3.

1.1. Main results

Let Y be an algebraic variety over $\mathbb{R}$. Denote by $H_n^{\text{alg}}(Y(\mathbb{R}), \mathbb{Z}/2)$ the subgroup of the homology group $H_n(Y(\mathbb{R}), \mathbb{Z}/2)$ generated by the fundamental classes of n-dimensional Zariski-closed subsets of $Y(\mathbb{R})$, see [BH] or [BCR]. We will say that these classes can be *represented by algebraic cycles*. The problem of determining these groups is still open for most classes of surfaces.

For a real rational surface X we always have $H_2^{\text{alg}}(X(\mathbb{C}), \mathbb{Z}) = H_2(X(\mathbb{C}), \mathbb{Z})$ and $H_1^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = H_1(X(\mathbb{R}), \mathbb{Z}/2)$, see [Si]. For real K3-surfaces, the situation is not so rigid. In most connected components of the moduli space of real K3-surfaces the points corresponding to a surface X with $\dim H_1^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2) \geqslant k$ form a countable union of real analytic subspaces of codimension k for any $k \leqslant \dim H_1(X_0(\mathbb{R}), \mathbb{Z}/2)$, where X_0 is any K3-surface corresponding to a point from that component. In some components this is only true for $k < \dim H_1(X_0(\mathbb{R}), \mathbb{Z}/2)$; these components do not contain any point corresponding to a surface X with $H_1^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2) = H_1(X(\mathbb{R}), \mathbb{Z}/2)$, see [Ma2]. For real Abelian surfaces the situation is similar, see [Hu] or [Ma1, Ch.V].

THEOREM 1.1. *Let Y be a real Enriques surface with $Y(\mathbb{R}) \neq \emptyset$. If all connected components of the real part $Y(\mathbb{R})$ are orientable, then*

$$H_1^{\text{alg}}(Y(\mathbb{R}), \mathbb{Z}/2) = H_1(Y(\mathbb{R}), \mathbb{Z}/2).$$

Otherwise,

$$\dim H_1^{\text{alg}}(Y(\mathbb{R}), \mathbb{Z}/2) = \dim H_1(Y(\mathbb{R}), \mathbb{Z}/2) - 1.$$

See Theorem 4.4 for more details.

In order to state further results we should mention that the set of connected components of the real part of a real Enriques surface Y has a natural decomposition into two parts $Y(\mathbb{R}) = Y_1 \sqcup Y_2$. Following [DKh1] we will refer to these two parts as the two *halves* of the real Enriques surface. In [Ni1] it is shown that Y is *GM* if both halves of $Y(\mathbb{R})$ are nonempty. It follows from [DKh2, Lem. 6.3.4] that if precisely one of the halves of $Y(\mathbb{R})$ is empty, then Y is *GM* if and only if $Y(\mathbb{R})$ is nonorientable. This result plays an important role in the proof of many of the main results of that paper (see Sect. 7 in *loc. cit.*).

In the present paper we will see in the course of proving Theorem 1.1 that a real Enriques surface with orientable real part is not a $\mathbb{Z}$-*GM*-variety. In Section 5 we also tackle the nonorientable case and combining our results with the results for coefficients in $\mathbb{Z}/2$ that were already known we obtain the following theorem.

THEOREM 1.2. *Let Y be a real Enriques surface with nonempty real part.*

(i) *Suppose the two halves Y_1 and Y_2 are nonempty. Then Y is GM. Moreover, Y is $\mathbb{Z}$-GM if and only if $Y(\mathbb{R})$ is nonorientable.*

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(ii) *Suppose one of the halves Y_1 or Y_2 is empty. Then Y is GM if and only if $Y(\mathbb{R})$ is nonorientable. Moreover, Y is $\mathbb{Z}$-GM if and only if $Y(\mathbb{R})$ has at least one component of odd Euler characteristic.*

There are examples of all cases described in the above theorem (see [DKh1, Figure 1]).

In Section 6 we study the Brauer group $\mathrm{Br}(Y)$ of a real Enriques surface Y using the fact that $\mathrm{Br}(Y)$ is isomorphic to the cohomological Brauer group $\mathrm{Br}'(Y) = H^2_{\text{ét}}(Y, \mathbb{G}_m)$, since Y is a nonsingular surface. In [NS] Nikulin and Sujatha gave various equalities and inequalities relating the dimension of the 2-torsion of $\mathrm{Br}(Y)$ to other topological invariants of a real Enriques surface Y. It was shown in [Ni1] that

$$\dim_{\mathbb{Z}/2} \mathrm{Tor}(2, \mathrm{Br}(Y)) \geqslant 2s - 1,$$

where s is the number of connected components of $Y(\mathbb{R})$, and that equality holds if Y is *GM*. Using the results in Section 5 on equivariant homology with integral coefficients we can compute the whole group $\mathrm{Br}(Y)$.

THEOREM 1.3. *Let Y be a real Enriques surface. Let s be the number of connected components of $Y(\mathbb{R})$. If $Y(\mathbb{R}) \neq \emptyset$ is nonorientable then*

$$\mathrm{Br}(Y) \simeq (\mathbb{Z}/2)^{2s-1}.$$

If $Y(\mathbb{R}) \neq \emptyset$ is orientable then

$$\mathrm{Br}(Y) \simeq \begin{cases} (\mathbb{Z}/2)^{2s-2} \oplus \mathbb{Z}/4 & \text{if both halves are nonempty,} \\ (\mathbb{Z}/2)^{2s} & \text{if one half is empty.} \end{cases}$$

If $Y(\mathbb{R}) = \emptyset$ then

$$\mathrm{Br}(Y) \simeq \mathbb{Z}/2.$$

We were informed by the referee that Theorems 1.2 and 1.3 and a result similar to Theorem 1.1 have independently been obtained by V.A. Krasnov. His paper will be published in Izv. Ross. Akad. Nauk Ser. Math. **60** (1996), No. 5.

2. Equivariant homology and cohomology

Since the group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts in a natural way on the complex points of an algebraic variety Y defined over $\mathbb{R}$, the best homology and cohomology theories for studying the topology of $Y(\mathbb{R})$ are the ones that take this group action into account. In [NS] étale cohomology $H^*_{\text{ét}}(Y, \mathbb{Z}/2)$ is used, and in [Ni1] the observation is made that this is essentially the same as equivariant cohomology $H^*(Y(\mathbb{C}); G, \mathbb{Z}/2)$. In [DKh2] Degtyarev and Kharlamov do not use equivariant cohomology as

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such, but instead a 'stabilized' form of the Hochschild-Serre spectral sequence $E^2_{p,q}(X;G,\mathbb{Z}/2) = H^p(G,H^q(X,\mathbb{Z}/2))$. This construction, due to I. Kalinin, is based on the fact that if $G=\mathbb{Z}/2$ then $H^{p+2}(G,M) = H^p(G,M)$ for any group M and any $p>0$, and if M is a $\mathbb{Z}/2$-module then even $H^{p+1}(G,M) = H^p(G,M)$ for any $p>0$, so it is possible to squeeze the Hochschild–Serre spectral sequence into 1, or at most 2 diagonals. They also use the analogue of this Kalinin spectral sequence in homology. In the present paper we stick to the original equivariant cohomology supplemented with a straightforward dual construction which we call equivariant Borel–Moore homology.

First we will recall some properties of equivariant cohomology for a space with an action of $G=\mathbb{Z}/2$. Then we will give the definition of equivariant Borel–Moore homology and list the properties that we are going to need. In Section 3 we give a short treatment of the fundamental class of G-manifolds and formulate Poincaré duality in the equivariant context.

