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Secular evolution of exoplanetary systems and close encounters

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Abstract. We investigate the secular evolution of non-resonant exoplanetary systems consisting of a central star and two co-planar planets using a semi-numerical averaging method of the first order in planetary masses (in this case equivalent to "averaging by scissors" or simply dropping the fast periodic terms). The resulting Hamiltonian level curves for different exoplanetary systems were compared to those obtained by direct numerical integration. Studying the dependence of the reliability of the averaging method (as well as chaoticity of numerically integrated trajectories) upon the initial conditions, we found that the averaging methods fails even for Hill stable systems. Based on the Hill stability criterion we introduced empirically a more restrictive stability condition, that enabled us to give an estimate for the region of validity of the averaging method in the plane of initial conditions.

Keywords. Celestial mechanics, exoplanets, stability

1. Introduction

There are several possibilities to study the secular behaviour of exoplanetary systems consisting of a star and two co-planar planets, which are not in a mean motion resonance. The simplest approach is the classical Laplace-Lagrange secular solution, which takes into account only terms up to the second order in eccentricities. The resulting equations of motion may be written as linear differential equations which can be solved easily. Very interesting insight into topology and artificial singularities of this problem, when reduced to one degree of freedom, was presented by Pauwels (1983), where the representation of motion is depicted on a sphere. But being limited to almost circular orbits this method will fail to describe correctly the secular evolution of most exoplanetary systems which have in general larger eccentricities.

A better possibility is the expansion of the perturbing function in powers of eccentricities as was done by Libert & Henrard (2005) who showed that an expansion to 12th order

Table 1. Elements of exosystems used in our calculations

System	$M\,[M_\odot]$	$M_P\,[M_J \sin I]$	$P[d]$	$a[AU]$	e	$\varpi[°]$
HD 12661	1.11	2.34	262.53	0.83	0.361	116.3
		1.83	1679	2.86	0.017	218.0
HD 169830	1.43	2.9	225.62	0.82	0.31	328.0
		4.1	2100	3.62	0.33	72.0
HD 108874	1.0	1.37	395.27	1.05	0.068	70.0
		1.02	1599	2.68	0.253	200

is able to describe correctly most of the exoplanetary systems. The third possibility, the so called averaging method, was used for instance by Michtchenko & Malhotra (2004) and for 3-D problem by Michtchenko *et al.* (2006). Here the Hamiltonian is numerically averaged over the mean anomalies and is therefore not restricted to low degrees in eccentricities. The dependence of variables may be obtained by means of level curves of the Hamiltonian. On the other hand this method does not give the transformation between osculating and mean elements. Finally numerical integration can be used to check, how well the aforementioned methods solve the problem.

In this study we try to understand how well the averaging method describes the secular evolution in dependence on the initial conditions to come to conclusions on the validity of this method.

2. Canonical variables and the averaged Hamiltonian

Let us consider a system consisting of a star of mass M and two co-planar planets of mass m_1 and m_2. Introducing the relative position vectors $\mathbf{r}_j, (j=1,2)$ of the planets to the star and their conjugate momenta $\mathbf{p}_j = m_j \dot{\boldsymbol{\rho}}_j$, where $\boldsymbol{\rho}_j$ are the position vectors relative to the centre of gravity, we get the four degree of freedom Hamiltonian (Michtchenko & Malhotra 2004) $H = H_0 + H_1$, with

$$H_0 = \sum_{j=1}^{2} \left(\frac{{\mathbf{p}_j}^2}{2\mu_j} - \frac{G(M+m_j)\mu_j}{r_j} \right), \tag{2.1}$$

and

$$H_1 = -G\frac{m_1 m_2}{\Delta} + \frac{\mathbf{p}_1 \mathbf{p}_2}{M}. \tag{2.2}$$

G is the gravitational constant, $\mu_j = M m_j/(M+m_j)$ is the reduced mass of the j-th body and $\Delta = |\mathbf{r}_2 - \mathbf{r}_1|$.

