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EXPONENTIAL ATTRACTORS FOR ABSTRACT PARABOLIC SYSTEMS WITH BOUNDED DELAY

DALIBOR PRAŽÁK

We show that a suitable adaptation of the so-called method of trajectories can be used to construct an exponential attractor for a very general class of nonlinear reaction-diffusion systems with a bounded delay.

In particular, we assume that the dependence on the past history is controlled via convolution with a possibly singular measure. Assuming a priori that the solutions are bounded, a simple proof of the existence of an exponential attractor is given under very little regularity requirements.

1. INTRODUCTION

We study a nonlinear abstract parabolic system of the form

$$\partial_t u - \operatorname{div} a(\nabla u) = F(u^t) \qquad \text{in } \Omega \times (0, \infty), \tag{1}$$

where $u(x,t) : \Omega \times [-r, \infty) \to \mathbb{R}^N$ is the unknown, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, and $a(\eta) : \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ captures the diffusion effects.

In our notation

$$[u^t](x,s) = u(x, t+s), \qquad s \in [-r, 0]. \tag{2}$$

Hence the right-hand side includes a general functional dependence on the past history up to time $t - r$, where $r > 0$ (the maximal delay) is fixed.

We consider Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \partial\Omega \times (0, \infty), \tag{3}$$

and the initial condition / initial history, written succinctly (in view of (2)) as

$$u^0 = \phi, \qquad \text{in } \Omega \times [-r, 0]. \tag{4}$$

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We assume

1. $a(\cdot)$ is strongly monotone with quadratic growth, that is,

$$
\begin{gathered}
a(0)=0\,, \\
\big(a(\eta)-a(\zeta)\big)\cdot(\eta-\zeta)\geqslant c_1|\eta-\zeta|^2\,, \\
\big|a(\eta)-a(\zeta)\big|\leqslant c_2|\eta-\zeta|\,.
\end{gathered}
\tag{5}
$$

2. Denoting $X=C\big([-r,0];L^2(\Omega)\big)$, we require that $F:X\to L^2(\Omega)$ be continuous, with

$$
\big\|F(\phi)-F(\psi)\big\|_2^2\leqslant L^2\int_{-r}^{0}\big\|\phi(s)-\psi(s)\big\|_2^2\,d\mu(s)\,,
\tag{6}
$$

where μ is a suitable Radon measure on $[-r,0]$. We shall see below that such an abstract assumption covers a wide class of meaningful examples; on the other hand, it seems optimal in view of our approach. Note that (6) is only slightly stronger than a mere Lipschitz continuity of $F:X\to L^2(\Omega)$.

3. We finally assume that the equation has a bounded invariant region; more specifically, there exists a closed, bounded set $\mathcal{B}\subset L^2(\Omega)$ such that

$$
\phi(\cdot,s)\in\mathcal{B}\quad\forall s\in[-r,0]\implies u(\cdot,t)\in\mathcal{B}\quad\forall t\geqslant 0\,,
\tag{7}
$$

where u is a solution in the sense defined below.

We shall only consider the dynamics of solutions living in $\mathcal{B}$; hence it is enough to assume (6) for ϕ, ψ with values in $\mathcal{B}$. In particular, L depends on $\mathcal{B}$ as the case may be.

The paper is organised as follows: we finish the present section by several bibliographical remarks; we also discuss the meaningfulness of our abstract model and of our assumptions.

Section 2 brings the mathematical preliminaries. Here we sketch the proof of existence of weak solutions, and prove its uniqueness. The solution semigroup is introduced, and the main result (Theorem 2.3) is formulated.

Section 3 consists of the technical part of the paper. Here we introduce the "dynamics of trajectories", and we prove that it has an exponential attractor. The Theorem 2.3 is then proved in the last, short Section 4.

Commentary and bibliographical remarks.

