ON THE DERIVATION ALGEBRAS OF LIE ALGEBRAS

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Let L be a Lie algebra over a field of characteristic 0 and let $D(L)$ be the derivation algebra of L, that is, the Lie algebra of all derivations of L. Then it is natural to ask the following questions: What is the structure of $D(L)$? What are the relations of the structures of $D(L)$ and L? It is the main purpose of this paper to present some results on $D(L)$ as the answers to these questions in simple cases.

Concerning the questions above, we give an example showing that there exist non-isomorphic Lie algebras whose derivation algebras are isomorphic (Example 3 in § 5). Therefore the structure of a Lie algebra L is not completely determined by the structure of $D(L)$. However, there is still some intimate connection between the structure of $D(L)$ and that of L.

Let $L^{[1]} = L\ D(L) = \{\sum x_i D_i : x_i \in L,\ D_i \in D(L)\}$ and define $L^{[n+1]} = L^{[n]}\ D(L)$ inductively. L is called *characteristically nilpotent* provided there exists an integer k such that $L^{[k]} = (0)$ **(4**,p. 157**)**. Then L is characteristically nilpotent if and only if $D(L)$ is nilpotent and L is not one-dimensional **(6**, Theorem 1**)**. As an analogue, we call L *characteristically solvable* provided $D(L)$ is solvable and the centre of L is contained in $[L, L]$. Then characteristically nilpotent Lie algebras are characteristically solvable. It is known that $D(L)$ is semi-simple if and only if L is semi-simple **(5**, Theorem 4.4**)** and that $D(L)$ is nilpotent if and only if L is characteristically nilpotent or one-dimensional. In § 2, we shall show that $D(L)$ is the direct sum of a semi-simple ideal and the radical if and only if either L is reductive or L is the direct sum of a semi-simple ideal, a characteristically solvable ideal and a central ideal whose dimension is at most one (Theorem 1). We also prove that $D(L)$ is the direct sum of a semi-simple ideal and the nilpotent radical if and only if either L is reductive or L is the direct sum of a semi-simple ideal and a characteristically nilpotent ideal (Theorem 2). It is known that, as an algebraic Lie algebra, $D(L)$ has the following structure: $D(L) = \mathfrak{S} + \mathfrak{A} + \mathfrak{N}$ with $[\mathfrak{S}, \mathfrak{A}] = (0)$ where $\mathfrak{S}$ is a maximal semi-simple subalgebra, $\mathfrak{A}$ is a maximal abelian subalgebra of the radical consisting of semi-simple elements, and $\mathfrak{N}$ is the ideal of all nilpotent elements in the radical **(1**, p. 144**)**. If $D(L)$ is especially the direct sum of ideals $\mathfrak{S}$, $\mathfrak{A}$, and $\mathfrak{N}$, then either $D(L) = \mathfrak{S} + \mathfrak{N}$ or $D(L) = \mathfrak{S} + \mathfrak{A}$ where $\mathfrak{A}$ is one-dimensional (Corollary 1 of Theorem 2).

In §§ 1 and 3, we study the derivation algebra of L when L is the direct

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sum of the ideals L_i $(i = 1, 2, \ldots, n)$. $D(L)$ is the direct sum of a semi-simple ideal and the non-abelian nilpotent radical (resp. non-abelian nilpotent, reductive) if and only if $D(L_i)$, for each i, is also; in the case that the dimension of the image of the centre of L in $L/[L, L]$ is at most one, $D(L)$ is the direct sum of a semi-simple ideal and the radical (resp. solvable) if and only if $D(L_i)$, for each i, is also (Theorem 4). In § 4 we show that if the nil-radical of L is characteristically solvable, then the radical of L is characteristically solvable and is the direct summand of L (Proposition 2). We also show some other properties of characteristically solvable Lie algebra (Propositions 3, 4, and 5) and give some examples of such Lie algebras.

Section 5 contains some remarks and the partial answers to the questions asked in the first paragraph (Theorems 5 and 6).

1. Throughout this paper we denote by L a Lie algebra over a field K of characteristic 0, by $D(L)$ the derivation algebra of L and by $Z(L)$ the centre of L. For any element x of L, the adjoint mapping $\operatorname{ad} x: y \to [y, x]$ is a derivation of L which is called inner. Given a subset M of L, we denote by $\operatorname{ad} M$ the set of all $\operatorname{ad} x$ with x in M. L is called reductive provided L is the direct sum of a semi-simple ideal and the centre $Z(L)$.

Let L be the direct sum of the ideals L_i $(i = 1, 2, \ldots, n)$. Let p_i denote the projection of L onto L_i. Let $E(L)$ be the set of all linear transformations of L into L and let $E(L_i, L_j)$ be the set of those of L_i into L_j. We shall identify an element T_{ij} of $E(L_i, L_j)$ with an element $p_i T_{ij}$ of $E(L)$. Thus we have $E(L_i, L_j) \subset E(L)$. Put $D(L_i, L_j) = D(L) \cap E(L_i, L_j)$. Then it is obvious that $D(L_i, L_i) = D(L_i)$. We prove the following

LEMMA 1. *Let L be the direct sum of the ideals L_i $(i = 1, 2, \ldots, n)$. Then*

$$D(L) = \sum_{i,j=1}^{n} D(L_i, L_j); \tag{1}$$

(2) *For $i \neq j$, $D(L_i, L_j)$ consists of all elements T_{ij} of $E(L_i, L_j)$ such that $L_i T_{ij} \subset Z(L_j)$ and $[L_i, L_i]T_{ij} = (0)$;*

(3) *For $i \neq j$, $D(L_i, L_j)$ is abelian and*

$$\left[D(L_i, L_j), \sum_{k=1}^{n} D(L_k) \right] \subset D(L_i, L_j).$$

Proof. We shall first prove (2). Let D_{ij} be an element of $D(L_i, L_j)$ with $i \neq j$. Then, for x_i in L_i and x_j in L_j, we have

$$0 = [x_i, x_j]D_{ij} = [x_i D_{ij}, x_j].$$

Therefore $L_i D_{ij} \subset Z(L_j)$. Furthermore, for elements x_i and y_i of L_i, we have

$$[x_i, y_i]D_{ij} = [x_i D_{ij}, y_i] + [x_i, y_i D_{ij}] = 0,$$

which shows that $[L_i, L_i]D_{ij} = (0)$. Conversely, suppose that T_{ij} is an

element of $E(L_i, L_j)$ satisfying the conditions in (2). Then it is immediate that

$$[x_k, x_l]T_{ij} = [x_k T_{ij}, x_l] = [x_k, x_l T_{ij}] = 0$$

for all x_k in L_k and all x_l in L_l. It follows that T_{ij} is a derivation of L, that is, that T_{ij} belongs to $D(L_i, L_j)$. Thus (2) is proved.

