above; *i.e.* for a, b, c, d we take 15, 25, 35, 45, and for a', b', c', d' we take 2345, 1345, 1245, 1235; so that a, b', c,$'$ d are now orthogonal to 145, and so on. We have therefore a circle orthogonal to 2345, 1345, 1245, 1235, 1234; this we denote by 12345.

At the next step we adjoin a new symbol 6, in addition to the 5 already adjoined, and use the same proof as before. The chain can in this way be extended without limit:

Theorem VIII. An unending chain of circles can be defined, such that each circle can be named by a finite number of different symbols, and such that any two circles are orthogonal to each other if the name of one can be got from that of the other by simply cutting out or adding one symbol. The chain starts from a single circle, whose symbol for the purposes of this statement is to be considered a blank; and there is a circle in the chain corresponding to every combination of different symbols.

Alternative Forms of Expression for Hermite's Determinant.

By Sir Thomas Muir.

(1) Apparently it was in 1854 that Hermite first drew attention to the special determinant which now bears his name. It may be defined as being such that every two of its elements that are conjugate in position are conjugate-complex in form: and as a consequence its matrix is the sum of two matrices one of which is axisymmetric and the other zero-axial skew.

(2) Although Hermite had clear evidence that the determinant was imaginary only in appearance, he does not seem to have made any effort to obtain an expression for it free of $\sqrt{-1}$. Such an expression is first met with, almost casually, in a paper of Clebsch's of 1859, in which he has occasion to consider the latent roots of a Hermitant. His result we may formulate for ourselves thus: *Any Hermitant of the third order is expressible as the difference of a determinant and a ternary quadric, the former being the determinant of the axisymmetric portion of the matrix and the latter having for its discriminant the same determinant*: for example:—

$$\begin{vmatrix} a & h+\gamma\iota & g-\beta\iota \\ h-\gamma\iota & b & f+\alpha\iota \\ g+\beta\iota & f-\alpha\iota & c \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} - \begin{vmatrix} \alpha & \beta & \gamma & \\ a & h & g & \alpha \\ h & b & f & \beta \\ g & f & c & \gamma \end{vmatrix}.$$

(3) The corresponding expression for the case of the 4th and higher orders, which was not obtained until 1897,[1] is not at all so simple: indeed, it soon dawns on one who investigates it that the new expression is really an expansion-in-series, the number of term-groups being $n-1$ when the Hermitant is of the nth order. All that is known regarding it is still to be found in the concluding eleven pages of the memoir of 1897: the only later writing on the subject, though going no farther than the 4th order, is useful as a corroboration.

(4) The other mode of expression to which we wish to call attention is a contrast in every way. Not only is it unequivocally an alternative form suitable for everyday use, but it is in one or two respects a preferable form to the original. Strange to say, too, it is the long-familiar Pfaffian, and to pass from the Hermitant to the equivalent Pfaffian we literally have only to strike out the imaginaries and rearrange the variables: for example

$$\left|\begin{array}{cc} a & h+\gamma\iota \\ h-\gamma\iota & b \end{array}\right| = \left|\begin{array}{ccc} \gamma & h & a \\ & b & h \\ & & -\gamma \end{array}\right| ,$$

$$\left|\begin{array}{ccc} a & h+\gamma\iota & g+\beta\iota \\ h-\gamma\iota & b & f+\alpha\iota \\ g-\beta\iota & f-\alpha\iota & c \end{array}\right| = \left|\begin{array}{ccccc} \gamma & \beta & g & h & a \\ & \alpha & f & b & h \\ & & c & f & g \\ & & & -\alpha & -\beta \\ & & & & -\gamma \end{array}\right| ,$$

$$\left|\begin{array}{cccc} a & h+\gamma\iota & g+\beta\iota & r+\zeta\iota \\ h-\gamma\iota & b & f+\alpha\iota & q+\epsilon\iota \\ g-\beta\iota & f-\alpha\iota & c & p+\delta\iota \\ r-\zeta\iota & q-\epsilon\iota & p-\delta\iota & d \end{array}\right| = \left|\begin{array}{ccccccc} \gamma & \beta & \zeta & r & g & h & a \\ & \alpha & \epsilon & q & f & b & h \\ & & \delta & p & c & f & g \\ & & & d & p & q & r \\ & & & & -\delta & -\epsilon & -\zeta \\ & & & & & -\alpha & -\beta \\ & & & & & & -\gamma \end{array}\right| ,$$

where, in the third example, the reader would do well to note how the horizontally-running elements

$$a,\ h,\ g,\ r,\ \zeta,\ \beta,\ \gamma$$

of the Pfaffian are got from the first row of the determinant exactly as the vertically-running

$$a,\ h,\ g,\ r,\ -\zeta,\ -\beta,\ -\gamma$$

are got from the first column : how, similarly, the elements

$$b,\ f,\ q,\ \epsilon,\ \alpha \ \text{and}\ b,\ f,\ q,\ -\epsilon,\ -\alpha$$

[1] *Transac. R. Soc. Edinburgh*, xxxix., pp. 209-230.