Recently, many authors have considered problems with diffusion and delay, see for example, [**16, 20, 23**]. One can say that almost any model with bounded delay fits into our abstract scheme.

Usually a linear diffusion is considered, that is, $a(\eta)=A\eta$, where A is some strictly monotone matrix. Here we allow for a nonlinear diffusion, since it is more general and brings no additional complication to our approach.

The assumption (6) incorporates a very general class of dependencies on the past, as for example

$$\{F(u^t)\}(x,t) = \widehat{F}\Big(u(t,x), u(t-\alpha_i,x), \dots, \int_{-\tau}^{0} b\big(u(x,t+s),s\big)\,ds, \dots\Big).$$

In such a case (6) holds with $\mu = \lambda + \delta_0 + \delta_{\alpha_i} + \dots$, where λ is the Lebesgue measure, and δ_α a Dirac measure located in $x = \alpha$, assuming that $\widehat{F}(\cdot)$, $b(\cdot)$ are Lipschitz continuous on the appropriate spaces.

Since (6) only requires the control of L^2 norm with respect to the x variable, one can even include a non-local dependence (via convolution, see [**20**]) on x.

We remark finally that our last requirement – the asymptotic boundedness – is a nontrivial problem in the presence of delay. For some models, it can be easily proved using the maximum principle. For example, if

$$F(u^t)(x,t) = u(x,t)\big[a - bu(x,t) - f(u^t)\big],$$

where a, $b > 0$ and $f(\cdot) \geqslant 0$, one proves easily that the solution never leaves the interval $[0, a/b]$. A more involved generalisation of this simple idea, leads to the concept of "mixed quasimonotonicity", as developed in [**14, 15**]. These results in particular give a class of sufficient conditions for the existence of a bounded invariant set $\mathcal{B}$ in the form

$$\mathcal{B} = \big\{u \in L^2(\Omega);\ u_i(x) \in [a_i, b_i] \text{ for almost everywhere } x \in \Omega\big\}.$$

The asymptotic boundedness is also quite easy to verify, when the diffusion dominates delay effects in an appropriate sense, see [**23, 8, 17**]. It also holds if the delay is small, or in one spatial dimension, see [**11**].

On the other hand, very interesting results of [**11, 8**] show that in the case of a seemingly simple model

$$\partial_t u(x,t) - \nu\Delta u(x,t) = u(x,t)\big[1 - u(x,t-\tau)\big],$$

with $x \in \Omega \subset \mathbb{R}^2$, the solutions might grow exponentially provided that ν/τ is small enough. This is rather surprising since, if $\nu = 0$ (that is, the ODE case), asymptotic boundedness holds. So the unexpected dynamics arises from a nontrivial interplay between the delay and diffusion.

This is the main reason why we stipulate the asymptotic boundedness simply as an axiom, instead of trying to verify it directly in terms of the properties of the equation in our general form.

Our construction of the attractor rests on the so-called "method of trajectories", originating from papers [**12, 13**]. The heart of our paper (Section 3) is an adaptation of the method, which was originally devised to study nonlinear parabolic problems with no delay. Yet, according to a number of recent publications, the phase space of trajectories is a natural setting for problems with time delays (see for example [**2, 19, 10**].)

The main feature of our approach is the simple proof of the smoothing property (and hence the existence of an exponential attractor); though the result is ultimately obtained in $C([-r,0];L^2(\Omega))$ topology, no other than mild, natural parabolic (essentially $L^2(0,T;W^{1,2}(\Omega))$ for u and $L^2(0,T;W^{-1,2}(\Omega))$ for $\partial_t u$) regularity of solutions is needed. A similar construction was used already in [**17**]; however, the significant improvement of the present paper is that we do not have to prove additional regularity of the solutions.