We may define the formal osculating Keplerian elements $a_j, e_j, \varpi_j, \lambda_j$ as the usual elements of the Keplerian motion with H_0 after the perturbation H_1 is switched off and $\mathbf{r}_j$ and $\mathbf{p}_j$ do not change at this moment. The final set of canonical variables is then:

$$\begin{aligned} L_j &= \mu_j \sqrt{G(M+m_j)a_j}, & \lambda_j &= \text{mean longitude}, \\ S_j &= L_j(1-\sqrt{1-e_j^2}), & s_j = -\varpi_j &= \text{minus longitude of periastron}. \end{aligned}$$

The averaged Hamiltonian of the problem is $H_s = H_0 + \bar{H}_1$, where

$$H_0 = -\sum_{j=1}^{2} \frac{G^2(M+m_j)^2 \mu_j^3}{2L_j^2}, \tag{2.3}$$

and

$$\bar{H}_1 = -\frac{1}{4\pi^2} G m_1 m_2 \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\Delta}\, d\lambda_1\, d\lambda_2, \tag{2.4}$$

as the indirect part $\mathbf{p}_1\mathbf{p}_2/M$ does not contribute any secular term.

Being independent of the mean longitudes the conjugate momenta L_j (and thus the semi-major axes) are now constant. Introducing a new canonical set:

$$\begin{aligned} K_1 &= S_1, & k_1 &= s_1 - s_2 = \varpi_2 - \varpi_1 = \Delta\varpi \text{ and} \\ K_2 &= S_1 + S_2, & k_2 &= s_2 \end{aligned}$$

it becomes obvious that the resulting Hamiltonian $H_s = H_s(K_1, k_1)$ depends only on

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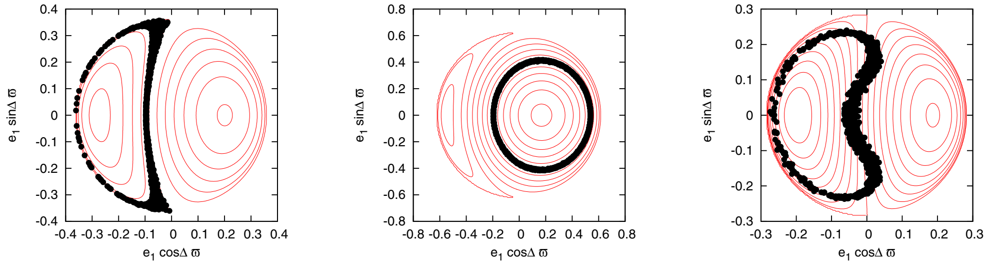

Figure 1. Level curves for the averaged Hamiltonian and numerical integration (dots) for HD 12661 - left, HD 169830 - centre and HD 108874 - right.

one coordinate, representing thus a problem with one degree of freedom. The second constant of motion K_2, introduced as Angular Momentum Deficit (AMD) by Laskar, (Laskar 2000) corresponds to total angular momentum conservation (for constant L_1 and L_2) and couples both eccentricities.

The Hamiltonian $\bar{H}_1$ may be obtained numerically for a fixed value of K_2 from (2.4) and its level curves drawn in the (x_j, y_j) plane, where $x_j = e_j \cos k_j$ and $y_j = e_j \sin k_j$. Since due to the conservation of AMD the eccentricities are not independent the level curves can be converted directly into each other.

Fig. 1 shows exemplarily the level curves of the Hamiltonian for 3 exoplanetary systems: HD 12226, HD 169830 and HD 108874. The calculations were done for a fixed value of K_2 defined by the initial conditions taken from the catalogue by Butler *et al.* (2006) and given in Table 1. In this figure we included also the nominal trajectories of these exoplanetary systems (shown as dots) obtained by numerical integration using the program *Mercury* by Chambers (1999).