are got from the first row and first column of the complementary minor of a in the determinant: how

$$c,\ p,\ \delta \text{ and } c,\ p,\ -\delta$$

are got from the complementary minor of ab in the determinant: and how, finally the d of the Pfaffian is got from the complementary minor of abc.

This could doubtless be proved directly by taking Zehfuss' determinant which equals the square of the determinant on the left and then transforming it into Cayley's determinant which equals the square of the Pfaffian on the right. It will, however, be better to prove a more general theorem and then draw the requisite deduction.

(5) *Any axisymmetric Pfaffian of $2m$ frame-lines is expressible as an m-line determinant.*

Taking as typical the case where m is 3 and the axisymmetric Pfaffian is

$$\left|\begin{array}{ccccc} e & d & c & b & a \\ & h & g & f & b \\ & & k & g & c \\ & & & h & d \\ & & & & e \end{array}\right| ,\text{ or } ff \text{ say,}$$

we have from Cayley

$$ff^2 = \left|\begin{array}{cccccc} . & e & d & c & b & a \\ -e & . & h & g & f & b \\ -d & -h & . & k & g & c \\ -c & -g & -k & . & h & d \\ -b & -f & -g & -h & . & e \\ -a & -b & -c & -d & -e & . \end{array}\right| = (-1)^3 \left|\begin{array}{cccccc} . & e & d & c & b & a \\ -e & . & h & g & f & b \\ -d & -h & . & k & g & c \\ c & g & k & . & -h & -d \\ b & f & g & h & . & -e \\ a & b & c & d & e & . \end{array}\right|$$

$$= (-1)^3 \left|\begin{array}{cccccc} a & e+b & d+c & c+d & b+e & a \\ -e+b & f & h+g & g+h & f & b-e \\ -d+c & -h+g & k & k & g-h & c-d \\ c & g & k & . & -h & -d \\ b & f & g & h & . & -e \\ a & b & c & d & e & . \end{array}\right|$$

$$= (-1)^3 \left|\begin{array}{cccccc} a & e+b & d+c & . & . & . \\ -e+b & f & h+g & . & . & . \\ -d+c & -h+g & k & . & . & . \\ c & g & k & -k & -h-g & -d-c \\ b & f & g & h-g & -f & -e-b \\ a & b & c & d-c & e-b & -a \end{array}\right|$$

$$= \left|\begin{array}{ccc} a & b+e & c+d \\ b-e & f & g+h \\ c-d & g-h & k \end{array}\right|^2 ;$$

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whence, as desired,

$$\left|\begin{array}{ccccc} e & d & c & b & a \\ & h & g & f & b \\ & & k & g & c \\ & & & h & d \\ & & & & e \end{array}\right| = \left|\begin{array}{ccc} a & b+e & c+d \\ b-e & f & g+h \\ c-d & g-h & k \end{array}\right|.$$

(6) Changing e, d, h in the foregoing into $e\sqrt{-1}$, $d\sqrt{-1}$, $h\sqrt{-1}$, we obtain

$$\left|\begin{array}{ccc} a & b+e\iota & c+d\iota \\ b-e\iota & f & g+h\iota \\ c-d\iota & g-h\iota & k \end{array}\right| = \left|\begin{array}{ccccc} e\iota & d\iota & c & b & a \\ & h\iota & g & f & b \\ & & k & g & c \\ & & & h\iota & d\iota \\ & & & & e\iota \end{array}\right|$$

$$= \left|\begin{array}{cccccc} . & e\iota & d\iota & c & b & a \\ -e\iota & . & h\iota & g & f & b \\ -d\iota & -h\iota & . & k & g & c \\ -c & -g & -k & . & h\iota & d\iota \\ -b & -f & -g & -h\iota & . & e\iota \\ -a & -b & -c & -d\iota & -e\iota & . \end{array}\right|^{\frac{1}{2}} = \left|\begin{array}{cccccc} . & e & d & c & b & a \\ -e & . & h & g & f & b \\ -d & -h & . & k & g & c \\ -c & -g & -k & . & -h & -d \\ -b & -f & -g & h & . & -e \\ -a & -b & -c & d & e & . \end{array}\right|^{\frac{1}{2}}$$

$$= \left|\begin{array}{ccccc} e & d & c & b & a \\ & h & g & f & b \\ & & k & g & c \\ & & & -h & -d \\ & & & & -e \end{array}\right|, \text{ as foretold in § 4,}$$

the alteration in the first six-line determinant here being due to multiplying the last three rows by ι and dividing the first three columns by the same.