One can compare this with the approach based on introducing a new variable, the so-called summed past history (see for example [**9**]). This method leads to a robust description of the large time dynamics. In particular, various relaxation/perturbation phenomena can be captured. On the other hand, an important assumption of this method is that the dependence on the past is given by a convolution with sufficiently regular function. One of the features of the present paper is that our method accommodates easily the convolution with a possibly singular measure if needed. See also [**1**] for the evaluation and comparison of the "past history approach" and the "trajectory approach".

From a more general perspective, the phase space of trajectories is a convenient tool in situations where one has not enough regularity to prove the asymptotic compactness (or even uniqueness) of solutions; see for example [**22, 4, 3, 7**] for further applications of this approach and its various generalisations.

2. Preliminaries

We denote by $(L^2(\Omega), \|\cdot\|_2)$, $(W^{1,2}(\Omega), \|\cdot\|_{1,2})$ the usual Lebesgue and Sobolev spaces; the scalar product in $L^2(\Omega)$ and the duality between $W^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$, respectively. For the sake of simplicity, we do not distinguish the vector-valued and scalar-valued functions notationally.

As said before, we set $X = C([-r,0];L^2(\Omega))$; this will be our phase-space. By weak solution we understand

$$\begin{aligned} u &\in L^\infty(-r,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)) \\ \partial_t u &\in L^2(0,T;W^{-1,2}(\Omega)) \end{aligned}$$

such that $u \restriction [-r,0] = \phi$ where $\phi \in X$, and (1) holds in the sense of distributions. It follows that $u \restriction [0,T]$ has a representative in $C([0,T];L^2(\Omega))$, and $u(0) = \phi(0)$.

Theorem 2.1. *Let $\phi \in X$ and $T > 0$ be given. Then there exists u a weak solution to* (1). *Moreover, one has*

$$\sup_{t\in[1,T]} \left\{ \|u(t)\|_2^2 + \int_{t-1}^{t} \|u\|_{1,2}^2 + \|\partial_t u\|_{-1,2}^2 \right\} \leqslant K\,, \tag{8}$$

where K is independent of T.

PROOF: (Outline.) Testing by u gives

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + c_1\|u\|_2^2 \leqslant (F(u^t), u) \leqslant \frac{1}{2}\|F(u^t)\|_2^2 + \frac{1}{2}\|u\|_2^2.$$

By (6),

$$\|F(u^t)\|_2^2 \leqslant c\left(1 + \int_{-r}^0 \|u(t+s)\|_2^2\, d\mu(s)\right),$$

and we deduce (replacing μ by $\mu + \delta_0$ if necessary)

$$\frac{d}{dt}\|u\|_2^2 + c_1\|u\|_{1,2}^2 \leqslant c\left(1 + \int_{-r}^0 \|u(t+s)\|_2^2\, d\mu(s)\right).$$

Integrating on $(0,t)$, and invoking Lemma 2.1 below, we arrive at

$$\|u(t)\|_2^2 + \int_0^t \|u\|_{1,2}^2 \leqslant \|u(0)\|_2^2 + \int_{-r}^t \|u\|_2^2 \leqslant \|u^0\|_X^2 + \int_0^t \|u\|_2^2.$$

From Gronwall's lemma one has

$$\int_0^T \|u\|_{1,2}^2 \leqslant K = K(T, \|u^0\|_X). \tag{9}$$

The estimate of $\int_0^T \|\partial_t u\|_{-1,2}^2$ is obtained directly from the equation (see Lemma 3.2 below).

The independence of (8) on T rests substantially on the existence of bounded invariant region $\mathcal{B}$ in (7). This gives directly that $\|u(t)\|_2$ is bounded for all $t \geqslant 0$, and the rest of (8) follows by bootstrapping of (9).