Let X be a topological space with an action of $G=\mathbb{Z}/2$. We denote the fixed point set of X by X^G. In [Gr1] the groups $H^*(X;G,\mathcal{F})$ are defined for a G-sheaf $\mathcal{F}$ on X, which is a sheaf with a G-action compatible with the G-action on X. Writing $G=\{1,\sigma\}$, this just means that we are given an isomorphism of sheaves $\varsigma\colon\, \mathcal{F}\to\sigma^*\mathcal{F}$ satisfying $\sigma^*(\varsigma)\circ\varsigma = \mathrm{id}$. Now define

$$H^p(X;G,-) = R^p\Gamma(X,-)^G$$

the pth right derived functor of the G-invariant global sections functor. We have natural mappings

$$e^p_{\mathcal{F}}\colon\, H^p(X;G,\mathcal{F}) \to H^p(X,\mathcal{F})^G,$$

which are the edge morphisms of the *Hochschild–Serre spectral sequence*

$$E^2_{p,q}(X;G,\mathcal{F}) = H^p(G,H^q(X,\mathcal{F})) \Rightarrow H^{p+q}(X;G,\mathcal{F})$$

For us, the most important G-sheaves will be the constant sheaf $\mathbb{Z}/2$ and the constant sheaves constructed from the G-modules $\mathbb{Z}(k)$ for $k\in\mathbb{Z}$. Here we define $\mathbb{Z}(k)$, to be the group of integers, equipped with an action of G defined by $\sigma\cdot z = (-1)^k z$. We will use the notation $A(k)$ to denote either $\mathbb{Z}/2$ or $\mathbb{Z}(k)$, and we will sometimes use A instead of $A(k)$ if k is even.

There is a cup-product

$$H^p(X;G,A(k))\otimes H^q(X;G,A(l)) \to H^{p+q}(X;G,A(k+l))$$

and a pull-back f^* for any continuous equivariant mapping $f\colon\, X\to Y$, which both have the usual properties.

If X is a point, $H^p(\mathrm{pt};G,M) = H^p(G,M)$, which is cohomology of the group G with coefficients in M. Recall that as a graded ring, $H^*(G,\mathbb{Z}/2)$ is isomorphic to the polynomial ring $\mathbb{Z}/2[\eta]$, where η is the nontrivial element in $H^1(G,\mathbb{Z}/2)$. By abuse of notation, we will also use the notation η for the nontrivial element in $H^1(G,\mathbb{Z}(1)) \simeq \mathbb{Z}/2$ and η^2 for the nontrivial element in $H^2(G,\mathbb{Z}) \simeq \mathbb{Z}/2$. This notation is justified by the fact that $\eta \in H^1(G,\mathbb{Z}(1))$ maps to $\eta \in H^1(G,\mathbb{Z}/2)$ under the reduction modulo 2 mapping and $\eta^2 \in H^2(G,\mathbb{Z})$ maps to $\eta^2 \in H^2(G,\mathbb{Z}/2)$.

The constant mapping $X \to \mathrm{pt}$ induces a mapping $H^*(G,\mathbb{Z}/2) \to H^*(X;G,\mathbb{Z}/2)$ and we have a natural injection $H^p(X^G,\mathbb{Z}/2) \hookrightarrow H^p(X^G;G,\mathbb{Z}/2)$, so cup-product gives us for any G-space X a mapping

$$H^*(X^G,\mathbb{Z}/2)\otimes H^*(G,\mathbb{Z}/2) \to H^*(X^G;G,\mathbb{Z}/2),$$

which is well-known to be an isomorphism. Taking the inverse of this isomorphism and sending η to the unit element in $H^*(X^G,\mathbb{Z}/2)$ we obtain a surjective homomorphism of rings $H^*(X^G;G,\mathbb{Z}/2) \to H^*(X^G,\mathbb{Z}/2)$ and we define for $A=\mathbb{Z}$ or $\mathbb{Z}/2$ and any $k \in \mathbb{Z}$ the homomorphism of rings

$$\beta\colon\ H^*(X;G,A(k)) \to H^*(X^G,\mathbb{Z}/2)$$

to be the composite mapping

$$H^*(X;G,A(k)) \xrightarrow{i^*} H^*(X^G;G,A(k))$$

$$\xrightarrow{\text{mod } 2} H^*(X^G;G,\mathbb{Z}/2) \longrightarrow H^*(X^G,\mathbb{Z}/2),$$

where i^* is induced by the inclusion $i\colon\ X^G \hookrightarrow X$. Note that β coincides with the mapping β' in [Kr3]. It is clear from the definition that

$$\beta(f^*\omega) = f^*\beta(\omega).$$

We use the notation

$$\beta^{n,p}\colon\ H^n(X;G,A(k)) \to H^p(X^G;\mathbb{Z}/2),$$

for the mapping induced by β.

In Section 5, we will need one technical lemma which can easily be proven using the Hochschild–Serre spectral sequence.

LEMMA 2.1. *Let X be a G-space with $X^G \neq \emptyset$. Then if $e^2_{A(k)}$ is not surjective on $H^2(X,A(k))^G$, there is a class $\omega \in H^1(X;G,A(k-1))$ such that $e^1_{A(k-1)}(\omega) \neq 0$, but $\beta(\omega)=0$.*

The homology theory we are going to use is the natural dual to equivariant cohomology. For an extensive treatment of its properties, see [vH]. Here we will give a short account without proofs.

In the rest of this section we assume X to be a locally compact space of finite cohomological dimension with an action of $G = \mathbb{Z}/2$, and $A(k)$ will be as above. We define the *equivariant Borel–Moore homology of X with coefficients in $A(k)$* by

$$H_p(X; G, A(k)) = R^{-p}\mathrm{Hom}_G(R\Gamma_c(X, \mathbb{Z}), A(k))$$

for $p \in \mathbb{Z}$, where Hom_G stands for homomorphisms in the category of G-modules and Γ_c stands for global sections with compact support; this is the natural equivariant generalization of the usual Borel–Moore homology in the context of sheaf theory (see, for example, [Iv, Ch.IX]).

If X is homeomorphic to an n-dimensional locally finite simplicial complex with a (simplicial) action of G, then we can determine $H_p(X; G, A(k))$ from a double complex analogous to the double complex (1-12) in [N1], which is used for the calculation of equivariant cohomology. Consider the oriented chain complex $\mathcal{C}_n^\infty \to \mathcal{C}_{n-1}^\infty \to \cdots \to \mathcal{C}_0^\infty$ with closed supports (i.e., the elements of $\mathcal{C}_p^\infty$ are p-chains that can be infinite). The chain complex with coefficients in $A(k)$ is defined by

$$\mathcal{C}_p^\infty(A(k)) = \mathcal{C}_p^\infty \otimes A(k),$$

and we give it the diagonal G-action. Then $H_p(X; G, A(k))$ is naturally isomorphic to the $(-p)$th homology group of the total complex associated to the double complex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{C}_{n-1}^\infty(A(k)) & \xrightarrow{1-\sigma} & \mathcal{C}_{n-1}^\infty(A(k)) & \xrightarrow{1+\sigma} & \mathcal{C}_{n-1}^\infty(A(k)) & \xrightarrow{1-\sigma} & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{C}_{n}^\infty(A(k)) & \xrightarrow{1-\sigma} & \mathcal{C}_{n}^\infty(A(k)) & \xrightarrow{1+\sigma} & \mathcal{C}_{n}^\infty(A(k)) & \xrightarrow{1-\sigma} & \cdots,
\end{array}$$

where the lower left-hand corner has bidegree $(-n, 0)$. Note that by construction $H_p(\mathrm{pt}; G, A(k)) = H^{-p}(G, A(k))$, so Poincaré duality holds trivially when X is a point (and the proof of Poincaré duality in higher dimensions, as stated in Proposition 3.1, is no more difficult than in the nonequivariant case). In particular, $H_p(X; G, A(k))$ need not be zero for $p < 0$.