(7) As an example of evaluation of the new form let us take the second Pfaffian of the three here, the result in this case having the advantage of being comparable with Clebsch's. Using the so-called "mixed" expansion at the start we obtain

$$\left|\begin{array}{ccccc} \gamma & \beta & g & h & a \\ & \alpha & f & b & h \\ & & c & f & g \\ & & & -\alpha & -\beta \\ & & & & -\gamma \end{array}\right| = \gamma\left|\begin{array}{ccc} c & f & g \\ & -\alpha & -\beta \\ & & -\gamma \end{array}\right| - \beta\left|\begin{array}{ccc} f & b & h \\ & -\alpha & -\beta \\ & & -\gamma \end{array}\right| + \alpha\left|\begin{array}{ccc} g & h & a \\ & -\alpha & -\beta \\ & & -\gamma \end{array}\right| - \left|\begin{array}{cc} g & h \\ f & b \\ c & f \end{array}\right.$$

$$= \gamma(-c\gamma + f\beta - g\alpha) - \beta(-f\gamma + b\beta - h\alpha)$$
$$+ \alpha(-g\gamma + h\beta - a\alpha) + \left|\begin{array}{ccc} a & h & g \\ h & b & f \\ g & f & c \end{array}\right|$$
$$= -(a\alpha^2 + b\beta^2 + c\gamma^2 - 2h\alpha\beta - 2f\beta\gamma + 2g\gamma\alpha) + \left|\begin{array}{ccc} a & h & g \\ h & b & f \\ g & f & c \end{array}\right|$$

which is seen to be identical with the expression in § 2.

Another very interesting example is the third Pfaffian of the three when $a = b = c = d = 0$, that is to say, the zero-axial 4-line Hermitant. This, if we write

$$F\begin{Bmatrix} l, & m, & n \\ x, & -y, & z \end{Bmatrix} \text{ for } \begin{matrix} l^2+m^2+n^2-2mn-2nl-2lm \\ +(x-y+z)^2 \end{matrix} + 2\begin{vmatrix} l & x & 1 \\ m & -y & 1 \\ n & z & 1 \end{vmatrix},$$

is found equal to

$$F\begin{Bmatrix} f\zeta, & g\epsilon, & h\delta \\ ra, & -q\beta, & p\gamma \end{Bmatrix} + F\begin{Bmatrix} ph, & qg, & rf \\ \gamma\delta, & -\beta\epsilon, & \alpha\zeta \end{Bmatrix},$$

a result which at the same time is the solution of a problem (No 16494) proposed without effect in the *Educational Times* over twenty years ago.

(8) In conclusion we note as being closely connected with the foregoing the following theorem in so-called "block" determinants: *If A, S be n-line square arrays, A axisymmetric and S zero-axial skew, then*

$$\begin{vmatrix} A & S \\ S' & A \end{vmatrix} = \begin{vmatrix} S' & A \\ -A & S' \end{vmatrix},$$

a curious feature of the identity being that one member is axisymmetric and the other skew.

For example, taking

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ for } A, \qquad \begin{vmatrix} . & z & y \\ -z & . & x \\ -y & -x & . \end{vmatrix} \text{ for } S,$$

we have

$$\begin{vmatrix} a & h & g & . & z & y \\ h & b & f & -z & . & x \\ g & f & c & -y & -x & . \\ . & -z & -y & a & h & g \\ z & . & -x & h & b & f \\ y & x & . & g & f & c \end{vmatrix} = \begin{vmatrix} . & -z & -y & a & h & g \\ z & . & -x & h & b & f \\ y & x & . & g & f & c \\ -a & -h & -g & . & -z & -y \\ -h & -b & -f & z & . & -x \\ -g & -f & -c & y & x & . \end{vmatrix}$$

Another identity having the same feature attracted a little unusual attention in the *Educational Times* for 1914. (See *Math. from Educ. Times*, (2) xxvi., pp. 69-71.)