The argument can be made rigorous using a suitable approximating scheme (see [**20**] where Galerkin method is used). Strong convergence of u in $L^2(0,T;L^2(\Omega))$ follows by Aubin–Lions lemma, and enables the passage limit in $F(u^t)$. For $a(\nabla u)$ one uses Minty's trick together with the monotonicity of $a(\cdot)$. We omit further details. ∎

THEOREM 2.2. *The weak solution is unique.*

PROOF: Let u, v be two solutions. Multiplying the equation for $w = u - v$ by w gives (in view of (5), (6))

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + c_1\|u\|_2^2 &\leqslant (F(u^t) - F(u^t), w)\\ &\leqslant \frac{1}{2}L^2\int_{-r}^0 \|w(t+s)\|_2^2\, d\mu(s) + \frac{1}{2}\|w\|_2^2.\end{aligned}$$

As above, one deduces

$$\frac{d}{dt}\|w\|_2^2 + c_1\|w\|_{1,2}^2 \leqslant L^2\int_{-r}^0 \|w(t+s)\|_2^2\, d\mu(s). \tag{10}$$

Invoking Lemma 2.1, and assuming $u = v$ on $[-r, 0]$ gives

$$\|w(t)\|_2^2 \leqslant c \int_0^t \|w\|_2^2 \, d\tau$$

for all $t \geqslant 0$. Hence $u = v$ by Gronwall's lemma. ∎

The following lemma (in fact a Fubini's theorem) is instrumental in handling the estimates of the memory term.

Lemma 2.1. *Let $f \in L^1(-r, T)$, and $F(t) = \int_{-r}^0 f(t+s)\, d\mu(s)$, where μ is a Radon measure on $[-r, 0]$. Then*

$$\int_0^T |F(t)| \, dt \leqslant c \int_{-r}^T |f(t)| \, dt,$$

where $c = \mu([-r, 0])$.

Proof: One computes

$$\begin{aligned}
\int_0^T |F(t)| \, dt &\leqslant \int_0^T \left(\int_{-r}^0 |f(t+s)| \, d\mu(s) \right) dt \\
&= \int_{-r}^0 \left(\int_0^T |f(t+s)| \, dt \right) d\mu(s) \\
&\leqslant \int_{-r}^0 \left(\int_{-r}^T |f(y)| \, dy \right) d\mu(s) \\
&= \mu([-r, 0]) \int_{-r}^T |f(y)| \, dy .
\end{aligned}$$

∎

Theorems 2.1, 2.2 enable us to define "solution semigroup" $S(t) : X \to X$ by the formula $S(t)\phi = u^t$, where u is the unique solution to (1)–(4).

Recall that $\mathcal{E} \subset X$ is called an exponential attractor, if $\mathcal{E}$ is compact, positively invariant, that is to say, $S(t)\mathcal{E} \subset \mathcal{E}$ for all $t > 0$), has finite fractal dimension, and attracts the dynamics exponentially fast:

$$\operatorname{dist}_X(S(t)X, \mathcal{E}) \leqslant c \exp(-\gamma t),$$

with some c, $\gamma > 0$. The fractal dimension of a set A is defined as

$$\limsup_{\varepsilon \to 0+} \frac{\ln N(A, \varepsilon)}{\ln \varepsilon},$$

where $N(A, \varepsilon)$ is the minimal number of ε-balls that cover A. See for example [**21**] to read more about this concept and its motivation. Regarding the exponential attractors, the literature about its theory and applications is abundant. See the basic monograph [**5**], and also [**6, 18**], for example.

Our main result reads as follows:

Theorem 2.3. *Let the assumptions (5)–(7) be in force. Then the dynamical system $(S(t), X)$, associated to the equation (1), has an exponential attractor.*

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3. Dynamics of trajectories

Let $\ell > 0$ be fixed. We set

$$X_\ell = \{\chi : [-r,\ell] \to L^2(\Omega);\ \chi \text{ is solution to (1) on } [0,\ell]\}.$$

The semigroup $L(t) : X_\ell \to X_\ell$ is defined by

$$\{L(t)\chi\}(s) = u(t+s) \qquad s \in [-r,\ell],$$

where for a given $\chi \in X_\ell$, u is the unique solution to (1) satisfying (4) with $\phi = \chi \restriction [-r,0]$.