Let D be any element of $D(L)$. Put $T_{ij} = p_i D p_j$. Then

$$D = \sum_{i,j=1}^{n} T_{ij}$$

where T_{ij} belongs to $E(L_i, L_j)$. It is easy to see that T_{ii} is a derivation of L_i and that, for $i \neq j$, T_{ij} satisfies the conditions in (2). Therefore it follows from (2) proved above that T_{ij} belongs to $D(L_i, l_j)$. Thus we have

$$D(L) \subset \sum_{i,j=1}^{n} D(L_i, L_j).$$

Since the converse inclusion is evident, we have

$$D(L) = \sum_{i,j=1}^{n} D(L_i, L_j)$$

and (1) is proved.

(3) is evident. Thus the lemma is proved.

Let $\bar{D}(L)$ denote the subalgebra of $D(L)$ consisting of all elements D of $D(L)$ such that $L\,D \subset Z(L)$. Then we have

LEMMA 2. *Let L be the direct sum of the ideals L_i $(i = 1, 2, \ldots, n)$. Suppose that $Z(L_j) \subset [L_j, L_j]$ for some j. Then*

(1) *$\bar{D}(L_j)$ is an abelian ideal of $D(L)$;*

(2) *$[D(L_i, L_j), D(L_j, L_i)] \subset \bar{D}(L_j)$ for all $i \neq j$.*

Proof. Let $\bar{D}_{jj}$ be any element of $\bar{D}(L_j)$. Then it is immediate that $[L_j, L_j]\bar{D}_{jj} = (0)$ and therefore that $Z(L_j)\bar{D}_{jj} = (0)$. By using the fact that the centre of L_j is stable under all derivations of L_j, it is easy to see that $\bar{D}(L_j)$ is an abelian ideal of $D(L_j)$. By Lemma 1 (2) it is clear that any element D_{ji} of $D(L_j, L_i)$ with $i \neq j$ satisfies $Z(L_j)D_{ji} = (0)$. Therefore it is immediate that

$$\left[\bar{D}(L_j), \sum_{i \neq j} D(L_i) + \sum_{i \neq k} D(L_i, L_k)\right] = (0).$$

We can now use Lemma 1 (1) to conclude that $\bar{D}(L_j)$ is an abelian ideal of $D(L)$, and (1) is proved.

For $i \neq j$, let D_{ij} and D_{ji} be any elements of $D(L_i, L_j)$ and $D(L_j, L_i)$ respectively. Then, by Lemma 1 (2), we have

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$$L_i[D_{ij}, D_{ji}] = L_i D_{ij} D_{ji} \subset Z(L_j) D_{ji} = (0),$$
$$L_j[D_{ij}, D_{ji}] = L_j D_{ji} D_{ij} \subset Z(L_i) D_{ij} \subset Z(L_j),$$

which shows that $[D_{ij}, D_{ji}]$ belongs to $\bar{D}(L_j)$. Thus we have (2), completing the proof.

2. In this section we determine the structure of the Lie algebra L such that $D(L)$ is the direct sum of a semi-simple ideal and the radical. We begin with

LEMMA 3. *Let L be a solvable Lie algebra such that $Z(L) \subset [L, L]$. If $D(L)$ is the direct sum of a semi-simple ideal and the radical, then L is characteristically solvable.*

Proof. It is clear that L is not abelian. Write $D(L) = \mathfrak{S} + \mathfrak{R}$ where $\mathfrak{S}$ is a semi-simple ideal and $\mathfrak{R}$ is the radical of $D(L)$. Since ad L is a solvable ideal of $D(L)$, it follows that ad $L \subset \mathfrak{R}$. Let D be any element of $\mathfrak{S}$. Then ad $LD = [\text{ad } L, D] = (0)$ by hypothesis. Therefore $L\ D \subset Z(L)$. Since $Z(L) \subset [L, L]$ by hypothesis, it follows that $D^2 = 0$, which shows that all elements of $\mathfrak{S}$ are nilpotent. By Engel's theorem, $\mathfrak{S}$ is nilpotent and therefore $\mathfrak{S} = (0)$. Thus $D(L)$ is solvable and L is characteristically solvable, completing the proof.

LEMMA 4. *Let L be a non-abelian solvable Lie algebra. If $D(L)$ is the direct sum of a semi-simple ideal and the radical, then $D(L)$ is solvable, and L is either characteristically solvable or the direct sum of a characteristically solvable ideal and a one-dimensional central ideal.*

Proof. By virtue of Lemma 3, it suffices to prove the lemma when $Z(L) \not\subset [L, L]$. Let L_1 and Z be subspaces of $Z(L)$ such that

$$Z(L) = L_1 + Z, \quad L_1 \cap [L, L] = (0), \text{ and } Z \subset [L, L].$$

Let L_2 be a subspace of L containing $[L, L]$ such that

$$L = L_1 + L_2, \qquad L_1 \cap L_2 = (0).$$

Then it is clear that L_1 is a non-zero central ideal of L and that L_2 is a non-zero ideal of L such that $Z(L_2) \subset [L_2, L_2]$.

By hypothesis, $D(L) = \mathfrak{S} + \mathfrak{R}$ where $\mathfrak{S}$ is a semi-simple ideal and $\mathfrak{R}$ is the radical of $D(L)$. Write $D(L_2) = \mathfrak{S}_2 + \mathfrak{R}_2$ with $\mathfrak{S}_2$ a semi-simple subalgebra and $\mathfrak{R}_2$ the radical of $D(L_2)$. Then, since $Z(L_2) \subset [L_2, L_2]$, it follows from Lemma 2 (1) that $\mathfrak{R}_2$ contains $\bar{D}(L_2)$. Let D_1 be the identity derivation of L_1 and let $\mathfrak{M}$ be the space spanned by D_1, $D(L_1, L_2)$, $D(L_2, L_1)$ and $\mathfrak{R}_2$. Then, by Lemma 1 (1), (3) and Lemma 2 (2), it is immediate that $\mathfrak{M}$ is an ideal of $D(L)$. We assert that $\mathfrak{M}$ is solvable. In fact, by Lemma 1 (3) and Lemma 2 (2), we have

$$\mathfrak{M}^{(i)} \subset \mathfrak{R}_2^{(i)} + (\bar{D}(L_2) + D(L_1, L_2) + D(L_2, L_1)).$$