The groups $H_p(X; G, A(k))$ are covariantly functorial in X with respect to equivariant proper mappings and the homomorphisms $\mathbb{Z}(k) \to \mathbb{Z}/2$ induce homomorphisms $H_p(X; G, \mathbb{Z}(k)) \to H_p(X; G, \mathbb{Z}/2)$ that fit into a long exact sequence

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$$\cdots \to H_p(X;G,\mathbb{Z}(k)) \xrightarrow{\times 2} H_p(X;G,\mathbb{Z}(k))$$
$$\to H_p(X;G,\mathbb{Z}/2) \to H_{p-1}(X;G,\mathbb{Z}(k)) \to \cdots. \quad (4)$$

As in the case of cohomology, there are natural homomorphisms

$$e_p^{A(k)}\colon\ H_p(X;G,A(k)) \to H_p(X,A(k))^G,$$

which are the edge morphisms of a Hochschild–Serre spectral sequence

$$E^2_{p,q}(X;G,A(k)) = H^{-p}(G,H_q(X,A(k))) \Rightarrow H_{p+q}(X;G,A(k)).$$

If no confusion is likely, we use e instead of $e_p^{A(k)}$; otherwise we will often write $e_p^+ = e_p^{\mathbb{Z}(2k)}$, $e_p^- = e_p^{\mathbb{Z}(2k+1)}$, and $e_p = e_p^{\mathbb{Z}/2}$, and we have similar conventions for the edge morphisms $e^p_{A(k)}$ in cohomology.

There is a cap-product between homology and cohomology

$$H_p(X;G,A(k)) \otimes H^q(X;G,A(l)) \to H_{p-q}(X;G,A(k-l)),$$
$$\gamma \otimes \omega \mapsto \gamma \cap \omega,$$

and of course we have

$$\gamma \cap (\omega \cup \omega') = (\gamma \cap \omega) \cap \omega', \quad (5)$$

$$e(\gamma \cap \omega) = e(\gamma) \cap e(\omega), \quad (6)$$

and for any proper equivariant mapping $f\colon X \to Y$

$$(f_*\gamma) \cap \omega = f_*(\gamma \cap f^*\omega). \quad (7)$$

Recall that η is the nontrivial element in $H^1(G,A(1))$. Cap-product with η considered as an element of $H^1(X;G,A(1))$ defines a map

$$s_p^{A(k)}\colon\ H_p(X;G,A(k)) \to H_{p-1}(X;G,A(k+1)),$$
$$\gamma \mapsto \gamma \cap \eta.$$

It can be shown, that the $e_p^{A(k)}$ and $s_p^{A(k)}$ fit into a long exact sequence

$$\cdots \xrightarrow{s_{p+1}^{A(k-1)}} H_p(X;G,A(k)) \xrightarrow{e_p^{A(k)}} H_p(X,A)$$
$$\longrightarrow H_p(X;G,A(k-1)) \xrightarrow{s_p^{A(k-1)}} H_{p-1}(X;G,A(k)) \longrightarrow \cdots. \quad (8)$$

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For $s_p^{A(k)}$ we adopt the same notational conventions as for $e_p^{A(k)}$.

The natural mapping $H_p(X^G, A) \to H_p(X^G; G, A)$ and the cap-product give us a homomorphism

$$H_*(X^G, \mathbb{Z}/2) \otimes H^*(G, \mathbb{Z}/2) \to H_*(X^G; G, \mathbb{Z}/2),$$

which is an isomorphism. Taking the inverse of this isomorphism and sending the nontrivial element $\eta \in H^1(G, \mathbb{Z}/2)$ to the unit element in $H^*(X^G, \mathbb{Z}/2)$ we obtain a surjective homomorphism

$$H_*(X^G; G, \mathbb{Z}/2) \to H_*(X^G, \mathbb{Z}/2).$$

Furthermore, the mapping $i_*\colon H_n(X^G; G, \mathbb{Z}/2) \to H_n(X; G, \mathbb{Z}/2)$ induced by the inclusion $i: X^G \to X$ is an isomorphism for any $n < 0$, so we can define a homomorphism

$$\rho\colon H_*(X; G, A(k)) \to H_*(X^G, \mathbb{Z}/2)$$

by taking the composite mapping

$$H_*(X; G, A(k)) \xrightarrow{\text{mod } 2} H_*(X; G, \mathbb{Z}/2) \xrightarrow{\cap \eta^N} H_{<0}(X; G, \mathbb{Z}/2)$$
$$\xrightarrow{(i_*)^{-1}} H_*(X^G; G, \mathbb{Z}/2) \longrightarrow H_*(X^G, \mathbb{Z}/2),$$

where N is any integer greater than the (cohomological) dimension of X. We use the notation ρ_n for the restriction of ρ to $H_n(X; G, A(k))$, we write $\rho_{n,p}$ for the composition of ρ_n with the projection $H_*(X^G, \mathbb{Z}/2) \to H_p(X^G, \mathbb{Z}/2)$, and similar definitions hold for $\rho_{n,\text{even}}$ and $\rho_{n,\text{odd}}$.

It is clear from the above that

$$\rho \circ s = \rho, \tag{9}$$

and that the mapping

$$\rho_n: H_n(X; G, \mathbb{Z}/2) \to H_*(X^G, \mathbb{Z}/2)$$

induced by ρ is surjective if $n < 0$. Note that, together with the Hochschild–Serre spectral sequence $E^r_{p,q}(X; G, \mathbb{Z}/2)$, this proves equation (1). Equations (2) and (3) can be derived from the Hochschild–Serre spectral sequence with coefficients in $\mathbb{Z}$ and the following proposition.

PROPOSITION 2.2. *Let X be a locally compact space of finite cohomological dimension with an action of $G = \mathbb{Z}/2$. Then*

$$\rho_{n,\text{even}}: H_n(X; G, \mathbb{Z}(k)) \to H_{\text{even}}(X^G, \mathbb{Z}/2)$$

is an isomorphism if $n < 0$ and $n + k$ is even, and

$$\rho_{n,\text{odd}}: H_n(X; G, \mathbb{Z}(k)) \to H_{\text{odd}}(X^G, \mathbb{Z}/2)$$

is an isomorphism if $n < 0$ and $n + k$ is odd.

Observe that it is not claimed that $\rho_n(H_n(X;G,\mathbb{Z}(k))) \subset H_*(X^G,\mathbb{Z}/2)$ is contained in $H_{\text{even}}(X^G,\mathbb{Z}/2)$ (resp. $H_{\text{odd}}(X^G,\mathbb{Z}/2)$). In fact this is often not the case: for any $\gamma \in H_n(X;G,\mathbb{Z}(k))$ there is a $p \equiv n+k \bmod 2$ such that

$$\rho(\gamma) = \rho_{n,p}(\gamma) + \delta(\rho_{n,p}(\gamma)) + \rho_{n,p-2}(\gamma) + \delta(\rho_{n,p-2}(\gamma)) + \cdots, \tag{10}$$

where δ is the Bockstein homomorphism $H_{p+1}(X^G,\mathbb{Z}/2) \to H_p(X^G,\mathbb{Z}/2)$ associated to the short exact sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$$

(compare [Kr3, Thm. 0.1]).

We will also use the symbol δ for the connecting homomorphism $H_{n+1}(X;G,\mathbb{Z}/2) \to H_n(X;G,\mathbb{Z}(k))$ of the long exact sequence (4), and we have

$$\rho_{n,\text{even}}(\delta(\gamma)) = \rho_{n+1,\text{even}}(\gamma) + \delta(\rho_{n+1,\text{odd}}(\gamma)) \quad \text{if } n+k \text{ is even}, \tag{11}$$

$$\rho_{n,\text{odd}}(\delta(\gamma)) = \rho_{n+1,\text{odd}}(\gamma) + \delta(\rho_{n+1,\text{even}}(\gamma)) \quad \text{if } n+k \text{ is odd}. \tag{12}$$

It is clear from the definition and the projection formula (7) that

$$\rho(\gamma) \cap \beta(\omega) = \rho(\gamma \cap \omega), \tag{13}$$

and for any proper mapping $f: X \to Y$ of G-spaces

$$\rho(f_*\gamma) = f_*\rho(\gamma). \tag{14}$$

There are canonical isomorphisms $H_0(\text{pt};G,A) \simeq A$ and $H_0(\text{pt},A) = A$, so the homomorphisms induced by the constant mapping φ: $X \to \text{pt}$ give us for every compact G-space X the *degree maps*

$$\deg_G\colon H_0(X;G,A) \to A$$

and

$$\deg\colon H_0(X,A) \to A,$$

which satisfy the equality

$$e \circ \deg_G = \deg \circ e. \tag{15}$$

Extending the degree map on $H_0(X^G,\mathbb{Z}/2)$ by 0 to the whole of $H_*(X^G,\mathbb{Z}/2)$, we have by equation (14) that

$$\deg_G(\gamma) \equiv \deg(\rho(\gamma)) \quad \bmod 2, \tag{16}$$

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for any $\gamma \in H_0(X;G,A)$.