The set of trajectories X_ℓ is considered with the topology $L^2(-r,\ell;L^2(\Omega))$; which – with a slight abuse of notation – will also be denoted X_ℓ.

We first prove a simple continuity result.

Lemma 3.1.

(1) *The mapping $\chi \mapsto L(t)\chi$ is Lipschitz continuous on X_ℓ, uniformly with respect to $t \in [0,T]$.*

(2) *The mapping $t \mapsto L(t)\chi$ is $1/2$-Hölder continuous on $[r,\infty)$, uniformly with respect to $\chi \in X_\ell$.*

Proof: (1) Given χ_1, $\chi_2 \in X_\ell$, let u_1, u_2 be the corresponding solutions to (1), that is, $u_i \restriction [-r,\ell] = \chi_i$. Denote $w = u_1 - u_2$. We have proved (see (10) above) that

$$\frac{d}{dt}\|w(\tau)\|_2^2 + c_1\|w(\tau)\|_{1,2}^2 \leqslant L^2 \int_{-r}^{0} \|w(\tau+s)\|_2^2\, d\mu(s)$$

holds for any $\tau \in [0,T]$. We integrate on (s,t), where $s \in (0,\ell)$ and $t \in (\ell,\ell+T)$ are fixed. In view of Lemma 2.1, we have

$$\|w(t)\|_2^2 + 2c_1\int_s^t \|w\|_{1,2}^2 \leqslant \|w(s)\|_2^2 + c\int_{s-r}^t \|w\|_2^2 \tag{11}$$

$$\|w(t)\|_2^2 + 2c_1\int_\ell^t \|w\|_{1,2}^2 \leqslant \|w(s)\|_2^2 + c\int_{-r}^\ell \|w\|_2^2 + \int_\ell^t \|w\|_2^2. \tag{12}$$

One more integration over $s \in (0,\ell)$ gives

$$\|w(t)\|_2^2 + 2c_1\int_\ell^t \|w\|_{1,2}^2 \leqslant c\int_\ell^t \|w\|_{1,2}^2 + c\int_{-r}^\ell \|w\|_2^2.$$

As $t \in (\ell,\ell+T)$ is arbitrary, we deduce by Gronwall's lemma

$$\sup_{t\in[\ell,\ell+T]} \|w(t)\|_2^2 + \int_\ell^{\ell+T} \|w\|_{1,2}^2 \leqslant c(T)\int_{-r}^\ell \|w\|_2^2. \tag{13}$$

Recalling the definitions of χ, $L(t)\chi$, one concludes easily that

$$\sup_{t\in[0,T]} \|L(t)\chi_1 - L(t)\chi_2\|_{X_\ell}^2 \leqslant c(T)\|\chi_1 - \chi_2\|_{X_\ell}^2.$$

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(2) Observe that

$$\left\|L(t+\tau)\chi - L(t)\chi\right\|_{X_\ell}^2 = \int_{-r}^{\ell} \left\|u(t+\tau+s) - u(t+s)\right\|_2^2 \, ds\,,$$

where, as above, u is the solution satisfying $u \restriction [-r,\ell] = \chi$. The trick is to write

$$\begin{aligned}\left\|u(t+\tau+s) - u(t+s)\right\|_2^2 &= \left\|u(t+\tau+s)\right\|_2^2 - \left\|u(t+s)\right\|_2^2 \\ &\quad - 2\big(u(t+\tau+s) - u(t+s), u(t+s)\big) \\ &= 2\int_0^{\tau} \left\langle \partial_t u(t+h+s), u(t+h+s)\right\rangle dh \\ &\quad - 2\int_0^{\tau} \left\langle \partial_t u(t+h+s), u(t+s)\right\rangle dh\,.\end{aligned}$$

Note that, thanks to (8), u has sufficient regularity to justify this if $t \geqslant r$. In particular, for the second integral we need $u(t+s) \in W^{1,2}(\Omega)$, which holds for almost every $s \in (-r,\ell)$.