Since $\mathfrak{R}_2^{(k)} = (0)$ for some integer k, it follows that

$$\mathfrak{M}^{(k)} \subset \bar{D}(L_2) + D(L_1, L_2) + D(L_2, L_1).$$

By Lemma 2 we have $\mathfrak{M}^{(k+1)} \subset \bar{D}(L_2)$. It now follows from Lemma 2 (1) that $\mathfrak{M}^{(k+2)} = (0)$, that is, that $\mathfrak{M}$ is solvable, as was asserted. Thus $\mathfrak{M}$ is a solvable ideal of $D(L)$ and therefore it is contained in $\mathfrak{R}$. Since $\mathfrak{S}$ is a unique maximal semi-simple subalgebra of $D(L)$, it contains $\mathfrak{S}_2$. Therefore $[\mathfrak{S}_2, \mathfrak{R}_2] = (0)$, which shows that $D(L_2)$ is the direct sum of a semi-simple ideal and the radical. Therefore we can use Lemma 3 to see that L_2 is characteristically solvable and we see that

$$\mathfrak{M} = (D_1) + D(L_1, L_2) + D(L_2, L_1) + D(L_2).$$

Furthermore, we assert that dim $L_1 = 1$. In fact, if dim $L_1 > 1$, then $D(L_1) = \mathfrak{S}_1 + (D_1)$ where $\mathfrak{S}_1$ is a non-zero semi-simple ideal of $D(L_1)$. Therefore $D(L) = \mathfrak{S}_1 + \mathfrak{M}$ and $[\mathfrak{S}_1, \mathfrak{M}] = (0)$ by hypothesis. Let D_{11} be any element of $\mathfrak{S}_1$. Then

$$D_{21}D_{11} = [D_{21}, D_{11}] = 0$$

for any element D_{21} of $D(L_2, L_1)$. But, since L_1 is abelian and $L_2 \neq (0)$, it follows from Lemma 1 (2) that $L_2D(L_2, L_1) = L_1$. Therefore we have $D_{11} = 0$, whence $\mathfrak{S}_1 = (0)$, which is a contradiction. Thus L_1 must be one-dimensional, as was asserted. We now see that $D(L_1) = (D_1)$ and therefore that $D(L) = \mathfrak{M}$. Thus $D(L)$ is solvable and the lemma is proved.

We can now prove the following

THEOREM 1. *$D(L)$ is the direct sum of a semi-simple ideal and the radical if and only if L is one of the following Lie algebras:*

(1) *L is reductive;*

(2) *L is the direct sum of a semi-simple ideal and a characteristically solvable ideal;*

(3) *L is the direct sum of a semi-simple ideal, a characteristically solvable ideal, and a one-dimensional central ideal.*

Proof. Suppose that $D(L)$ is the direct sum of a semi-simple ideal $\mathfrak{S}$ and the radical $\mathfrak{R}$. Write $L = S + R$ where S is a semi-simple subalgebra and R is the radical of L. Then it is clear that ad S and ad R are contained in $\mathfrak{S}$ and $\mathfrak{R}$ respectively. Therefore

$$\text{ad } [S, R] = [\text{ad } S, \text{ad } R] = (0),$$

from which it follows that $[S, [S, R]] = (0)$. Since ad S is completely reducible, it follows that $[S, R] = (0)$. Thus L is the direct sum of the ideals S and R. Since $Z(S) = (0)$ and $S = [S, S]$, by Lemma 1 (2) it is clear that $D(S, R) = D(R, S) = (0)$. Therefore by Lemma 1(1) we have $D(L) = D(S) + D(R)$. It now follows that $\mathfrak{R}$ is the radical of $D(R)$ and therefore that $D(R) = \mathfrak{S} \cap D(R) + \mathfrak{R}$. Since $\mathfrak{S} \cap D(R)$ is semi-simple as an ideal of $\mathfrak{S}$, $D(R)$ is the

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direct sum of a semi-simple ideal and the radical. We can use Lemma 4 to see that R is abelian or characteristically solvable or the direct sum of a characteristically solvable ideal and a one-dimensional central ideal. Thus the necessity of the condition is proved.

To prove the sufficiency of the condition, if L is reductive, write $L = S + A$ with S a semi-simple ideal and A an abelian ideal. Then by Lemma 1 we have $D(L) = D(S) + D(A)$. Since $D(A)$ is the direct sum of a semi-simple ideal and the one-dimensional central ideal, so is $D(L)$. If L is the Lie algebra as in (2), then $D(L)$ is clearly the direct sum of a semi-simple ideal and the radical. If L is the Lie algebra as in (3), write $L = S + R + Z$ where S is a semi-simple ideal, R is a characteristically solvable ideal, and Z is a one-dimensional central ideal. Then $D(L)$ is the direct sum of the ideals $D(S)$ and $D(R + Z)$, the latter being the radical of $D(L)$ (cf. the fact that $\mathfrak{M}$ is solvable in the proof of Lemma 4). Thus the theorem is proved.

As an immediate consequence of Theorem 1, we have

COROLLARY 1. *$D(L)$ is solvable if and only if L is characteristically solvable or one-dimensional or the direct sum of a characteristically solvable ideal and a one-dimensional central ideal.*

The following corollary is remarked in **(6, § 3)**.

COROLLARY 2. *If $D(L)$ consists of semi-simple elements, then L is a reductive Lie algebra whose centre is at most one-dimensional.*

Proof. Since the radical of $D(L)$ consists of semi-simple elements, it follows from the structure theorem on algebraic Lie algebras **(1**, p. 144**)** that $D(L)$ is reductive. By Theorem 1 we see that L is reductive. If dim $Z(L) > 1$, then it is evident that L has a non-zero nilpotent derivation. Therefore dim $Z(L) \leqslant 1$, completing the proof.

Let $D_0(L) = L$, $D_1(L) = D(L)$ and let $D_n(L)$ be the derivation algebra of $D_{n-1}(L)$. Then we have the following corollary correcting **(7**, Theorem 4**)**.

