WHITEHEAD GROUPS OF SEMIDIRECT PRODUCTS OF FREE GROUPS II

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Let G be a group. We denote the Whitehead group of G by Wh G and the projective class group of the integral group ring $\mathbb{Z}(G)$ of G by $\tilde{K}_0\mathbb{Z}(G)$. For undefined terminologies used in the paper, we refer to [**3**] and [**6**].

First we recall the definition of semidirect product of groups. Let G be a group, α an automorphism of G and F a free group generated by $\{t_\lambda\}_{\lambda\in\Lambda}$. If w is a word in t_λ defining an element of F, we denote by $|w|$ the total exponent sum of the t_λ appearing in w. The semidirect product $G\times_\alpha F$ of G and F with respect to α is defined as follows: $G\times_\alpha F = G\times F$ as sets and multiplication in $G\times_\alpha F$ is given by

$$(g, w)(g', w') = (g\alpha^{-|w|}(g'), ww'),$$

for any (g, w), (g', w') in $G\times_\alpha F$. In particular, if F is an infinite cyclic group $T=\langle t\rangle$ generated by t, we have the semidirect product $G\times_\alpha T$ of G and T with respect to α.

Next we recall the following definition of a group of type n [**6**, p. 214]: Any group possessing only a single element will be called a group of type 0. Inductively, we define G to be a group of type n if $G=H\times_\alpha T$ where H is a group of type $n-1$. In particular, any free abelian group of finite rank n is a group of type n.

Now let G be a group of type n, $F_1, F_2, \ldots$ a set of free groups, each of rank at least two, and let

$$H_k = G\times F_1\times\ldots\times F_k \tag{1}$$

be the direct product of $G, F_1, \ldots, F_k$ $(k=1,2,\ldots)$. Let α be an automorphism of H_k which leaves all but one of the F_j $(j=1,\ldots,k)$ pointwise fixed. Then, in [**5**], we have shown:

THEOREM 1 [**5**, Theorem 1]. *For each* $k=1,2,\ldots,$

$$\text{Wh}\,H_k=0,\qquad \tilde{K}_0\mathbb{Z}(H_k)=0\quad\textit{and}\quad \tilde{C}(\mathbb{Z}(H_k).\,\text{id})=0.$$

THEOREM 2 [**5**, Theorem 2]. *For each* $k=1,2,\ldots,$

$$\text{Wh}(H_k\times_\alpha T)=0,\qquad \tilde{K}_0\mathbb{Z}(H_k\times_\alpha T)=0\quad\textit{and}\quad \tilde{C}(\mathbb{Z}(H_k\times_\alpha T),\text{id})=0,$$

where T is any infinite cyclic group.

Let $F=\langle t_\lambda\rangle_{\lambda\in\Lambda}$ be another free group and $H_k\times_\alpha F$ the semidirect product of H_k and F with respect to α $(k=1,2,\ldots)$. Then the purpose of this paper is to extend these results to:

MAIN THEOREM. *For each* $k=1,2,\ldots,$

$$\text{Wh}(H_k\times_\alpha F)=0,\qquad \tilde{K}_0\mathbb{Z}(H_k\times_\alpha F)=0\quad\textit{and}\quad \tilde{C}(\mathbb{Z}(H_k\times_\alpha F),\text{id})=0.$$

REMARK. The results clearly extend those of [**3**]. Moreover, since the group D defined in [**4**] is just one of our H_k, these results also give an affirmative answer to the question mentioned at the end of [**4**].

As noted in [**4**, p. 29], we have for $k=1,2,\ldots,$

$$H_k \times_\alpha F \cong (H_k \times F') *_{H_k} (H_k \times_\alpha T), \tag{2}$$

where $T=\langle t_{\lambda_0}\rangle$ for some λ_0 and F' is the free group generated by $\langle t_\lambda^{-1} t_{\lambda_0} \mid \lambda \neq \lambda_0\rangle$.

The following direct sum decomposition for free product of groups with amalgamation is due to Waldhausen (cf. [**9**, §5] or [**10**]).

THEOREM 3. *Let A and B be two groups and C a common subgroup of A and B. Let $A *_C B$ be the free product with amalgamation and $\gamma: C \to A$, $\beta: C \to B$ the inclusions defining $A *_C B$. Then there is a natural direct sum splitting for* $\mathrm{Wh}(A *_C B)$:

$$\mathrm{Wh}(A *_C B) \cong \mathrm{Wh}(A, B; C) \oplus \tilde{K}_0(C; A, B) \oplus \tilde{C}(C; A, B), \tag{3}$$

where $\mathrm{Wh}(A, B; C)$ *and* $\tilde{K}_0(C; A, B)$ *are given by the following exact sequences*

$$\mathrm{Wh}\, C \xrightarrow{\gamma_* \oplus -\beta_*} \mathrm{Wh}\, A \oplus \mathrm{Wh}\, B \longrightarrow \mathrm{Wh}(A, B; C) \longrightarrow 0;$$

$$0 \longrightarrow \tilde{K}_0(C; A, B) \longrightarrow \tilde{K}_0\mathbb{Z}(C) \xrightarrow{\gamma_* \oplus -\beta_*} \tilde{K}_0\mathbb{Z}(A) \oplus \tilde{K}_0\mathbb{Z}(B);$$

and $\tilde{C}(C; A, B)$ *is given in* [**9**, p. 2.2] (*we refer the readers to* [**9**] *for a detailed description*).