Finally, define

$$H_*(X^G,A)^0 = \ker\{\deg\colon H_*(X^G,A) \to A\},$$

and $H_{\text{even}}(X^G,\mathbb{Z}/2)^0 = H_{\text{even}}(X^G,\mathbb{Z}/2) \cap H_*(X^G,\mathbb{Z}/2)^0$. We will record three technical lemmas for use in Section 5. They can be proven by a careful inspection of either the Hochschild–Serre spectral sequence $E_{p,q}(X;G,A(k))$ or the long exact sequence (8) with the appropriate coefficients.

LEMMA 2.3. *Let X be a compact connected G-space with $X^G \neq \emptyset$. Then*

$$\rho_2\colon H_2(X;G,\mathbb{Z}/2) \to H_*(X^G,\mathbb{Z}/2)^0$$

is surjective if and only if the composite mapping

$$H_1(X;G,\mathbb{Z}/2) \xrightarrow{e_1} H_1(X,\mathbb{Z}/2)^G \xrightarrow{\cup\eta^2} H^2(G,H_1(X,\mathbb{Z}/2))$$

is zero.

LEMMA 2.4. *Let X be a compact connected G-space. Then*

$$\rho_{2,\text{even}}\colon H_2(X;G,\mathbb{Z}) \to H_{\text{even}}(X^G,\mathbb{Z}/2)^0$$

is surjective if and only if the composite mapping

$$H_1(X;G,\mathbb{Z}(1)) \xrightarrow{e_1^-} H_1(X,\mathbb{Z}(1))^G \xrightarrow{\cup\eta^2} H^2(G,H_1(X,\mathbb{Z}(1))$$

is zero.

LEMMA 2.5. *Let X be a locally compact connected G-space with $X^G \neq \emptyset$. Then the mapping*

$$\rho_{2,\text{odd}}\colon H_2(X;G,\mathbb{Z}(1)) \to H_{\text{odd}}(X^G,\mathbb{Z}/2)$$

is surjective if and only if the composite mapping

$$H_1(X;G,\mathbb{Z}) \xrightarrow{e_1^+} H_1(X,\mathbb{Z})^G \xrightarrow{\cap\eta^2} H^2(G,H_1(X,\mathbb{Z}))$$

is zero.

3. The fundamental class of a *G*-manifold

Let again A be $\mathbb{Z}/2$ or $\mathbb{Z}$. Let X be an A-oriented topological manifold of finite dimension d with an action of $G = \{1, \sigma\}$. We will define the fundamental class of X in equivariant homology with coefficients in $A(k)$ for k even or odd.

It is well-known, that $H_d(X, A) = A$, and the A-orientation determines a generator μ_X of $H_d(X, A)$. Observe that we do not need to require X to be compact, since we use Borel–Moore homology. If G acts via an A-orientation preserving involution, then $\mu_X \in H_d(X, A)^G$, otherwise $\mu_X \in H_d(X, A(1))^G$. By the Hochschild–Serre spectral sequence (2) we have for $k \in \mathbb{Z}$ an isomorphism $H_d(X; G, A(k)) \simeq H_d(X, A(k))^G$, given by the edge morphisms $e_d^{A(k)}$, so we have the fundamental class

$$\mu_X \in H_d(X; G, A(k)),$$

where k must have the right parity.

PROPOSITION 3.1 (Poincaré duality). *Let X be a G-manifold with fundamental class $\mu_X \in H_d(X; G, A(k))$. Then the mapping*

$$H^i(X; G, A(l)) \to H_{d-i}(X; G, A(k-l)),$$

$$\omega \mapsto \mu_X \cap \omega,$$

is an isomorphism.

Assuming that the action of G is *locally smooth* (i.e., each fixed point has a neighbourhood that is equivariantly homeomorphic to $\mathbb{R}^d$ with an orthogonal G-action), the fixed point set of X^G is again a topological manifold, but it need not be A-orientable and it need not be equi-dimensional. However, if V is a connected component of X^G and V has dimension d_0, then it has a fundamental class $\mu_V \in H_{d_0}(V, \mathbb{Z}/2)$, and we have that the restriction of $\rho_{d,d_0}(\mu_X) \in H_{d_0}(X^G, \mathbb{Z}/2)$ to V equals μ_V (see [vH]). If X is a closed sub-G-manifold of a G-manifold Y, then the embedding $j\colon X \to Y$ is proper, so it induces a mapping in equivariant homology. We define the class in $H_d(Y; G, A(k))$ *represented by* X to be $j_*\mu_X$.

Now let X be an algebraic variety defined over $\mathbb{R}$. We want to define the class in $H_{2d}(X; G, \mathbb{Z}(d))$ represented by a subvariety of dimension d. As in [Fu], we will distinguish two kinds of subvarieties, the *geometrically irreducible subvarieties*, which are varieties over $\mathbb{R}$ themselves, and the *geometrically reducible subvarieties*, which are irreducible over $\mathbb{R}$, but which split into two components when tensored with $\mathbb{C}$. Then the complex conjugation exchanges these two components.

Any complex algebraic variety V of dimension d has a fundamental class $\mu_V \in H_{2d}(V(\mathbb{C}), \mathbb{Z})$, and the complex conjugation on $\mathbb{C}^d$ preserves orientation if d is even, and reverses orientation if d is odd. This implies that if $j\colon Z \hookrightarrow X$ is the

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inclusion of a subvariety of dimension d defined over $\mathbb{R}$, then $\mu_{Z_{\mathbb{C}}}$ is a generator of $H_d(Z(\mathbb{C}), \mathbb{Z}(d))^G$ if $Z_{\mathbb{C}}$ is irreducible, and $H_d(Z(\mathbb{C}), \mathbb{Z}(d))^G$ is generated by $\mu_{Z_1} + \mu_{Z_2}$ if $Z_{\mathbb{C}}$ is the union of two distinct complex varieties Z_1 and Z_2 of dimension d. Hence we define the fundamental class $\mu_Z \in H_{2d}(Z(\mathbb{C}); G, \mathbb{Z}(d))$ of Z to be the inverse image of $\mu_{Z_{\mathbb{C}}}$ (resp. of $\mu_{Z_1} + \mu_{Z_2}$) under $e_{2d}^{\mathbb{Z}(d)}$. The class $[Z] \in H_{2d}(X(\mathbb{C}); G, \mathbb{Z}(d))$ represented by Z is of course defined to be $j_*\mu_Z$. If we use the notation $[Z(\mathbb{R})] \in H_d(X(\mathbb{R}), \mathbb{Z}/2)$ for the homology class represented by $Z(\mathbb{R})$, as defined in [BH], then indeed

$$\rho_{2d,d}([Z]) = [Z(\mathbb{R})]. \tag{17}$$

If Z, Z' are subvarieties of X defined over $\mathbb{R}$ which are rationally equivalent over $\mathbb{R}$ (see [Fu] for a definition), then $[Z] = [Z']$, so we get for every $d \leqslant \dim X$ a well-defined cycle map

$$CH_d(X) \to H_{2d}(X(\mathbb{C}); G, \mathbb{Z}(d))$$

from the Chow group in dimension d to equivariant homology. The image will be denoted by $H_{2d}^{\text{alg}}(X(\mathbb{C}); G, \mathbb{Z}(d))$, and we see by equation (17), that

$$\rho_{2d,d}(H_{2d}^{\text{alg}}(X(\mathbb{C}); G, \mathbb{Z}(d))) = H_d^{\text{alg}}(X(\mathbb{R}), \mathbb{Z}/2). \tag{18}$$

For X nonsingular projective of dimension n, this map coincides with the composition of the mapping

$$CH_d(X) \to H^{2(n-d)}(X(\mathbb{C}); G, \mathbb{Z}(n-d))$$

as defined in [Kr2] and the Poincaré duality isomorphism. As a consequence we can use the following description of the image of the cycle map in codimension 1, where we use the notation $H^2_{\text{alg}}(X(\mathbb{C}); G, \mathbb{Z}(1))$ for the image of $CH_{n-1}(X)$ in cohomology.