COROLLARY 3. *For any integers m, $n \geqslant 0$, $D_m(L)$ is reductive if and only if $D_n(L)$ is reductive. Then all the $D_n(L)$'s with $n \geqslant 1$ are completely reducible and isomorphic to each other.*

Proof. It follows from Theorem 1 that $D_n(L)$ is reductive if and only if $D_{n-1}(L)$ is reductive. Therefore the first part of the corollary is evident. If some $D_m(L)$ is reductive, then all the $D_n(L)$ with $n \geqslant 1$ are completely reducible. Since the centre of $D(L)$ is at most one-dimensional, it is immediate that all the $D_n(L)$'s with $n \geqslant 1$ are isomorphic to each other, completing the proof.

In Theorem 1, if L is not reductive, then the maximal semi-simple subalgebra of $D(L)$ is ad S with S the maximal semi-simple ideal of L. We here

note the following proposition which is an easy consequence of **(5**, Theorem 4.3**)**.

PROPOSITION 1. *Let S be a maximal semi-simple subalgebra of L. Let R be the radical of L and let $\mathfrak{M}$ be the subalgebra of $D(R)$ consisting of all derivations D of R which can be trivially extended to the derivation of L, that is, such that, by putting $S\,D = (0)$, D is a derivation of L. Then* ad S *is a maximal semi-simple subalgebra of $D(L)$ if and only if $\mathfrak{M}$ is solvable.*

Proof. We identify an element of $\mathfrak{M}$ with the trivially extended derivation of L. Therefore $\mathfrak{M} \subset D(L)$. Let D be any element of $D(L)$. Then, as is well known, there exists an element x of L such that the restriction of D to S is equal to the restriction of ad x to S as the derivations of S into L. Put $D' = D - \text{ad } x$. Then it is clear that D' belongs to $\mathfrak{M}$, which shows that $D(L) = \text{ad } L + \mathfrak{M}$. If we write $\mathfrak{M}_1 = \text{ad } R + \mathfrak{M}$, then it is immediate that $\mathfrak{M}_1$ is an ideal of $D(L)$ and ad $S \cap \mathfrak{M}_1 = (0)$. Let $\mathfrak{R}$ be the radical of $D(L)$. Then, since $D(L)/\mathfrak{M}_1$ is semi-simple, it follows that $\mathfrak{R}$ is contained in $\mathfrak{M}_1$.

If $\mathfrak{M}$ is solvable, then $\mathfrak{M}_1$ is solvable and therefore ad S is a maximal semi-simple subalgebra of $D(L)$. Conversely, if ad S is such a subalgebra of $D(L)$, then it is clear that dim $\mathfrak{R}$ = dim $\mathfrak{M}_1$. Since $\mathfrak{R} \subset \mathfrak{M}_1$, we have $\mathfrak{R} = \mathfrak{M}_1$. Therefore $\mathfrak{M}$ is solvable, completing the proof.

Before we state the second theorem, we prove

LEMMA 5. *Let L be a non-abelian nilpotent Lie algebra such that $Z(L)$ is not contained in $[L, L]$. Then $D(L)$ is not nilpotent. $D(L)$ actually contains a solvable non-nilpotent ideal.*

Proof. Let L_1 and Z be the subspaces of $Z(L)$ such that

$$Z(L) = L_1 + Z, \quad L_1 \cap [L, L] = (0), \text{ and } Z \subset [L, L].$$

Let L_2 be a subspace of L, complementary to L_1 and containing $[L, L]$. Then it is clear that $Z(L_2) \subset [L_2, L_2]$. Let D_1 be the identity derivation of L_1 and let $\mathfrak{M}$ be the space spanned by D_1, $D(L_1, L_2)$, $D(L_2, L_1)$, and $\bar{D}(L_2)$. We assert that $\mathfrak{M}$ is a solvable non-nilpotent ideal of $D(L)$. In fact, by Lemma 1 (1), (3) and Lemma 2 (2), we see that $\mathfrak{M}$ is an ideal of $D(L)$. It is obvious that

$$\mathfrak{M}^{(1)} \subset \bar{D}(L_2) + D(L_1, L_2) + D(L_2, L_1).$$

Therefore it follows from Lemma 2 that $\mathfrak{M}^{(3)} = (0)$, that is, that $\mathfrak{M}$ is solvable. By the hypothesis that L is non-abelian and nilpotent, we have $D(L_1, L_2) \neq (0)$. Since

$$[D_1, D(L_1, L_2)] = D(L_1, L_2),$$

it follows that $\mathfrak{M}$ is not nilpotent. Thus $\mathfrak{M}$ is a solvable non-nilpotent ideal of $D(L)$, as was asserted. The proof is complete.

We can now prove the following

THEOREM 2. *$D(L)$ is the direct sum of a semi-simple ideal and the nilpotent radical if and only if either L is reductive or L is the direct sum of a semi-simple ideal and a characteristically nilpotent ideal.*

Proof. The sufficiency of the condition is immediate by Lemma 1. To prove the necessity, suppose that $D(L)$ is the direct sum of a semi-simple ideal and the nilpotent radical. Then, by Theorem 1, we have that (1) L is reductive or (2) L is the direct sum of a semi-simple ideal S and a characteristically solvable ideal R, or (3) L is the direct sum of a semi-simple ideal S, a characteristically solvable ideal R, and a one-dimensional ideal Z. In the case (2), $D(R)$ must be nilpotent and R is not one-dimensional. Therefore by **(6**, Theorem 1**)** R is characteristically nilpotent. It now suffices to show that the case (3) does not happen. If L is the Lie algebra in (3), then it follows from Lemma 1 that $D(L) = D(S) + D(R + Z)$. Since $D(R + Z)$ is a solvable ideal of $D(L)$ by Theorem 1, it must be nilpotent by our assumption. Therefore $R + Z$ is a non-abelian nilpotent Lie algebra. Then Lemma 5 tells us that $D(R + Z)$ is not nilpotent, which is a contradiction. Therefore we cannot have the case (3). Thus the theorem is proved.

COROLLARY 1. *If $D(L)$ is the direct sum of a semi-simple ideal and the nilpotent radical, then the radical of $D(L)$ is either one-dimensional and consists of semi-simple elements or consists of nilpotent elements.*

Proof. This is immediate from Theorem 2 and the fact that N is a characteristically nilpotent Lie algebra if and only if all the derivations of N are nilpotent.