We can see that for the first two systems the agreement between the averaging method and numerical integration is good in comparison to the poor agreement for HD 108874. The reason here is the proximity of the system to the 4:1 mean motion resonance. If one changes here the initial semi-major axis of the outer planet from 2.68 AU towards the nominal value of the resonance at 2.66 AU one can observe a growing disagreement between the averaging method and the real behaviour of the system. The reason for the failure of the averaging method is clear: the double averaging over mean longitudes is of course impossible in case of mean motion resonances.

Therefore in the following we are interested in systems such as HD 12226 and HD 169830, which are not in or close to such mean motion resonances. However it is clear that, if the planets during the system evolution will be close enough for the mean semi-major axes to be changed during such encounters the averaging method must fail as well. In the remaining part we will define the area for which the averaging method is working well to predict the secular evolution.

3. Stability map in the plane of initial conditions

Following Michtchenko & Malhotra (2004) we introduce the plane of initial conditions as follows: from Fig. 1 it is clear that each trajectory, no matter whether librating or circulating, goes through the line defined by $y_1 = 0$. Therefore each set of initial conditions can be represented by a point in the (x_1, e_2) plane, where the initial value of $\Delta\varpi$ is fixed to either $0°$ or $180°$.

Fig. 2 shows the plane of initial conditions for HD 12661 and HD 169830 with energy levels (solid lines) and AMD levels (dashed lines). Let us remark that for the linear

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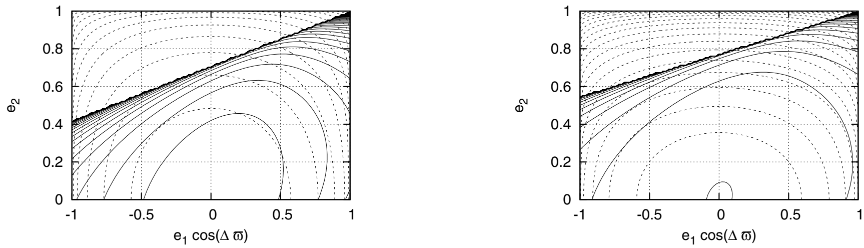

Figure 2. The plane of initial conditions with energy levels (solid lines) and AMD levels (dashed lines) for HD 12661 - left and HD 169830 - right.

approximation both of these level curves are ellipses, where the inclination of the axes for energy level ellipse would be 45°. The orbits which are initially crossing are above the line defined by $a_1(1 - e_1 \cos \Delta\varpi) = a_2(1 - e_2)$.

Using this plane we calculated with the initial condition of the therewith defined planetary systems the maximum Lyapunov Characteristic Exponent (LCE). The masses and semi-major axis were taken from Table 1. As numerical integrator we used the *ODEX* code - an extrapolation method proposed by Hairer & Wanner (1995). Fig. 3 shows the logarithm of the LCE (dark areas refer to stable regions) in the plane of initial conditions. The line of initially crossing orbits is given additionally in each of the figures.

To see how well the averaging method is working in dependence on the stability, three different points were chosen from Fig. 3: for the same e_1 one with $e_2 = 0.1$, 0.3 and 0.4 (shown as light dots in the left part of the figure). The predicted behaviour by the averaging method in comparison to the direct numerical integration for all three points can be found in Fig. 4. While for $e_2 = 0.3$ there is merely a shift to the contour lines, there is complete disagreement for $e_2 = 0.4$. The reason for the latter becomes clear if one looks at the semi-major axis as a function of time, as is shown in Fig. 5. For an initial value of $e_2 = 0.4$ the semi-major axis of the inner planet has small jumps of about 1% as the result of closer approaches between the planets. Numerical experiments show that the averaged Hamiltonian (2.4) gives the correct picture only for log LCE $\leqslant$ -5.5 (or Lyapunov time larger 100 000 years).

