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ON ADELIC AUTOMORPHIC FORMS WITH RESPECT TO A QUADRATIC EXTENSION

ZE-LI DOU

Let E/F be a totally real quadratic extension of a totally real algebraic number field. The author has in an earlier paper considered automorphic forms defined with respect to a quaternion algebra B_E over E and a theta lift from such quaternionic forms to Hilbert modular forms over F. In this paper we construct adelic forms in the same setting, and derive explicit formulas concerning the action of Hecke operators. These formulas give an algebraic foundation for further investigations, in explicit form, of the arithmetic properties of the adelic forms and of the associated zeta and L-functions.

0. INTRODUCTION

Let E/F be a totally real quadratic extension of a totally real algebraic number field. In an earlier paper [**1**], the author has considered automorphic forms defined with respect to a quaternion algebra B_E over E, and a theta lift of such quaternionic forms to Hilbert modular forms defined over F. The Fourier coefficients of the lifted form are expressed, in explicit formulas, in terms of certain periods of the original form. In order to further investigate this theta correspondence, it is necessary to consider the Hecke operators, and to work in the adelic setting. It is the purpose of this paper to do so, again in an explicit fashion, so as to prepare adequately for the study of the arithmetic aspects of such adelic forms and their associated zeta and L-functions.

We give a few words of elaboration. In a series of deep papers mainly appearing in the 1980's, Shimura investigated several period invariants of automorphic forms, which manifest themselves in diverse settings such as the Fourier expansion, the cohomology theory, the zeta and L-functions, *et cetera,* that are attached to the automorphic forms in question. Their mutual relations and their behaviour under various liftings are summarised in a sequence of conjectures in [**5**]. In his important paper [**8**], Yoshida has achieved a breakthrough towards proving these conjectures. At the same time, his paper also reveals a need for more precise knowledge in the setting of a totally real quadratic extension of a totally real algebraic number field. This is the setting chosen in [**1**] as

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well as in this paper, and we insist on explicit formulation of everything. From this broader point of view, this paper provides the algebraic foundation, upon which further investigations in arithmetic can be developed. In fact, some of the results in this paper have already been applied, in [**2**], to establish the precise relation between the actions of Hecke operators on adelic automorphic forms linked by a theta correspondence, which generalises naturally the one considered in [**1**]. The author hopes to present results on the relations of period invariants of these forms in the near future. See also [**3**]. It may be worthwhile to note that such results are not automatically inferred by the machinery of theta correspondence in the representation theoretic language. The amount of real work, so to speak, would be about the same regardless of the language chosen. We have adopted the more direct approach, since the setting of interest for us is a very specific one.

Many of the results (and even some definitions) in this article are rather technical. In order to keep the paper as short as possible, detailed proofs of many results have been suppressed, when such proofs can be obtained by modifying arguments found either elsewhere in this article, or elsewhere in the literature. Also, unless the statement of an assertion is fairly long, it is not displayed as a proposition *per se*. Copious cross references have been provided, however, to highlight the analogous situations, and also to compensate for the somewhat terse style of our presentation.

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1. Hilbert modular forms

In this section we deal with Hilbert modular forms and Hecke operators in the integral case. Since we eventually need to keep both fields F and E in view, the material in this section is necessary. We have adopted the approach of Shimura, and attempt to formulate everything in terms of SL_2 whenever possible. In fact, in this section we are essentially specialising some of the development in [**7**] to our case of interest. Consequently we shall mostly state the results without proof. The framework established here will serve as the model for the next section as well.

Let F be a totally real algebraic number field. We denote the set of archimedean primes in F by $\mathbf{a}$, the set of finite primes by $\mathbf{h}$, and write $G = \mathrm{SL}_2(F)$ and $\widetilde{G} = \mathrm{GL}_2(F)$. The adelisation of $\widetilde{G}$ (or G) is denoted by a subscript $\mathbf{A}$; its finite and infinite parts are denoted by subscripts $\mathbf{a}$ and $\mathbf{h}$. We further define

$$\widetilde{G}_{\mathbf{a}+} = \{x \in \widetilde{G}_{\mathbf{a}} \mid \det(x) \gg 0\},\ \widetilde{G}_{\mathbf{A}+} = \widetilde{G}_{\mathbf{a}+}\widetilde{G}_{\mathbf{h}},\ \text{and}\ \widetilde{G}_{+} = \widetilde{G} \cap \widetilde{G}_{\mathbf{A}+}.$$