Interchanging the order of integration gives

$$\left\|L(t+\tau)\chi - L(t)\chi\right\|_{X_\ell}^2 = 2\int_0^{\tau} \left\{ \int_{-r}^{\ell} \left\langle \partial_t u(t+h+s), u(t+h+s) - u(t+s)\right\rangle ds \right\} dh\,.$$

The inner integral, however, is bounded (uniformly with respect to t, h) thanks to the estimates (8). Hence

$$\left\|L(t+\tau)\chi - L(t)\chi\right\|_{X_\ell}^2 \leqslant c\tau\,. \qquad \square$$

We now deduce the key observation of the method of trajectories, that is, the smoothing property for $L(t)$.

Lemma 3.2. *Set $t^* = r+\ell$. Then $L(t^*)$ is Lipschitz continuous from X_ℓ into W_ℓ, where*

$$\|\chi\|_{W_\ell}^2 = \int_{-r}^{\ell} \|\chi\|_{1,2}^2 + \int_{-r}^{\ell} \|\partial_t \chi\|_{-1,2}^2\,.$$

Proof: Taking $T = \ell + r$ in (13) at once gives

$$\left\|L(t^*)\chi_1 - L(t^*)\chi_2\right\|_{W_\ell}^2 \leqslant c\left\|\chi_1 - \chi_2\right\|_{X_\ell}^2\,. \tag{14}$$

To estimate the time derivative, we use the duality argument

$$\left\|\partial_t w\right\|_{L^2(\ell,2\ell+r;W^{-1,2}(\Omega))} = \sup_{\varphi} \int_{\ell}^{2\ell+r} \left\langle \partial_t w, \varphi\right\rangle,$$

where φ is taken from $\varphi \in L^2\big(\ell, 2\ell+r; W^{1,2}(\Omega)\big)$ with $\|\varphi\| = 1$. From the equation we have

$$\begin{aligned}\int_{\ell}^{2\ell+r} \left\langle \partial_t w, \varphi\right\rangle &= \int_{\ell}^{2\ell+r} \big(a(\nabla u) - a(\nabla v), \nabla\varphi\big) + \int_{\ell}^{2\ell+r} \big(F(u^t) - F(u^t), \varphi\big) \\ &= I_1 + I_2\,.\end{aligned}$$

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One deduces from (5) that

$$I_1 \leqslant c_1 \int_\ell^{2\ell+r} \int_\Omega |\nabla w|\,|\nabla\varphi| \leqslant c\|w\|_{L^2(\ell,2\ell+r;W^{1,2}(\Omega))}.$$

Similarly we have

$$I_2 \leqslant \left(\int_\ell^{2\ell+r} \|F(u^t) - F(u^t)\|_2^2\, dt \right)^{1/2}.$$

Invoking (6) and Lemma 2.1,

$$\int_\ell^{2\ell+r} \|F(u^t) - F(u^t)\|_2^2\, dt \leqslant c \int_\ell^{2\ell+r} \int_{-r}^0 \|w(t+s)\|_2^2\, d\mu(s)\, dt \leqslant c\|w\|^2_{L^2(\ell-r,2\ell+r;L^2(\Omega))}.$$

Rewriting in terms of $L(t)$, we deduce

$$\begin{aligned} c^{-1}\|\partial_t(L(t^*)\chi_1 - L(t^*)\chi_2)\|_{L^2(-r,\ell;W^{-1,2}(\Omega))} \leqslant \|L(t^*)\chi_1 - L(t^*)\chi_2\|_{L^2(-r,\ell;W^{1,2}(\Omega))} \\ + \|\chi_1 - \chi_2\|_{X_\ell}. \end{aligned} \tag{15}$$

Combining (14), (15) gives the conclusion of lemma. ∎

We can now formulate the main result of this section.