PROPOSITION 3.2. *Let X be a nonsingular projective algebraic variety over $\mathbb{R}$. Let $\mathcal{O}_h$ be the sheaf of germs of holomorphic functions on $X(\mathbb{C})$. Then $H^2_{\text{alg}}(X(\mathbb{C}); G, \mathbb{Z}(1))$ is the kernel of the composite mapping*

$$H^2(X(\mathbb{C}); G, \mathbb{Z}(1)) \xrightarrow{e^2_-} H^2(X(\mathbb{C}), \mathbb{Z}) \to H^2(X(\mathbb{C}), \mathcal{O}_h).$$

Proof. This follows immediately from Proposition 1.3.1 in [Kr2], which states that $H^2_{\text{alg}}(X(\mathbb{C}); G, \mathbb{Z}(1))$ is the image of the connecting morphism

$$H^1(X(\mathbb{C}); G, \mathcal{O}_h^*) \to H^2(X(\mathbb{C}); G, \mathbb{Z}(1))$$

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in the long exact sequence induced by the exponential sequence of G-sheaves

$$0 \to \mathbb{Z}(1) \to \mathcal{O}_h \to \mathcal{O}_h^* \to 0. \qquad \square$$

4. Algebraic cycles

The following facts about real Enriques surfaces can be found in [Ni2] or [DKh1]. Let Y be a real Enriques surface. Let $X \to Y_{\mathbb{C}}$ be the double covering of $Y_{\mathbb{C}}$ by a complex K3-surface X. Since $X(\mathbb{C})$ is simply connected, $X(\mathbb{C})$ is the universal covering space of $Y(\mathbb{C})$ and $H_1(Y(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2$. The complex conjugation σ on $Y(\mathbb{C})$ can be lifted to the covering $X(\mathbb{C})$ by [Si, Th.A$_8$.5], and this can obviously be done in two different ways. Hence we can give X the structure of a variety over $\mathbb{R}$ in two different ways, which we will denote by X_1 and X_2. The two halves Y_1 and Y_2 of $Y(\mathbb{R})$ mentioned in the introduction consist of the components covered by $X_1(\mathbb{R})$ and $X_2(\mathbb{R})$, respectively. All connected components of $X_1(\mathbb{R})$ and $X_2(\mathbb{R})$ are orientable, as is the case for the real part of any real K3-surface. If a connected component of a half Y_i is orientable, then it is covered by two components of $X_i(\mathbb{R})$, which are interchanged by the covering transformation of X. A nonorientable component of Y_i is covered by just one component of $X_i(\mathbb{R})$; this is the orientation covering.

Since for an Enriques surface $H^2(Y(\mathbb{C}), \mathcal{O}_h) = 0$ (see [BPV, V.23]), we see by Proposition 3.2 and Poincaré duality that $H_2^{\mathrm{alg}}(Y(\mathbb{C}); G, \mathbb{Z}(1)) = H_2(Y(\mathbb{C}); G, \mathbb{Z}(1))$, so $H_1^{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2)$ is the image of the mapping

$$\alpha_2 = \rho_{2,1}\colon\ H_2(Y(\mathbb{C}); G, \mathbb{Z}(1)) \to H_1(Y(\mathbb{R}), \mathbb{Z}/2).$$

In order to determine the image of α_2 we will define α_n for any $n \in \mathbb{Z}$ by

$$\alpha_n = \rho_{n,1}\colon\ H_n(Y(\mathbb{C}); G, \mathbb{Z}(n-1)) \to H_1(Y(\mathbb{R}), \mathbb{Z}/2).$$

Observe, that $\alpha_n = \alpha_{n-1} \circ s_n^{+/-}$.

LEMMA 4.1. *For a real Enriques surface Y the codimension of* Im α_2 *in* $H_1(Y(\mathbb{R}), \mathbb{Z}/2)$ *does not exceed* 1.

Proof. We may assume that $Y(\mathbb{R}) \neq \emptyset$. Using the fact that α_{-1} is an isomorphism by Proposition 2.2, and both s_0^- and s_1^+ are surjective by the long exact sequence (8), we see that α_1 is surjective. Since $\alpha_2 = \alpha_1 \circ s_2^-$, it suffices to remark that if the cokernel of $s_2^-\colon\ H_2(Y(\mathbb{C}); G, \mathbb{Z}(1)) \to H_1(Y(\mathbb{C}); G, \mathbb{Z})$ is nonzero, it is isomorphic to $H_1(Y(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2$.

PROPOSITION 4.2. *Let Y be a real Enriques surface. A class $\gamma \in H_1(Y(\mathbb{R}), \mathbb{Z}/2)$ is contained in the image of α_2 if and only if*

$$\deg(\gamma \cap w_1(Y(\mathbb{R}))) = 0,$$

where $w_1(Y(\mathbb{R})) \in H^1(Y(\mathbb{R}), \mathbb{Z}/2)$ is the first Stiefel–Whitney class of $Y(\mathbb{R})$.

Proof. Again we may assume that $Y(\mathbb{R}) \neq \emptyset$. Denote by Ω the subspace of $H_1(Y(\mathbb{R}), \mathbb{Z}/2)$ whose elements γ verify $\deg(\gamma \cap w_1(Y(\mathbb{R}))) = 0$.

If $Y(\mathbb{R})$ is orientable, $w_1(Y(\mathbb{R})) = 0$ and $\Omega = H_1(Y(\mathbb{R}), \mathbb{Z}/2)$. Furthermore, we have a surjective morphism

$$H_1(X_1(\mathbb{R}), \mathbb{Z}/2) \oplus H_1(X_2(\mathbb{R}), \mathbb{Z}/2) \to H_1(Y(\mathbb{R}), \mathbb{Z}/2),$$

where the X_1 and X_2 are the two real K3-surfaces covering Y (see the beginning of this section). This morphism fits in a commutative diagram

$$\begin{array}{ccc} H_2(X_1(\mathbb{C}); G, \mathbb{Z}(1)) \oplus H_2(X_2(\mathbb{C}); G, \mathbb{Z}(1)) & \longrightarrow & H_2(Y(\mathbb{C}); G, \mathbb{Z}(1)) \\ \Big\downarrow {\scriptstyle \alpha_2^{X_1} \oplus \alpha_2^{X_2}} & & \Big\downarrow {\scriptstyle \alpha_2} \\ H_1(X_1(\mathbb{R}), \mathbb{Z}/2) \oplus H_1(X_2(\mathbb{R}), \mathbb{Z}/2) & \longrightarrow & H_1(Y(\mathbb{R}), \mathbb{Z}/2). \end{array}$$

Here the $\alpha_n^{X_i}: H_n(X_1(\mathbb{C}); G, \mathbb{Z}(n-1)) \to H_1(X_1(\mathbb{R}), \mathbb{Z}/2)$ are defined in the same way as α_n. As $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$ for a real K3-surface X, it follows from Lemma 2.5, that $\alpha_2^{X_1}$ and $\alpha_2^{X_2}$ are surjective, which implies the surjectivity of α_2. In other words, $\operatorname{Im} \alpha_2 = \Omega$.

Now assume that $Y(\mathbb{R})$ is nonorientable. Then $w_1(Y(\mathbb{R})) \neq 0$, and by non-degeneracy of the cap-product pairing $\operatorname{codim} \Omega = 1$. First we will prove that $\operatorname{Im} \alpha_2 \subset \Omega$.

Let $K = -cw_1(Y(\mathbb{C})) \in H^2(Y(\mathbb{C}); G, \mathbb{Z}(1))$, where $cw_1(Y(\mathbb{C}))$ is the first mixed characteristic class of the tangent bundle of $Y(\mathbb{C})$ as defined in [Kr2, 3.2]. Then $e(K) \in H^2(Y(\mathbb{C}), \mathbb{Z})$ is the first Chern class of the canonical line bundle of Y, so $2e(K) = 0$ (see [BPV, V.32]). This means that for any $\gamma \in H_2(Y(\mathbb{C}); G, \mathbb{Z}(1))$ we have $\deg_G(\gamma \cap K) = \deg(e(\gamma) \cap e(K)) = 0$, so $\deg(\rho(\gamma) \cap \beta(K)) = 0$ by Equations (16) and (13).