For most exoplanetary systems discovered by radial velocity measurement only lower limits for the planetary masses are known. To account and test also for this indetermination we multiplied both planetary masses by κ, keeping the mass ratio of the planets constant and we calculated stability maps as a function of κ. It can be recognized that

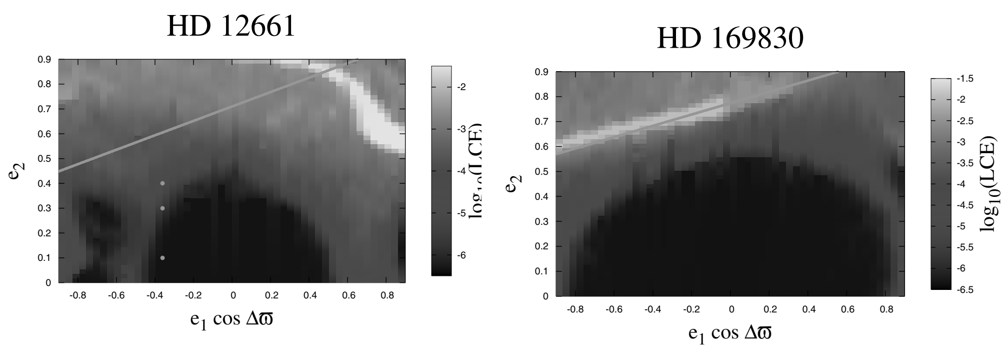

Figure 3. The stability map for HD 12661 and HD 169830.

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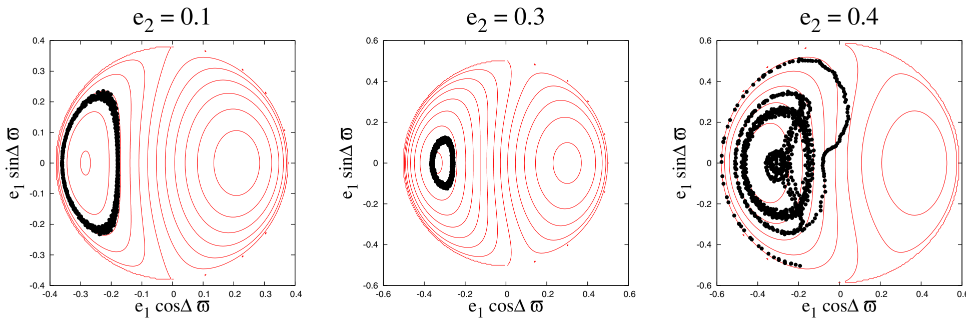

Figure 4. Comparison between the averaging method and numerical integration for different values of e_2. See text for explanation.

the region of applicability of the averaging method slowly shrinks as expected with increasing κ. But the general features remain the same: a stable area for nearly circular orbits enclosed by strongly chaotic orbits and in between some sort of fuzzy layer where the semi-major axis has short periodic variations with growing amplitude but its mean value still does not show any apparent jumps. Here the level curves given by the averaging method are slightly shifted compared to the numerical integration as already mentioned. Our boundary curve will be fitted to the region where the mean semi-major axes starts to jump.

4. Hill stability

The Hill stability condition for the elliptic case of one host star with mass M and two planets with masses m_i is given by Gladman (1993) and based on results of Marchal & Bozis (1982). It reads to lowest order as

$$C\alpha^{-3}\left(\mu_1 + \frac{\mu_2}{\delta^2}\right)(\mu_1\gamma_1 + \mu_2\gamma_2\delta)^2 > 1 + \mu_1\mu_2\left(\frac{3}{\alpha}\right)^{4/3}, \tag{4.1}$$

where $C = 1$, $\mu_i = m_i/M$, $\gamma_i = \sqrt{1-e_i^2}$ for $i = 1,2$. $\alpha = \mu_1 + \mu_2$, $\delta = \sqrt{1+\Delta}$ and $\Delta = (a_2 - a_1)/a_1$.