Theorem 3.1. *The dynamical system $(L(t), X_\ell)$ has an exponential attractor $\mathcal{E}_\ell \subset X_\ell$.*

Proof: The proof follows a standard scheme. The smoothing property from the above lemma entails the existence of an exponential attractor $\mathcal{E}_\ell^*$ for the (discrete) subsystem $L(t^*n)$, $n \in \mathbb{N}$. See for example [**6**].

We then set

$$\mathcal{E}_\ell = \bigcup_{t\in[0,t^*]} L(t)\mathcal{E}_\ell^*.$$

In view of the continuity properties of $L(t)$ (Lemma 3.1) one verifies that $\mathcal{E}_\ell$ is the desired exponential attractor. See [**5**] for details. ∎

Remark. There is a minor technical obstacle in the fact that X_ℓ – which consists of solutions, and hence is a subset to $C([-r,\ell]; L^2(\Omega))$ – is not a complete space (it is not even closed) when considered with the topology $L^2(-r,\ell; L^2(\Omega))$.

A closer look on the construction of an exponential attractor shows that one only needs that ω-limit points remain in X_ℓ, that is to say, X_ℓ is asymptotically compact. Yet the estimates (8) ensure that any ω-limit point has a regularity of the weak solution. In particular, it belongs to $C([-r,\ell]; L^2(\Omega))$ as required. See [**10**] for details of this argument.

4. Proof of Theorem 2.3

Let us define the mapping $E : X_\ell \to X$ by $E(\chi) := \{L(\ell + r)\chi\} \restriction [-r, 0]$. It follows at once from (13) that E is Lipschitz continuous from $L^2(-r, \ell; L^2(\Omega))$ into the $C([-r, 0]; L^2(\Omega))$ topology. Also, observe that

$$S(t)\big(E(\chi)\big) = E\big(L(t)\chi\big). \tag{16}$$

We can now prove our main result. Recall that $\mathcal{E}_\ell \subset X_\ell$ is an exponential attractor for $(L(t), X_\ell)$. Set

$$\mathcal{E} := E(\mathcal{E}_\ell).$$

We claim that $\mathcal{E}$ is an exponential attractor for $(S(t), X)$.

1. $\mathcal{E}$ is compact and has finite fractal dimension, since $\mathcal{E}_\ell$ has these properties and E is Lipschitz (see [**21**, Proposition 13.2].)

2. $\mathcal{E}$ is positively invariant, because (by (16))

$$S(t)\mathcal{E} = S(t)E(\mathcal{E}_\ell) = E\big(L(t)\mathcal{E}_\ell\big) \subset E(\mathcal{E}_\ell) = \mathcal{E}.$$

3. Observe that $E(X_\ell) = S(\ell + r)X$; hence

$$\begin{aligned}
\operatorname{dist}_X\big(S(t+\ell+r)X, \mathcal{E}\big) &= \operatorname{dist}_X\big(S(t)S(\ell+r)X, \mathcal{E}\big)\\
&= \operatorname{dist}_X\Big(S(t)\big(E(X_\ell)\big), \mathcal{E}\Big)\\
&= \operatorname{dist}_X\Big(E\big(L(t)X_\ell\big), E(\mathcal{E}_\ell)\Big)\\
&\leqslant c \operatorname{dist}_{X_\ell}\big(L(t)X_\ell, \mathcal{E}_\ell\big)
\end{aligned}$$

(The last step uses the Lipschitz continuity of E). Thus $\mathcal{E}$ is exponentially attracting, since $\mathcal{E}_\ell$ is. The proof is finished. ∎

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Department of Mathematical Analysis
Charles University
Sokolovská, 83
CZ-18675 Prague 8
Czech Republic
e-mail: prazak@karlin.mff.cuni.cz