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A note on order topologies on ordered tensor products

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If E, F are regularly ordered vector spaces the tensor product $E \otimes F$ can be ordered by the conic hull K^{π} of tensors, $x \otimes y$ with $x \geq 0$ in E and $y \geq 0$ in F, or by the cone $K^{\otimes}$ of tensors $\Phi \in E \otimes F$ such that $\Phi(x', y') \geq 0$ for positive linear functionals x', y' on E, F.

If E, F are locally convex spaces the tensor product can be given the π-topology which is defined by seminorms $p_{\alpha} \otimes q_{\beta}$ where $\{p_{\alpha}\}$, $\{q_{\beta}\}$ are classes of seminorms defining the topologies on E, F. The tensor product can also be given the ε-topology which is the topology of uniform convergence on equicontinuous subsets $J \times H$ of $E' \times F'$. The main result of this note is that if the regularly ordered vector spaces E, F carry their order topologies then the order topology on $E \otimes F$ is the π-topology when $E \otimes F$ is ordered by K^{π}, and the ε-topology when $E \otimes F$ is ordered by $K^{\otimes}$.

1.

The purpose of this note is to give a description of topologies on completed tensor products of certain ordered vector spaces. The principal result is that if two regularly ordered vector spaces with order units carry their order topologies then the π-topology on the tensor product is just the order topology for the projective ordering, and the ε-topology

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is the order topology for the biprojective ordering.

An explanation of the undefined terms in ordered vector space theory can be found in Peressini [2] and those concerning ordered tensor products in Peressini and Sherbert [3].

2.

Suppose that E, F are ordered vector spaces. The *projective wedge*, K^{π}, in the tensor product $E \otimes F$ is the set consisting of all tensors $\sum_i x_i \otimes y_i$ with $x_i \geq \theta$ in E and $y_i \geq \theta$ in F. If E and F are regularly ordered (that is, the order duals of E, F separate points of E, F) then K^{π} is a cone in $E \otimes F$. A larger cone $K^{\otimes}$, the biprojective cone, can also be defined in this case: a tensor $\sum_i x_i \otimes y_i$ belongs to $K^{\otimes}$ if for $x' \geq \theta$ in E', $y' \geq \theta$ in F' (for the dual ordering of these spaces)

$$\sum_i \left\langle x_i, x' \right\rangle \left\langle y_i, y' \right\rangle \geq 0 .$$

Assume now that E, F are almost-Archimedean spaces with order units 1_E, 1_F respectively. The reducible tensor $1_E \otimes 1_F$ is then an order unit for $E \otimes F$ ordered by the projective cone K^{π}. The order topologies on E, F are generated by the gauge functionals p_E, p_F of the intervals $[-1_E, 1_E]$, $[-1_F, 1_F]$. Denote by $p_{E \otimes F}$ the gauge functional of the projective interval $[-1_E \otimes 1_F, 1_E \otimes 1_F]$.

LEMMA. $p_{E \otimes F} \leq p_E \otimes p_F$.

Proof. Firstly recall that for $\Phi \in E \otimes F$

$$p_{E \otimes F}(\Phi) = \inf\{\lambda > 0 : -\lambda 1_E \otimes 1_F \leq \Phi \leq \lambda 1_E \otimes 1_F\}$$

and

$$p_E \otimes p_F(\Phi) = \inf\left\{\sum_i p_E(x_i) p_F(y_i) \ : \ \Phi = \sum_i x_i \otimes y_i\right\} .$$

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For any $\varepsilon > 0$, $\left(p_E(x_i)+\varepsilon\right)1_E \geq x_i$ and $\left(p_F(y_i)+\varepsilon\right)1_F \geq y_i$, for all i. Thus,

$$\sum_i \left(p_E(x_i)+\varepsilon\right)\left(p_F(y_i)+\varepsilon\right)1_E \otimes 1_F - \sum_i x_i \otimes y_i$$
$$= \sum_i \left(p_E(x_i)+\varepsilon\right)1_E \otimes \left(p_F(y_i)+\varepsilon\right)1_F - x_i \otimes y_i \in K^\pi .$$

That is, for any $\varepsilon > 0$,

$$\sum_i \left(p_E(x_i)+\varepsilon\right)\left(p_F(y_i)+\varepsilon\right)1_E \otimes 1_F \geq \sum_i x_i \otimes y_i$$

and similarly,

$$-\sum_i \left(p_E(x_i)+\varepsilon\right)\left(p_F(y_i)+\varepsilon\right)1_E \otimes 1_F \leq \sum_i x_i \otimes y_i$$

which means that $\sum_i \left(p_E(x_i)+\varepsilon\right)\left(p_F(y_i)+\varepsilon\right)$ is one of those $\lambda > 0$ for which $-\lambda 1_E \otimes 1_F \leq \Phi \leq \lambda 1_E \otimes 1_F$. From this it follows that $p_{E\otimes F}(\Phi) \leq p_E \otimes p_F(\Phi)$.

Now consider the spaces E, F, as before, equipped with their order topologies which are norm topologies generated by the norms p_E, p_F respectively.