The projection $\rho_{2,2}(\gamma)$ of $\rho(\gamma) \in H_*(Y(\mathbb{R}), \mathbb{Z}/2)$ to $H_2(Y(\mathbb{R}), \mathbb{Z}/2)$ is zero by equation (10) and the projection $\beta^{2,0}(K)$ of $\beta(K) \in H^*(Y(\mathbb{R}), \mathbb{Z}/2)$ to $H^0(Y(\mathbb{R}), \mathbb{Z}/2)$ is zero by [Kr3, Th. 0.1]. This implies

$$\deg(\rho(\gamma) \cap \beta(K)) = \deg(\rho_{2,1}(\gamma) \cap \beta^{2,1}(K)),$$

but $\beta^{2,1}(K) = w_1(Y(\mathbb{R}))$ by [Kr2, Th. 3.2.1], and $\rho_{2,1}(\gamma) = \alpha_2(\gamma)$ by definition, so $\deg(\alpha_2(\gamma) \cap w_1(Y(\mathbb{R}))) = 0$. In other words, $\operatorname{Im} \alpha_2 \subset \Omega$. Lemma 4.1 now gives us that $\operatorname{Im} \alpha_2 = \Omega$. □

COROLLARY 4.3. *With the above notation, α_2 is surjective if and only if $Y(\mathbb{R})$ is orientable.*

Theorem 1.1 in the introduction is an immediate consequence of Proposition 4.2. We can even give an explicit description of $H_1^{\mathrm{alg}}(Y(\mathbb{R}), \mathbb{Z}/2)$.

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THEOREM 4.4. *Let Y be a real Enriques surface. A class $\gamma \in H_1(Y(\mathbb{R}), \mathbb{Z}/2)$ can be represented by an algebraic cycle if and only if* $\deg(\gamma \cap w_1(Y(\mathbb{R}))) = 0$.

5. Galois-maximality

The aim of this section is to describe which Enriques surfaces are $\mathbb{Z}$-*GM*-varieties and/or *GM*-varieties in terms of the orientability of the real part and the distribution of the components over the halves. See the introduction for the definition of Galois-maximality and Section 4 for the definition of 'halves'.

The proof of Theorem 1.2 will consist of a collection of technical results and explicit constructions of equivariant homology classes. For completeness we also prove the parts of Theorem 1.2 concerning coefficients in $\mathbb{Z}/2$, although these results are not new (see the Introduction).

LEMMA 5.1. *Let Y be an algebraic variety over $\mathbb{R}$. Then*

(i) *Y is $\mathbb{Z}$-GM if and only if e_p^+ is surjective on $H_p(Y(\mathbb{C}), \mathbb{Z})^G$ and e_p^- is surjective onto $H_p(Y(\mathbb{C}), \mathbb{Z}(1))^G$ for all p.*
(ii) *Y is GM if and only e_p is surjective onto $H_p(Y(\mathbb{C}), \mathbb{Z}/2)^G$ for all p.*

Proof. This follows from the fact that Y is *GM* (resp. $\mathbb{Z}$-*GM*) if and only if the Hochschild–Serre spectral sequence $E^r_{p,q}(Y(\mathbb{C}); G, A)$ is trivial for $A = \mathbb{Z}/2$ (resp. $\mathbb{Z}$), and this can be checked by looking at the edge morphisms, since we have for every $k \geqslant 0$ and every G-module M natural surjections $H^k(G, M) \to H^{k+2}(G, M)$, and $H^k(G, M) \to H^{k+1}(G, M(1))$, which are isomorphisms for $k > 0$.

LEMMA 5.2. *Let Y be a real Enriques surface with $Y(\mathbb{R}) \neq \emptyset$. Then*

(i) *for any $p \in \{0, 2, 3, 4\}$, $e_p^{+/-}$ is surjective onto $H_p(Y(\mathbb{C}), \mathbb{Z}(k))^G$,*
(ii) *for any $p \in \{0, 3, 4\}$, e_p is surjective onto $H_p(Y(\mathbb{C}), \mathbb{Z}/2)^G$.*

Proof. This can be seen from the Hochschild–Serre spectral sequences (cf. [Krl, Sect. 5]).

COROLLARY 5.3. *Let Y be a real Enriques surface with $Y(\mathbb{R}) \neq \emptyset$. Then Y is $\mathbb{Z}$-GM if and only if $e_1^{+/-}$ is surjective onto $H_1(Y(\mathbb{C}), \mathbb{Z}(k))^G$ for $k = 0, 1$. Moreover, Y is GM if and only if e_1 and e_2 are surjective onto $H_1(Y(\mathbb{C}), \mathbb{Z}/2)^G$, resp. $H_2(Y(\mathbb{C}), \mathbb{Z}/2)^G$.*

LEMMA 5.4. *Let Y be a real Enriques surface with $Y(\mathbb{R}) \neq \emptyset$. If e_2 is not surjective onto $H_2(Y(\mathbb{C}), \mathbb{Z}/2)^G$, then e_1 is not surjective onto $H_1(Y(\mathbb{C}), \mathbb{Z}/2)^G$.*

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Proof. By Poincaré duality we see that if e_2 is not surjective onto $H_2(Y(\mathbb{C}), \mathbb{Z}/2)^G$, then e^2 is not surjective onto $H^2(Y(\mathbb{C}), \mathbb{Z}/2)^G$. Let us assume that e^2 is not surjective. Then by Lemma 2.1 there exists an $\omega \in H^1(Y(\mathbb{C}); G, \mathbb{Z}/2)$ such that $e^1(\omega) \neq 0$, but $\beta(\omega) = 0$.

Now suppose e_1 is surjective onto $H_1(Y(\mathbb{C}), \mathbb{Z}/2)^G$, then there exists a $\gamma \in H_1(Y(\mathbb{C}); G, \mathbb{Z}/2)$ such that

$$\deg(e_1(\gamma) \cap e^1(\omega)) \neq 0.$$

This means that $\deg_G(\gamma \cap \omega) \neq 0$, but this contradicts

$$\deg_G(\gamma \cap \omega) = \deg(\rho(\gamma) \cap \beta(\omega)) = \deg(\rho(\gamma) \cap 0) = 0.$$

Hence e_1 is not surjective.

PROPOSITION 5.5. *Let Y be a real Enriques surface with $Y(\mathbb{R}) \neq \emptyset$. Then*

(i) *Y is $\mathbb{Z}$-GM if and only if e_1^+ and e_1^- are nonzero.*
(ii) *Y is GM if and only if e_1 is nonzero.*
(iii) *If e_1 is zero then e_1^+ and e_1^- are zero. In particular, if Y is $\mathbb{Z}$-GM, then Y is also GM.*

Proof. If Y is an Enriques surface,

$$H_1(Y(\mathbb{C}), \mathbb{Z}) = H_1(Y(\mathbb{C}), \mathbb{Z}/2) = \mathbb{Z}/2,$$

so $e_1^{+/-}$ and e_1 are surjective if and only if they are nonzero. By Lemma 5.4, e_2 is surjective if $e_1 \neq 0$, so we obtain the first two assertions from Corollary 5.3. The last assertion follows from the commutative diagram

$$\begin{array}{ccc} H_1(Y(\mathbb{C}); G, \mathbb{Z}(k)) & \xrightarrow{e_1^{+/-}} & H_1(Y(\mathbb{C}), \mathbb{Z}(k)) \\ \downarrow & & \downarrow \\ H_1(Y(\mathbb{C}); G, \mathbb{Z}/2) & \xrightarrow{e_1} & H_1(Y(\mathbb{C}), \mathbb{Z}/2). \end{array}$$

□

LEMMA 5.6. *Let Y be a real Enriques surface with $Y(\mathbb{R}) \neq \emptyset$. Then $e_1^+ = 0$ if and only if $Y(\mathbb{R})$ is orientable.*

Proof. We know from Corollary 4.3, that α_2 is surjective if and only if $Y(\mathbb{R})$ is orientable. Since $H_1(Y(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2$, the mapping $H_1(Y(\mathbb{C}), \mathbb{Z})^G \xrightarrow{\cup \eta^2} H^2(G, H_1(Y(\mathbb{C}), \mathbb{Z}))$ is an isomorphism, so Lemma 2.5 gives us that α_2 is surjective if and only if $e_1^+ = 0$.