COROLLARY 2. *Let R and N be the radical and the nil-radical of L respectively. Then the following conditions are equivalent:*

(1) *$D(L)$ is the direct sum of a semi-simple ideal and the radical consisting of nilpotent elements;*

(2) *R is characteristically nilpotent;*

(3) *N is characteristically nilpotent;*

(4) *$N\,D(L)^n = (0)$ for some integer n.*

If L satisfies one of these conditions, then $R = N$.

Proof. (1) $\rightarrow$ (2) is an immediate consequence of Theorem 2. (2) $\rightarrow$ (3) is evident, since (2) implies that $R = N$. (3) $\rightarrow$ (4) is immediate by the fact that N is stable under all derivations of L. Therefore it suffices to prove that (4) $\rightarrow$ (1). Suppose that L satisfies the condition (4). Let S be a maximal semi-simple subalgebra of L. Then $L = S + R$. Since $N(\mathrm{ad}\ S)^n = (0)$ and $[R, S] \subset N$, it follows that $R(\mathrm{ad}\ S)^{n+1} = (0)$. Since ad S is completely reducible, we have $R(\mathrm{ad}\ S) = (0)$, that is, $[R, S] = (0)$. Then, by Lemma 1, $D(L)$ is the direct sum of the ideals $D(S)$ and $D(R)$. It is obvious that $D(S)$ is semi-simple. From the fact that $R\,D \subset N$ for any D in $D(R)$, it follows that $R\,D(R)^{n+1} = (0)$ and therefore that $D(R)$ consists of nilpotent elements. Thus we see that (1) is satisfied by L. The proof is complete.

3. This section is devoted to the study of $D(L)$ in the case that L is the direct sum of the ideals. By using Lemma 1, we can first prove

THEOREM 3. *Let L be the direct sum of the ideals L_i $(i = 1, 2, \ldots, n)$. Then $D(L) = D(L_1) + D(L_2) + \ldots + D(L_n)$ if and only if L satisfies one of the following conditions:*

(1) $Z(L) = (0)$;

(2) $L = [L, L]$;

(3) *All the L_i's except one are such that $Z(L_i) = (0)$ and $L_i = [L_i, L_i]$.*

Proof. If $Z(L) = (0)$ (resp. $L = [L, L]$), then it is clear that $Z(L_i) = (0)$ (resp. $L_i = [L_i, L_i]$) for all i. Therefore, if one of the three conditions in the statement is satisfied by L, it follows from Lemma 1 (2) that $D(L_i, L_j) = (0)$ for all $i \neq j$. By Lemma 1 (1) we have $D(L) = \sum_{i=1}^{n} D(L_i)$. Conversely, suppose that $D(L) = \sum_{i=1}^{n} D(L_i)$. If $Z(L) \neq (0)$ and $L \neq [L, L]$, let i and j be respectively any integers such that $Z(L_i) \neq (0)$ and such that $L_j \neq [L_j, L_j]$. If $i \neq j$, then by Lemma 1 (2) we have $D(L_j, L_i) \neq (0)$, contrary to our assumption. Therefore $i = j$. This shows that there exists only one L_i such that $Z(L_i) \neq (0)$ and $L_i \neq [L_i, L_i]$, and that all the other L_k's satisfy the conditions $Z(L_k) = (0)$ and $L_k = [L_k, L_k]$. The proof is complete.

LEMMA 6. *If $D(L)$ is abelian, then L is one-dimensional.*

Proof. If $D(L)$ is abelian, then we have

$$\mathrm{ad}[L, L] = [\mathrm{ad}\, L, \mathrm{ad}\, L] = (0),$$

from which it follows that $L^3 = (0)$. Then it is easy to construct a non-zero semi-simple derivation of L, whence L is not characteristically nilpotent. By **(6**, Theorem 1**)** we see that L is one-dimensional.

We now prove the following

THEOREM 4. *Let L be the direct sum of the ideals L_i $(i = 1, 2, \ldots, n)$. Then*

(1) *$D(L)$ is reductive (resp. semi-simple) if and only if $D(L_i)$, for each i, is reductive (resp. semi-simple);*

(2) *$D(L)$ is the direct sum of a semi-simple ideal and the non-abelian nilpotent radical if and only if $D(L_i)$, for each i, is such a direct sum;*

(3) *$D(L)$ is non-abelian nilpotent if and only if $D(L_i)$, for each i, is non-abelian nilpotent.*

If $\dim (Z(L) + [L, L]/[L, L]) \leqslant 1$, *then*

(4) *$D(L)$ is the direct sum of a semi-simple ideal and the radical if and only if $D(L_i)$, for each i, is such a direct sum;*

(5) *$D(L)$ is solvable if and only if $D(L_i)$, for each i, is solvable.*

Proof. (1) is immediate from Corollary 3 of Theorem 1, Lemma 1 (1), (2), and the fact that L is reductive (resp. semi-simple) if and only if L_i, for each i, is reductive (resp. semi-simple).

(3) is a consequence of **(6**, Theorem 6**)**, but for completeness we write the proof in a slightly different way. Suppose that all $D(L_i)$'s are non-abelian

nilpotent. Then all L_i's are characteristically nilpotent, whence we have $Z(L_i) \subset [L_i, L_i]$ for all i. Therefore $L\,D(L_i, L_j)D(L_j, L_k) = (0)$ for all i, j, k such that $i \neq j$, $j \neq k$. Let m_i and l_i be the integers such that

$$D(L_i)^{m_i} = (0) \qquad \text{and} \qquad L_i^{[l_i]} = (0),$$

and let m be the maximal integer of all m_i and l_i. By Lemma 1 (1), we have

$$D(L)^{2m} = \sum_{i=1}^{n} D(L_i)^{2m} + \sum_{p=0}^{2m-1} \sum_{i \neq j} [\ldots[[D(L_i)^p, D(L_i, L_j)], D(L_j)], \ldots, D(L_j)]$$

$$+ \ldots + \sum_{i \neq j,\, j \neq k, \ldots,\, l \neq q} [\ldots[D(L_i, L_j), D(L_j, L_k)], \ldots, D(L_l, L_q)],$$

where $D(L_i)^0$ means the identity transformation of L_i into L_i for each i. It is clear that all the terms except the ones

$$\mathfrak{M} = [\ldots[[D(L_i)^p, D(L_i, L_j)], D(L_j)], \ldots, D(L_j)] \text{ with } i \neq j \text{ and } p < m_i$$

are equal to (0). But we have

$$L\,\mathfrak{M} \subset L_j D(L_j)^{2m-p-1} \subset L_j^{[2m-p-1]} = (0),$$

since $2m - p - 1 \geqslant m \geqslant l_j$. Therefore $\mathfrak{M} = (0)$. Thus we see that $D(L)^{2m} = (0)$. Since $m > 1$, $D(L)$ is non-abelian nilpotent. Conversely, suppose that $D(L)$ is non-abelian nilpotent. Then it is clear that all $D(L_i)$'s are nilpotent. If some $D(L_j)$ is abelian, it follows from Lemma 6 that L_j is one-dimensional and therefore from Lemma 5 that $D(L)$ is not nilpotent, contrary to our supposition. Therefore all $D(L_i)$'s are not abelian. Thus (3) is proved.