Consider then the completed π-product $E \hat{\otimes}_\pi F$ and the closure $\hat{K}^\pi$ of (the image of) K^π in $E \hat{\otimes}_\pi F$:

THEOREM 1. *$\hat{K}^\pi$ is a normal cone in $E \hat{\otimes}_\pi F$, $1_E \otimes 1_F$ is an order unit in $E \hat{\otimes}_\pi F$ ordered by $\hat{K}^\pi$, and the topology on $E \hat{\otimes}_\pi F$ is the order topology.*

Proof. The previous lemma shows that the projective cone K^π is normal in $E \otimes_\pi F$ and therefore in $E \hat{\otimes}_\pi F$. Since the closure of a normal cone is again a normal cone (Peressini [2], p. 63), it follows that $\hat{K}^\pi$ is normal in $E \hat{\otimes}_\pi F$. If $\psi \in E \hat{\otimes}_\pi F$ then there are null sequences

(x_i) in E, (y_i) in F and an element $(\lambda_i) \in l'$ such that $\psi = \sum_{i=1}^{\infty} \lambda_i x_i \otimes y_i$ where the series is absolutely convergent.

For any $\varepsilon > 0$, and for all i,

$$-\left(p_{E\otimes F}(x_i \otimes y_i)+\varepsilon/2^i\right)1_E \otimes 1_F \le x_i \otimes y_i \le \left(p_{E\otimes F}(x_i \otimes y_i)+\varepsilon/2^i\right)1_E \otimes 1_F$$

for the order on $E \hat{\otimes}_\pi F$ given by K^π and thus also for the order given by $\hat{K}^\pi$.

Then for any $N \ge 1$,

$$-\sum_{i=1}^{N} \left(p_{E\otimes F}(x_i \otimes y_i)+\varepsilon/2^i\right)\lambda_i 1_E \otimes 1_F \le \sum_{i=1}^{N} \lambda_i x_i \otimes y_i$$
$$\le \sum_{i=1}^{N} \left(p_{E\otimes F}(x_i \otimes y_i)+\varepsilon/2^i\right)\lambda_i 1_E \otimes 1_F .$$

Since $p_{E\otimes F} \le p_E \otimes p_F$ and (x_i), (y_i) are null sequences (hence bounded) it follows that for some $M > 0$, $p_{E\otimes F}(x_i \otimes y_i) \le M$ for all i. Thus $\left(p_{E\otimes F}(x_i \otimes y_i)\lambda_i\right) \in l'$ so that for some $\lambda > 0$, $-\lambda 1_E \otimes 1_F \le \psi \le \lambda 1_E \otimes 1_F$ for the order on $E \hat{\otimes}_\pi F$ given by $\hat{K}^\pi$.

That is, $1_E \otimes 1_F$ is an order unit in $E \otimes_\pi F$. The last assertion follows from the fact that in a Banach space with an order unit and a closed normal cone, the norm topology is just the order topology.

A similar, but simpler, result holds for $E \otimes F$ with the ε-topology and biprojective ordering:

THEOREM 2 (Ellis [1]). *With E, F as in the previous theorem the ε-topology on $E \otimes F$ is the order topology when $E \otimes F$ is biprojectively ordered, and $1_E \otimes 1_F$ is an order unit for $E \hat{\otimes}_\varepsilon F$ ordered by the closure, $\hat{K}^\otimes$, of the image of $K^\otimes$ in $E \hat{\otimes}_\varepsilon F$.*

Proof. Write

$$S = \left\{x' \in E' : x' > 0, \left\langle 1_E, x' \right\rangle = 1\right\}$$

and

$$T = \left\{y' \in F' : y' > 0, \left\langle 1_F, y' \right\rangle = 1\right\} .$$

Then S, T are equicontinuous subsets of E', F', and

$$-\lambda 1_E \otimes 1_F \le \Phi \le \lambda 1_E \otimes 1_F \Longleftrightarrow [\Phi(x', y')] \le \lambda \quad \text{for all} \quad (x', y') \in S \times T$$
$$\Longleftrightarrow \Phi \in U(S, T; \lambda)$$

so that the order topology agrees with the ε-topology. The statement about $1_E \otimes 1_F$ is a consequence of the following proposition, the proof of which is short and easily established:

PROPOSITION. *Let A be an ordered vector space with an order unit 1 and with a locally convex topology for which the positive cone is normal. Then 1 is an order unit for the topological completion $\hat{A}$ of A, ordered by the closure of the positive cone in A, if and only if Cauchy nets in A are order bounded.*

The order topology on $E \hat{\otimes}_\pi F$ ordered by $\hat{K}^\pi$ is generated by the gauge $\hat{p}_{E\otimes F}$ of the interval

$$[-1_E \otimes 1_F, 1_E \otimes 1_F] = \left\{\psi \in E \hat{\otimes}_\pi F : 1_E \otimes 1_F - \psi \in \hat{K}^\pi \ni 1_E \otimes 1_F + \psi\right\}$$

so that $p_{E\otimes F}(\Phi) \le \hat{p}_{E\otimes F}(\Phi)$ for $\Phi \in E \otimes F$ and thus the following result holds:

THEOREM 3. *Let E, F be regularly ordered vector spaces with order units $1_E, 1_F$ equipped with their order topologies. The π-topology on $E \otimes F$ is just the order topology when $E \otimes F$ is ordered by the projective cone K^π, and the ε-topology on $E \otimes F$ is the order topology when $E \otimes F$ is ordered by the biprojective cone $K^\otimes$.*

A similar result for regularly ordered vector spaces without order units can be established by making use of the fact that a regularly ordered vector space with its order topology is an inductive limit of regularly ordered vector spaces with order units, each carrying its topology.

References

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