Moreover, if C is such that $\mathbb{Z}(C)$ is coherent and of finite global dimension, then the exotic summand $\tilde{C}(C; A, B) = 0$, so that in this case,

$$\mathrm{Wh}(A *_C B) \cong \mathrm{Wh}(A, B; C) \oplus \tilde{K}_0(C; A, B).$$

Finally, we recall

THEOREM 4 [**5**, Theorem 4]. *Let G be a group of some particular form such that G and $G \times T$ are of the same form, where T is any infinite cyclic group. Suppose that the Whitehead group of any such group is trivial. Let $H'_k = G \times F_1 \times \ldots \times F_k$, where $F_1, \ldots, F_k$ are free groups each of rank at least two. Then, for each $k = 1, 2, \ldots,$*

$$\mathrm{Wh}\, H'_k = 0, \qquad \tilde{K}_0\mathbb{Z}(H'_k) = 0 \quad \textit{and} \quad \tilde{C}(\mathbb{Z}(H'_k), \mathrm{id}) = 0.$$

Proof of the main theorem. First, we prove the theorem for $k=1$. As noted in (2), we have,

$$H_1 \times_\alpha F \cong (H_1 \times F') *_{H_1} (H_1 \times_\alpha T),$$

where $T=\langle t_{\lambda_0}\rangle$ for some λ_0 and $F' = \langle t_\lambda^{-1} t_{\lambda_0} \mid \lambda \neq \lambda_0\rangle$.

We have pointed out in [**5**] that if G is a group of type n, then $H_1 = G \times F_1$ is coherent and of finite global dimension (cf. [**6**, Theorem 25] and [**2**, **7**]). Thus the exotic summand $\tilde{C}(H_1; H_1 \times F', H_1 \times_\alpha T) = 0$ and so the Waldhausen direct sum decomposition for $\mathrm{Wh}(H_1 \times_\alpha F)$ becomes

$$\mathrm{Wh}(H_1 \times_\alpha F) \cong \mathrm{Wh}(H_1 \times F', H_1 \times_\alpha T; H_1) \oplus \tilde{K}_0(H_1; H_1 \times F', H_1 \times_\alpha T)$$

and the following sequences are exact:

$$\mathrm{Wh}\, H_1 \to \mathrm{Wh}(H_1 \times F') \oplus \mathrm{Wh}(H_1 \times_\alpha T) \to \mathrm{Wh}(H_1 \times F', H_1 \times_\alpha T; H_1) \to 0;$$

$$0 \to \tilde{K}_0(H_1; H_1 \times F', H_1 \times_\alpha T) \to \tilde{K}_0\mathbb{Z}(H_1) \to \tilde{K}_0\mathbb{Z}(H_1 \times F') \oplus \tilde{K}_0\mathbb{Z}(H_1 \times_\alpha T).$$

But $\mathrm{Wh}(H_1 \times F') = 0$, $\tilde{K}_0\mathbb{Z}(H_1) = 0$ by Theorem 1 and $\mathrm{Wh}(H_1 \times_\alpha T) = 0$ by Theorem 2. Thus $\mathrm{Wh}(H_1 \times F', H_1 \times_\alpha T; H_1) = 0$ and $\tilde{K}_0(H_1; H_1 \times F', H_1 \times_\alpha T) = 0$. Hence $\mathrm{Wh}(H_1 \times_\alpha F) = 0$ and this proves the theorem for $k = 1$.

Next, since G is a group of type n, $G \times T$ is a group of type $n+1$. Thus G and $G \times T$ are of the same form and so $H_1 \times_\alpha F = (G \times F_1) \times_\alpha F$ and $(H_1 \times_\alpha F) \times T = (G \times T \times F_1) \times_\alpha F$ are of the same form. We have shown above that the Whitehead group of any such group $H_1 \times_\alpha F$ is trivial. Hence the assertions follow immediately from Theorem 4.

This completes the proof.

In addition to the results in the main theorem, we see that

$$\tilde{C}(H_k; H_k \times F', H_k \times_\alpha T) = 0 \qquad (k = 2, 3, \ldots),$$

by Theorem 3, although $\mathbb{Z}(H_k)$ is not coherent (cf. [**7**, p. 42]).

We conclude the paper by the following remark.

Remark. Let H_k, α and F be as given above. Let $\mathcal{A} = \{\alpha^{n_\lambda}\}_{\lambda \in \Lambda}$ a set of automorphisms α^{n_λ} of H_k, where each n_λ is an integer, for all $\lambda \in \Lambda$. Let $H_k \times_{\mathcal{A}} F$ be the semidirect product of H_k and F with respect to $\mathcal{A}$ (cf. [**5**]). Then a slight modification of arguments, as given above, will give

$$\mathrm{Wh}(H_k \times_{\mathcal{A}} F) = 0, \qquad \tilde{K}_0\mathbb{Z}(H_k \times_{\mathcal{A}} F) = 0 \quad \text{and} \quad \tilde{C}(\mathbb{Z}(H_k \times_{\mathcal{A}} F), \mathrm{id}) = 0.$$

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