To prove (5), suppose that $\dim (Z(L) + [L, L]/[L, L]) \leqslant 1$. Then either there exists only one suffix i_0 such that

$$Z(L_{i_0}) \not\subset [L_{i_0}, L_{i_0}],$$

or $Z(L_i) \subset [L_i, L_i]$ for all i. In the first (resp. second) case, let $\mathfrak{M}$ be

$$\sum_{i \neq j} D(L_i, L_j) + \sum_{i \neq i_0} \bar{D}(L_i) \qquad \left(\text{resp. } \sum_{i \neq j} D(L_i, L_j)\right).$$

For $i \neq k$ and $j \neq k$, by Lemma 1 (2) and Lemma 2 (2) we have

$$[D(L_i, L_k), D(L_k, L_j)] \begin{cases} = (0) & \text{if } i \neq j \text{ and } k \neq i_0, \\ \subset D(L_i, L_j) & \text{if } i \neq j \text{ and } k = i_0, \\ \subset \bar{D}(L_i) & \text{if } i = j \text{ and } k = i_0, \\ \subset \bar{D}(L_k) & \text{if } i = j = i_0, \\ = (0) & \text{if } i = j \neq i_0 \text{ and } k \neq i_0, \end{cases}$$

and

$$[D(L_i, L_k), \bar{D}(L_i) + \bar{D}(L_k)] = (0) \quad \text{if } i \neq i_0 \text{ and } k \neq i_0.$$

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In the first case, we have

$$\mathfrak{M}^{(1)} \subset \sum_{i,\, j \neq i_0,\, i \neq j} D(L_i, L_j) + \sum_{i \neq i_0} \bar{D}(L_i)$$

and therefore $\mathfrak{M}^{(2)} = (0)$. In the second case, we have $\mathfrak{M}^{(1)} = (0)$. Thus $\mathfrak{M}$ is a solvable subalgebra of $D(L)$. Furthermore, by Lemma 1 (3) and Lemma 2 (1), it is immediate that $\mathfrak{M}$ is an ideal of $D(L)$. Therefore, if $D(L_i)$ is solvable for each i, then $\sum_{i=1}^{n} D(L_i)$ is solvable. Since $D(L) = \sum_{i=1}^{n} D(L_i) + \mathfrak{M}$, it follows that $D(L)$ is solvable. The converse is evident and (5) is proved.

To prove (2) (resp. (4)), let L be any Lie algebra (resp. a Lie algebra such that $\dim (Z(L) + [L, L]/[L, L]) \leqslant 1$). If $D(L_i)$, for each i, is the direct sum of a semi-simple ideal and the non-abelian nilpotent radical (resp. the radical), then it follows from Theorem 2 (resp. Theorem 1) that L_i, for each i, is the direct sum of a semi-simple ideal S_i and the radical R_i with $D(R_i)$ non-abelian nilpotent (resp. solvable). Put $S = \sum_{i=1}^{n} S_i$ and $R = \sum_{i=1}^{n} R_i$. Then L is the direct sum of a semi-simple ideal S and the radical R. Then $D(R)$ is non-abelian nilpotent by (3) (resp. solvable by (5), since it is clear that $\dim (Z(R) + [R, R]/[R, R]) \leqslant 1$). By Lemma 1 we see that $D(L)$ is the direct sum of a semi-simple ideal $D(S)$ and the non-abelian nilpotent radical (resp. the radical) $D(R)$.

Conversely, if $D(L)$ is the direct sum of a semi-simple ideal and the non-abelian nilpotent radical (resp. the radical), then it follows from Theorem 2 (resp. Theorem 1) that L is the direct sum of a semi-simple ideal S and the radical R with $D(R)$ non-abelian nilpotent (resp. solvable). Then L_i, for each i, is the direct sum of a semi-simple ideal S_i and the radical R_i, and we have $S = \sum_{i=1}^{n} S_i$ and $R = \sum_{i=1}^{n} R_i$. Therefore it follows from (3) that $D(R_i)$ is non-abelian nilpotent (resp. solvable) for all i. By Lemma 1 $D(L_i)$, for each i, is the direct sum of a semi-simple ideal $D(S_i)$ and the non-abelian nilpotent radical (resp. the radical) $D(R_i)$. Thus (2) and (4) are proved. The proof is complete.

We note that, by Lemma 6 and **(6**, Theorem 1**)**, Theorem 4 (2) is equivalent to the following statement: *$D(L)$ is the direct sum of a semi-simple ideal and the radical consisting of nilpotent elements if and only if $D(L_i)$, for each i, is such a direct sum.*

4. In this section we show some properties and some examples of characteristically solvable Lie algebras. We first prove the following

PROPOSITION 2. (1) *If the radical of L is characteristically solvable (resp. the direct sum of a characteristically solvable ideal and a one-dimensional central ideal), then it is a direct summand of L.*

(2) *If the nil-radical of L is characteristically solvable, then the radical of L is also characteristically solvable.*

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Proof. Let S and R be respectively a maximal semi-simple subalgebra and the radical of L. If R satisfies the assumption in (1), then $D(R)$ is solvable, whence the image of the restriction homomorphism of ad S into $D(R)$ is semi-simple and solvable. Therefore the image is (0), which shows that $[R, S] = (0)$, that is, that R is the direct summand of L, proving (1).

To prove (2), suppose that the nil-radical N of L is characteristically solvable. Let $\mathfrak{S}$ be a maximal semi-simple subalgebra of $D(R)$. Then, since N is stable under all derivations of L, it is immediate that the image of the restriction homomorphism of $\mathfrak{S}$ into $D(N)$ is equal to (0), which shows that $N\mathfrak{S} = (0)$. Since $RD \subset N$ for any D in $D(R)$, we have $R\mathfrak{S}^2 = (0)$. Since $\mathfrak{S}$ is completely reducible, it follows that $R\mathfrak{S} = (0)$. Thus $\mathfrak{S} = (0)$, that is, $D(R)$ is solvable. If R is not characteristically solvable, then by Lemma 4 we see that R contains a one-dimensional ideal Z as a direct summand. Therefore N contains Z as its direct summand, whence $Z(N) \not\subset [N, N]$, contrary to the characteristic solvability of N. Thus we conclude that R is characteristically solvable.The proof is complete.