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LEMMA 5.7. *If the two halves Y_1 and Y_2 of a real Enriques surface Y are nonempty, then $e_1^- \neq 0$.*

Proof. Let X be the K3-covering of $Y_{\mathbb{C}}$, let τ be the deck transformation of this covering and denote by σ_1 and σ_2 the two different involutions of $X(\mathbb{C})$ lifting the involution σ of $Y(\mathbb{C})$. Let $X_i(\mathbb{R})$ be the set of fixed points under σ_i and let p_i be a point in $X_i(\mathbb{R})$ for $i = 1, 2$.

Let l be an arc in $X(\mathbb{C})$ connecting p_1 and p_2 without containing any other point of $X_1(\mathbb{R})$ or $X_2(\mathbb{R})$. Then the union L of the four arcs $l, \sigma_1(l), \sigma_2(l), \tau(l)$ is homeomorphic to a circle, and we have that $\tau(L) = L$. This implies that the image λ of L in $Y(\mathbb{C})$ is again homeomorphic to a circle; we choose an orientation on λ.

Now G acts on λ via an orientation reversing involution, so λ represents a class $[\lambda]$ in $H_1(Y(\mathbb{C}); G, \mathbb{Z}(1))$. Since $X(\mathbb{C}) \to Y(\mathbb{C})$ is the universal covering, and the inverse image of λ is precisely L, hence homeomorphic to a circle, the class of λ is nonzero in $H_1(Y(\mathbb{C}), \mathbb{Z})$, so $e_1^-([\lambda]) \neq 0$.

LEMMA 5.8. *If exactly one of the halves Y_1, Y_2 of a real Enriques surface Y is empty, then $e_1 = 0$ if and only if $Y(\mathbb{R})$ is orientable.*

Proof. If $e_1 = 0$, we have $e_1^+ = 0$ by Proposition 5.5 and then $Y(\mathbb{R})$ is orientable by 5.6. Conversely, if $Y(\mathbb{R})$ is orientable and $X_2(\mathbb{R}) = \emptyset$, then $X_1(\mathbb{R}) \to Y(\mathbb{R})$ is the trivial double covering, so it induces a surjection $H_*(X_1(\mathbb{R}), \mathbb{Z}/2)^0 \to H_*(Y(\mathbb{R}), \mathbb{Z}/2)^0$, where $H_*(-, \mathbb{Z}/2)^0$ denotes the kernel of the degree map as defined in Section 2. Since $H_1(X(\mathbb{C}), \mathbb{Z}/2) = 0$, the mapping ρ: $H_2(X_1(\mathbb{C}); G, \mathbb{Z}/2) \to H_*(X_1(\mathbb{R}), \mathbb{Z}/2)^0$ is surjective by Lemma 2.3. Now the functoriality of ρ with respect to proper equivariant mappings (Equation (14)) implies

$$\rho_2\colon\ H_2(Y(\mathbb{C}); G, \mathbb{Z}/2) \to H_*(Y(\mathbb{R}), \mathbb{Z}/2)$$

is surjective, and Lemma 2.3 then gives that e_1 is zero.

LEMMA 5.9. *If exactly one of the halves Y_1, Y_2 of a real Enriques surface Y is empty, then $e_1^- \neq 0$ if and only if $Y(\mathbb{R})$ has components of odd Euler characteristic.*

Proof. Assume $Y_2 = \emptyset$. By Lemma 2.4, it suffices to show that

$$\rho_{2,\text{even}}\colon\ H_2(Y(\mathbb{C}); G, \mathbb{Z}) \to H_{\text{even}}(Y(\mathbb{R}), \mathbb{Z}/2)^0$$

is surjective if and only if $Y(\mathbb{R})$ has no components of odd Euler characteristic. Although $Y(\mathbb{R})$ need not be orientable, we can apply the K3-covering as in the previous lemma and prove that the image of $\rho_{2,\text{even}}$ contains a basis for the subgroup $H_0(Y(\mathbb{R}), \mathbb{Z}/2) \cap H_{\text{even}}(Y(\mathbb{R}), \mathbb{Z}/2)^0$, so $\rho_{2,\text{even}}$ is surjective if and only if

$$\rho_{2,2}\colon\ H_2(Y(\mathbb{C}); G, \mathbb{Z}) \to H_2(Y(\mathbb{R}), \mathbb{Z}/2)$$

is surjective. We will use that $H_2(Y(\mathbb{R}), \mathbb{Z}/2)$ is generated by the fundamental classes of the connected components of $Y(\mathbb{R})$.

Pick a component V of $Y(\mathbb{R})$. If V is orientable, it gives a class in $H_2(Y(\mathbb{C}); G, \mathbb{Z})$, which maps to the fundamental class of V in $H_2(Y(\mathbb{R}), \mathbb{Z}/2)$. Now assume V is nonorientable. Let $[V]$ be the fundamental class of V in $H_2(Y(\mathbb{R}), \mathbb{Z}/2)$, let $[V]_G$ be the class represented by V in $H_2(Y(\mathbb{C}); G, \mathbb{Z}/2)$, and let $\gamma = \delta([V]_G)$ be the Bockstein image in $H_1(Y(\mathbb{C}); G, \mathbb{Z}(1))$. Then $\rho_{1,2}(\gamma) = \rho_{2,2}([V]_G) = [V]$ by equation (11), so $[V]$ is in the image of $H_2(Y(\mathbb{C}); G, \mathbb{Z})$ under $\rho_{2,2}$ if and only if $e_1^-(\gamma) = 0$.

From the construction of γ we see that $e_1^-(\gamma) = i_*\delta([V])$, where $i\colon V \to Y(\mathbb{C})$ is the inclusion and $\delta([V]) \in H_1(V, \mathbb{Z})$ is the Bockstein image of $[V]$. Therefore $e_1^-(\gamma)$ can be represented by a circle λ embedded in V. Since $X(\mathbb{C}) \to Y(\mathbb{C})$ is the universal covering, $e_1^-(\gamma)$ is zero if and only if the inverse image L of λ in $X(\mathbb{C})$ has two connected components. Let W be the component of $X_1(\mathbb{R})$ covering V. Then W is the orientation covering of V and $L \subset W$. If V has odd Euler characteristic, then it is the connected sum of a real projective plane and an orientable compact surface. We see by elementary geometry that L is connected. If V has even Euler characteristic, it is the connected sum of a Klein bottle and an orientable compact surface, and we see that L has two connected components.

Proof of Theorem 1.2. By Proposition 5.5, the first part of the theorem follows from Lemma 5.6 and Lemma 5.7, and the second part of the theorem follows from Lemma 5.8 and Lemma 5.9.