Gladman showed that when this inequality is fulfilled, the planets are forbidden to undergo close approaches for all the time. As a close approach he defines the separation, when one planet is inside the sphere of influence, $2\mu^{2/5}$, of the more massive one. But even for Hill stable systems, one has still a chaotic region close to the boundary defined by (4.1) with short Lyapunov times. Here one finds no crossing of the planetary orbits during several 10^5 conjunctions, which gives the impression of bounded semi-major axes preventing close approaches. But still jumps in the mean value of the semi-major

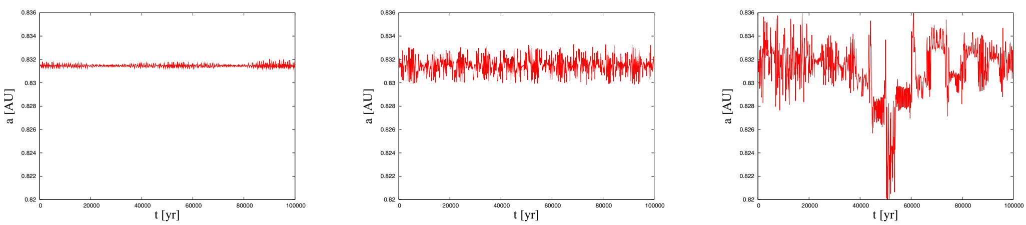

Figure 5. The evolution of the semi-major axis of the inner planet for the same trajectories as in Fig. 4.

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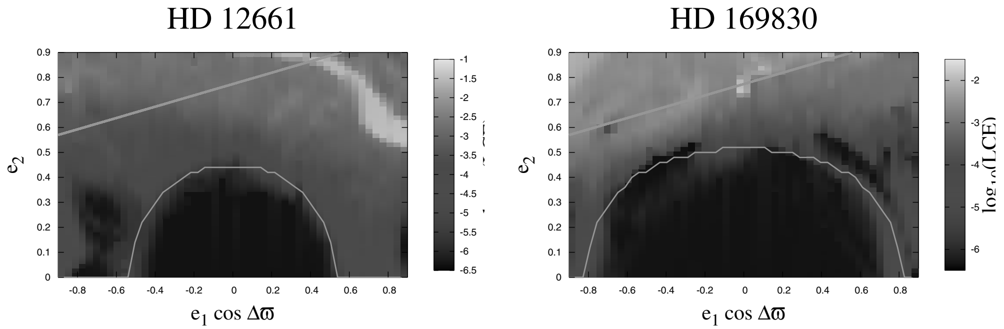

Figure 6. The stability map for HD 12661 for $\kappa = 1$ - left and for HD 169830 $\kappa = 3$ - right. The region of applicability of averaging method is roughly defined by boundary curve given by modified Gladman formula.

axes as shown in the right plot of Figure 5 can occur. The reason for these are called *small* encounters by Gladman. Further away from the boundary curve the amplitudes of short-periodic variations in the semi-major axis decrease and one finds quasi-periodic behaviour.

5. Conclusions

As already shown the Hill stability condition is not sufficient for the averaging method to work due to small encounters. Therefore such condition has to be stronger and so we introduced the constant C into (4.1) to scale the left hand side. We started with finding numerically the value of e_2, for which there is a jump in the semi-major axis for both exoplanetary systems and with different κ. Hereafter we fitted the value of C with least square method. The resulting value $C = 0.89$ leads to the required sharpening of (4.1). Fig. 6 shows the resulting boundary curves for both systems κ. These curves approximate the boundary of applicability of the averaging method.

This condition is useful to get very fast a first impression on the stability of some orbital configurations avoiding expensive (from the point of CPU time) calculations of dynamical maps. But of course for more detailed analysis more sophisticated tools have to be employed, since this method is working only in regions where strong mean-motion resonances are not important.

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