We remark that, in Proposition 2 (2), we cannot assert that the nil-radical of L is the radical of L, though it is true for characteristic nilpotence case (cf. Example 2).

As a generalization of **(6**, Theorem 4**)**, we prove

PROPOSITION 3. *If a Cartan subalgebra of L is characteristically solvable, then L is solvable.*

Proof. Let S and R be a maximal semi-simple subalgebra and the radical of L respectively. Then a Cartan subalgebra H of L is the sum of a Cartan subalgebra H_1 of S and a subalgebra of R, and H_1 is a central ideal of H **(2**, Proposition 1**)**. Therefore, if H is characteristically solvable, then we have $H_1 = (0)$, whence $S = (0)$, that is, L is solvable.

We here remark that it is easy to construct a solvable Lie algebra which is not nilpotent and whose Cartan subalgebras are characteristically solvable.

PROPOSITION 4. *Let L be the direct sum of the ideals L_i $(i = 1, 2, \ldots, n)$. Then L is characteristically solvable if and only if L_i, for each i, is characteristically solvable.*

Proof. This is immediate from Theorem 4 (5) and from the fact that $Z(L) \subset [L, L]$ if and only if $Z(L_i) \subset [L_i, L_i]$ for all i.

PROPOSITION 5. *Let L be a Lie algebra which has no proper subalgebra whose derived algebra is equal to $[L, L]$. If $[L, L]$ is characteristically solvable, then L is characteristically solvable.*

Proof. Let $\mathfrak{S}$ be a maximal semi-simple subalgebra of $D(L)$ and suppose that $\mathfrak{S} \neq (0)$. Then there exists a non-zero semi-simple derivation D in $\mathfrak{S}$. Let H be the set of all elements of L annihilated by D. Then H is an ideal of

L containing $[L, L]$, since we have $[L, L]\,\mathfrak{S} = (0)$ by the characteristic solvability of $[L, L]$. There exists a non-empty subspace U of L such that

$$L = U + H \quad \text{and} \quad UD \subset U.$$

We assert that $[U, H] = (0)$. In fact, let $\bar{K}$ be the algebraic closure of the basic field K, let $L^{\bar{K}} = L \otimes_{\bar{K}} \bar{K}$ and $U^{\bar{K}} = U \otimes_{\bar{K}} \bar{K}$. As usual, we consider that $U^{\bar{K}} \subset L^{\bar{K}}$ and we identify D with its extended derivation of $L^{\bar{K}}$. Let λ be an eigenvalue of D and let x be an element of $U^{\bar{K}}$ corresponding to λ. Then, for any element y of H, we have

$$[x, y]D = [xD, y] = \lambda[x, y] = 0.$$

Since $\lambda \neq 0$, we have $[x, y] = 0$. Since $U^{\bar{K}}$ is spanned by those elements x, it follows that $[U^{\bar{K}}, H] = (0)$ and therefore $[U, H] = (0)$, as was asserted. It now follows that

$$[[U, U], L] \subset [[U, L], U] = [[U, U], U] \subset [H, U] = (0).$$

Therefore we have

$$[L, L] = [U, U] + [H, H],$$

where $[U, U]$ is a central ideal of $[L, L]$. Since $[L, L]$ is characteristically solvable, it follows that $[U, U] \subset [H, H]$ and therefore that $[L, L] = [H, H]$. This contradicts our hypothesis since H is a proper subalgebra of L. Thus we see that $\mathfrak{S} = (0)$, that is, that $D(L)$ is solvable. By our hypothesis, L cannot contain a central ideal as its direct summand. Therefore we conclude that L is characteristically solvable. The proof is complete.

EXAMPLE 1. Let L be the Lie algebra over K with a basis x_1, x_2 such that

$$[x_1, x_2] = x_2, \quad [x_2, x_1] = -x_2.$$

As is well known, L is a solvable Lie algebra whose derivation algebra is isomorphic to L. Therefore L is characteristically solvable.

EXAMPLE 2. Let L be the algebra over K described in terms of a basis x_1, $x_2, \ldots, x_5$ by the following multiplication table:

$$\begin{aligned} &[x_1, x_2] = x_2, \quad [x_1, x_3] = x_3, \quad [x_1, x_4] = 2x_4, \\ &[x_1, x_5] = 3x_5, \quad [x_2, x_3] = x_4, \quad [x_2, x_4] = x_5. \end{aligned}$$

In addition $[x_i, x_j] = -[x_j, x_i]$ and for $i < j$ $[x_i, x_j] = 0$ if it is not in the table above. Then L is a solvable Lie algebra and $[L, L] = (x_2, \ldots, x_5)$. Let D be a derivation of L and let $x_iD = \sum_{j=1}^{5} \lambda_{ij}x_j$ $(i = 1, 2, \ldots, 5)$. Then the matrix of D is

$$\begin{bmatrix} 0 & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ 0 & \lambda_{22} & \lambda_{23} & \lambda_{13} & \frac{1}{2}\lambda_{14} \\ 0 & 0 & \lambda_{33} & -\lambda_{12} & 0 \\ 0 & 0 & 0 & \lambda_{22} + \lambda_{33} & -\lambda_{12} \\ 0 & 0 & 0 & 0 & 2\lambda_{22} + \lambda_{33} \end{bmatrix}.$$

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Let $[L, L] = (y_1, \ldots, y_4)$ with $y_i = x_{i+1}$ $(i = 1, \ldots, 4)$. Let D' be a derivation of $[L, L]$ and let $y_i D' = \sum_{j=1}^{4} \mu_{ij} y_j$ $(i = 1, \ldots, 4)$. Then the matrix of D' is

$$\begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} & \mu_{14} \\ 0 & \mu_{22} & \mu_{23} & \mu_{24} \\ 0 & 0 & \mu_{11} + \mu_{22} & \mu_{23} \\ 0 & 0 & 0 & 2\mu_{11} + \mu_{22} \end{bmatrix}.$$

Therefore L and $[L, L]$ are characteristically solvable Lie algebras. The nil-radical of L is $[L, L]$ and there is no proper subalgebra of L whose derived algebra is equal to $[L, L]$.