6. The Brauer group

Let Y be a nonsingular projective algebraic variety defined over $\mathbb{R}$. Let

$$\mathrm{Br}'(Y) = H^2_{ét}(Y, \mathbb{G}_m)$$

be the cohomological Brauer group of Y, and let $\mathrm{Tor}(n, \mathrm{Br}'(Y))$ be the n-torsion of $\mathrm{Br}'(Y)$. We have a canonical isomorphism

$$\mathrm{Tor}(n, \mathrm{Br}'(Y)) \simeq \mathrm{Coker}\{H^2_{\mathrm{alg}}(Y(\mathbb{C}); G, \mathbb{Z}(1)) \xrightarrow{\text{mod } n} H^2(Y(\mathbb{C}); G, \mathbb{Z}/n(1))\}, \tag{19}$$

as can be deduced from the Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \longrightarrow 1,$$

and the well-known identifications

$$H^k_{ét}(Y, \mu_n) \simeq H^k(Y(\mathbb{C}); G, \mathbb{Z}/n(1))$$

and

$$H^1(Y, \mathbb{G}_m) \simeq \mathrm{Pic}(Y).$$

It can be checked, that the mapping

$$\beta^{2,0}\colon H^2(Y(\mathbb{C});G,\mathbb{Z}/2)\to H^0(Y(\mathbb{R}),\mathbb{Z}/2)$$

induces a well-defined homomorphism

$$\text{Tor}(2,\text{Br}'(Y))\to H^0(Y(\mathbb{R}),\mathbb{Z}/2). \tag{20}$$

If $\dim Y\leqslant 2$, in particular if Y is a real Enriques surface, we may identify $\text{Br}'(Y)$ with the classical Brauer group $\text{Br}(Y)$ (see [Gr2, II, Th. 2.1]). Two of the main problems considered in [NS] and [Ni1] are the calculation of $\dim_{\mathbb{Z}/2}$ Tor(2, Br(Y)) and the question whether the mapping (20) is surjective for every real Enriques surface Y. Both problems were solved for certain classes of real Enriques surfaces. The second problem has been completely solved in [Kr3], where it is shown that the mapping (20) is surjective for any nonsingular projective surface Y defined over $\mathbb{R}$ (see Remark 3.3 in *loc. cit.*). The results in Section 5 will help us to solve the first problem for every Enriques surface Y by determining the whole group $\text{Br}(Y)$.

LEMMA 6.1. *Let Y be a nonsingular projective algebraic variety defined over $\mathbb{R}$ such that*

$$H^2_{\text{alg}}((Y(\mathbb{C});G,\mathbb{Z}(1))=H^2(Y(\mathbb{C});G,\mathbb{Z}(1)).$$

Then

$$\text{Tor}(\text{Br}'(Y))\simeq\text{Tor}(H^3(Y(\mathbb{C});G,\mathbb{Z}(1))).$$

Proof. By the hypothesis and the isomorphism (19) there is for every integer $n>0$ a short exact sequence

$$H^2(Y(\mathbb{C});G,\mathbb{Z}(1))\otimes\mathbb{Z}/n\to H^2(Y(\mathbb{C});G,\mathbb{Z}/n(1))\to\text{Tor}(n,\text{Br}'(Y)),$$

hence we deduce from the long exact sequence in equivariant cohomology associated to the short exact sequence

$$0\to\mathbb{Z}(1)\stackrel{\times n}{\to}\mathbb{Z}(1)\to\mathbb{Z}/n(1)\to 0,$$

that we have for every $n>0$ a natural isomorphism

$$\text{Tor}(n,\text{Br}'(Y))\simeq\text{Tor}(n,H^3(Y(\mathbb{C});G,\mathbb{Z}(1))).$$

Proof of Theorem 1.3. By [Gr2, 1.2 and II, Thm. 2.1] we have $\text{Br}(Y)=\text{Tor}(\text{Br}(Y))=\text{Tor}(\text{Br}'(Y))$. On the other hand, $\text{Tor}(H^3(Y(\mathbb{C});G,\mathbb{Z}(1)))=$

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$H^3(Y(\mathbb{C}); G, \mathbb{Z}(1))$ since $H^3(Y(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}/2$. Hence, by Lemma 6.1 and Poincaré duality

$$\mathrm{Br}(Y) \simeq H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)).$$

Now consider the long exact sequence (8) for $A(k) = \mathbb{Z}$

$$\cdots \xrightarrow{e_1^+} H_1(Y(\mathbb{C}), \mathbb{Z}) \to H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) \xrightarrow{s_1^-} H_0(Y(\mathbb{C}); G, \mathbb{Z}) \to \cdots$$

It follows from Proposition 2.2 and the long exact sequence (8) for $A(k) = \mathbb{Z}(1)$ that $\rho\colon H_*(Y(\mathbb{C}); G, \mathbb{Z}) \to H_*(Y(\mathbb{R}), \mathbb{Z}/2)$ induces an isomorphism

$$\operatorname{im} s_1^- \xrightarrow{\sim} H_{\mathrm{even}}(Y(\mathbb{R}), \mathbb{Z}/2)^0.$$

We obtain an exact sequence

$$\cdots \xrightarrow{e_1^+} \mathbb{Z}/2 \to H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) \to H_{\mathrm{even}}(Y(\mathbb{R}), \mathbb{Z}/2)^0 \to 0. \tag{21}$$

If $Y(\mathbb{R}) \neq \emptyset$ is nonorientable, then $e_1^+ \neq 0$ by Lemma 5.6, so $H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) \simeq (\mathbb{Z}/2)^{2s-1}$, which proves the first part of the theorem.

Now assume $Y(\mathbb{R}) \neq \emptyset$ is orientable. Then $e_1^+ = 0$ by Lemma 5.6, so we get from (21) an exact sequence

$$0 \to \mathbb{Z}/2 \to H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) \to (\mathbb{Z}/2)^{2s-1} \to 0.$$

Hence $H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) \simeq (\mathbb{Z}/2)^{2s}$ or $(\mathbb{Z}/2)^{2s-2} \oplus (\mathbb{Z}/4)$.

In order to decide between these two possibilities, consider the following commutative diagram with exact rows

$$\begin{array}{ccccc}
H_2(Y(\mathbb{C}); G, \mathbb{Z}/2) & \xrightarrow{\delta^-} & H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) & \xrightarrow{\times 2} & H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) \\
\downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_1^-} & & \downarrow{\scriptstyle e_1^-} \\
H_2(Y(\mathbb{C}), \mathbb{Z}/2) & \xrightarrow{\delta} & H_1(Y(\mathbb{C}), \mathbb{Z}) & \xrightarrow{\times 2} & H_1(Y(\mathbb{C}), \mathbb{Z}) \\
\uparrow{\scriptstyle e_2} & & \uparrow{\scriptstyle e_1^+} & & \uparrow{\scriptstyle e_1^+} \\
H_2(Y(\mathbb{C}); G, \mathbb{Z}/2) & \xrightarrow{\delta^+} & H_1(Y(\mathbb{C}); G, \mathbb{Z}) & \xrightarrow{\times 2} & H_1(Y(\mathbb{C}); G, \mathbb{Z}).
\end{array}$$

We have that $H_1(Y(\mathbb{C}); G, \mathbb{Z}(1))$ is pure 2-torsion if and only if δ^- is surjective. We claim that δ^- is surjective if and only if $e_1^- = 0$. Together with Lemmas 5.9 and 5.7 this would prove the second part of the theorem.

Let us prove the claim. Since $e_1^+ = 0$, we have $\delta \circ e_2 = 0$. If $e_1^- \neq 0$, an easy diagram chase shows that δ^- is not surjective. On the other hand the following diagram can be shown to be commutative.

$$\begin{array}{ccc} H_2(Y(\mathbb{C}); G, \mathbb{Z}) & \xrightarrow{\quad s_2^+ \quad} & \\ \downarrow{\scriptstyle \mathrm{mod} 2} & \searrow & \\ H_2(Y(\mathbb{C}); G, \mathbb{Z}/2) & \xrightarrow{\quad \delta^- \quad} & H_1(Y(\mathbb{C}); G, Z(1)) \end{array}$$

In other words, $\mathrm{Im}\, s_2^+ \subset \mathrm{Im}\, \delta^-$. Now $\ker e_1^- = \mathrm{Im}\, s_2^+$, so if $e_1^- = 0$, then δ^- is surjective.

Finally, we will consider the short exact sequence (21) for the case $Y(\mathbb{R}) = \emptyset$. Then G acts freely on $Y(\mathbb{C})$, so we have for all k that $H_k(Y(\mathbb{C}); G, \mathbb{Z}/2) = H_k(Y(\mathbb{C})/G, \mathbb{Z}/2)$. By the remarks made in the introduction of Section 4, this means that $H_1(Y(\mathbb{C}); G, \mathbb{Z}/2) = \mathbb{Z}/2 \times \mathbb{Z}/2$, and we can see from the long exact sequence (8) for $A(k) = \mathbb{Z}/2$ that $e_1 = 0$. This implies that $e_1^+ = 0$ (see Proposition 5.5.iii), hence $H_1(Y(\mathbb{C}); G, \mathbb{Z}(1)) = \mathbb{Z}/2$.

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