5. In this section, we summarize some obtained results and give some remarks as the partial answers to the questions stated in the beginning of the introduction.

For the first question in the introduction, we have the following

THEOREM 5. *We have the following statements for the derivation algebras of Lie algebras*:

(1) *An abelian derivation algebra is one-dimensional and consists of semi-simple elements;*

(2) *A non-abelian nilpotent derivation algebra consists of nilpotent elements;*

(3) *A reductive derivation algebra is the direct sum of a semi-simple ideal and a one-dimensional ideal consisting of semi-simple elements;*

(4) *A derivation algebra, which is the direct sum of a semi-simple ideal and a non-abelian nilpotent ideal, is the direct sum of a semi-simple ideal and an ideal which is another derivation algebra consisting of nilpotent elements.*

It would be interesting to know

(1) whether or not there exists a characteristically nilpotent derivation algebra;

(2) whether or not there exists a derivation algebra whose radical consists of nilpotent elements and is not a direct summand.

In connection with (1), we note that there exists a characteristically solvable derivation algebra, for instance, the derivation algebra of the Lie algebra in Example 1. In connection with (2), we note that, if $D(L)$ is such a derivation algebra of a solvable Lie algebra L, then L must be nilpotent, $L^3 \neq (0)$ and $\dim L \geqslant 6$. In fact, it is clear that L is nilpotent. Write $D(L) = \mathfrak{S} + \mathfrak{N}$ where $\mathfrak{S}$ is a semi-simple subalgebra and $\mathfrak{N}$ is the radical of $D(L)$. If $L^3 = (0)$, then there exists a subspace U of L such that

$$L = U + L^2, \quad U \cap L^2 = (0), \quad \text{and} \quad U\,\mathfrak{S} \subset U.$$

Define a derivation D of L in the following way:

$$\begin{aligned} x\,D &= x && \text{for } x \text{ in } U, \\ y\,D &= 2y && \text{for } y \text{ in } L^2. \end{aligned}$$

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Then it is immediate that $[D, \mathfrak{S}] = (0)$. Therefore D is a semi-simple derivation of L which does not belong to $\mathfrak{S}$. Write $D = D_1 + D_2$ with D_1 in $\mathfrak{S}$ and D_2 in $\mathfrak{N}$. Let $D_1 = S_1 + N_1$ be the Jordan sum decomposition of D_1: S_1 and N_1 are respectively semi-simple and nilpotent derivations of L and $[S_1, N_1] = 0$. Since $[D, D_1] = 0$, we see that $[D, S_1] = 0$ and $[D, N_1] = 0$. Therefore $D - S_1$ is a semi-simple derivation of L and $[D - S_1, N_1] = 0$, which shows that $D_2 = (D - S_1) + (-N_1)$ is the Jordan sum decomposition of D_2. Since D_2 is nilpotent, it follows that $D - S_1 = 0$, that is, that $D = S_1$. Since $\mathfrak{S}$ is splittable, D belongs to $\mathfrak{S}$, which is a contradiction. Therefore $L^3 \neq (0)$. All the nilpotent Lie algebras whose dimensions are $\leqslant 5$ are determined in **(3**, Proposition 1**)**. Therefore we can calculate the derivation algebras of those Lie algebras to see that their radicals contain non-zero semi-simple derivations. Thus dim $L \geqslant 6$.

As for the second question in the introduction, we have the following

THEOREM 6. *Let $D(L)$ be the derivation algebra of a Lie algebra L. Then:*

(1) *$D(L)$ is abelian if and only if L is one-dimensional;*

(2) *$D(L)$ is non-abelian nilpotent if and only if L is characteristically nilpotent;*

(3) *$D(L)$ is non-nilpotent solvable if and only if either L is characteristically solvable and not characteristically nilpotent or L is the direct sum of a characteristically solvable ideal and a one-dimensional central ideal;*

(4) *$D(L)$ is reductive (resp. semi-simple) if and only if L is reductive (resp. semi-simple);*

(5) *$D(L)$ is the direct sum of a semi-simple ideal and the non-abelian nilpotent radical if and only if L is the direct sum of a semi-simple ideal and a characteristically nilpotent ideal;*

(6) *$D(L)$ is the direct sum of a semi-simple ideal and the non-nilpotent radical if and only if either L is the direct sum of a semi-simple ideal and a characteristically solvable ideal which is not characteristically nilpotent or L is the direct sum of a semi-simple ideal, a characteristically solvable ideal and a one-dimensional central ideal.*

Finally, we note the following example, which shows that non-isomorphic Lie algebras can have isomorphic derivation algebras:

EXAMPLE 3. Let A_1, A_2 be abelian Lie algebras such that dim $A_1 \neq$ dim A_2. Then $D(A_i)$ is the direct sum of a semi-simple ideal S_i and the one-dimensional ideal Z_i $(i = 1, 2)$. Let L_1 (resp. L_2) be the direct sum of S_2 and A_1 (resp. S_1 and A_2). Then, by using Lemma 1, we see that $D(L_1)$ (resp. $D(L_2)$) is the direct sum of ideals $D(S_2)$, S_1, and Z_1 (resp. $D(S_1)$, S_2, and Z_2). Since $D(S_i)$ is isomorphic to S_i $(i = 1, 2)$, it follows that $D(L_1)$ is isomorphic to $D(L_2)$. But L_1 is not isomorphic to L_2.

References

1. C. Chevalley, *Théorie des groupes de Lie*, tome III, Act. Sci. Ind., no. 1226 (Paris, 1955).
2. J. Dixmier, *Sous-algèbres de Cartan et décompositions de Levi dans les algèbres de Lie*, Trans. Roy. Soc. Canada, Series III, Section III, *20* (1956), 17–21.
3. ———, *Sur les représentations unitaires des groupes de Lie nilpotents III*, Can. J. Math., *10* (1958), 321–348.
4. J. Dixmier and W. G. Lister, *Derivations of nilpotent Lie algebras*, Proc. Amer. Math. Soc., *8* (1957), 155–158.
5. G. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math., *64* (1942), 677–694.
6. G. Leger and S. Tôgô, *Characteristically nilpotent Lie algebras*, Duke Math. J., *26* (1959), 623–628.
7. S. Tôgô, *On the derivations of Lie algebras*, J. Sci. Hiroshima Univ., Ser. A, *19* (1955), 71–77.

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