Let $\mathfrak{g}$ denote the ring of integers in F. For two fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ in F

such that $\mathfrak{x}\mathfrak{y} \subset \mathfrak{g}$, define

$$\widetilde{D}[\mathfrak{x},\mathfrak{y}] = \widetilde{G}_{\mathbf{a}+} \prod_{v\in\mathbf{h}} \widetilde{D}_v[\mathfrak{x},\mathfrak{y}], \qquad \widetilde{D}_v[\mathfrak{x},\mathfrak{y}] = \mathfrak{o}[\mathfrak{x},\mathfrak{y}]_v^\times, \quad \forall v \in \mathbf{h},$$

where $\mathfrak{o}[\mathfrak{x},\mathfrak{y}] = \left\{x \in \mathrm{M}_2(F) \mid a_x \in \mathfrak{g},\ b_x \in \mathfrak{x},\ c_x \in \mathfrak{y},\ d_x \in \mathfrak{g}\right\}$. We further define

$$D[\mathfrak{x},\mathfrak{y}] = G_{\mathbf{A}} \cap \widetilde{D}[\mathfrak{x},\mathfrak{y}], \qquad D_v[\mathfrak{x},\mathfrak{y}] = G_v \cap \widetilde{D}_v[\mathfrak{x},\mathfrak{y}],$$
$$\widetilde{\Gamma}[\mathfrak{x},\mathfrak{y}] = \widetilde{G} \cap \widetilde{D}[\mathfrak{x},\mathfrak{y}], \qquad \Gamma[\mathfrak{x},\mathfrak{y}] = G \cap D[\mathfrak{x},\mathfrak{y}].$$

If $\alpha \in \mathrm{GL}_2(\mathbb{R})$ with $\det(\alpha) > 0$ and $w \in \mathbb{C}$, then we write as usual $aw = (a_\alpha w + b_\alpha)/(c_\alpha w + d_\alpha)$ and $j(\alpha, w) = \det(\alpha)^{-1/2}(c_\alpha w + d_\alpha)$. For $\tau \in \widetilde{G}_{\mathbf{A}+}$ and $z \in H^{\mathbf{a}}$, where H denotes the upper half complex plane, we define the multi-index notations $\tau z = (\tau_v z_v)_{v\in\mathbf{a}}$ and $j(\tau, z) = \big(j(\tau_v, z_v)\big)_{v\in\mathbf{a}}$.

If $k \in \mathbb{Z}^{\mathbf{a}}$ is an integral weight, then for function $f : H^{\mathbf{a}} \to \mathbb{C}$ and an element $\tau \in \widetilde{G}_{\mathbf{A}+}$, we define another function $f\|_k\tau$ of the same type by the formula

$$(f\|_k\tau)(z) = j(\tau, z)^{-k} f(\tau z), \qquad \forall \in H^{\mathbf{a}}.$$

For a congruence subgroup Γ of either G or $\widetilde{G}_+$, the vector space of all holomorphic functions $f : H^{\mathbf{a}} \to \mathbb{C}$ such that $f\|_k\gamma = f$ for all $\gamma \in \Gamma$ is denoted by $\mathcal{M}_k(\Gamma)$, except for the case $F = \mathbb{Q}$, where we also assume the usual condition at the cusps. The subspace of cusp forms in $\mathcal{M}_k(\Gamma)$ is denoted by $\mathcal{S}_k(\Gamma)$. Finally we define $\mathcal{M}_k = \bigcup_\Gamma \mathcal{M}_k(\Gamma)$ and $\mathcal{S}_k = \bigcup_\Gamma \mathcal{S}_k(\Gamma)$.

If Γ is a congruence subgroup and two functions f and g on $H^{\mathbf{a}}$ are both invariant under the operation $\|_k\gamma$ for all $\gamma \in \Gamma$, then the Petersson inner product is defined by $\langle f, g\rangle = \mathrm{vol}(D)^{-1}\int_D \overline{f(z)} g(z)\,\mathrm{Im}(z)^k d_H z$ whenever the integral is convergent. Here $D = \Gamma\backslash H^{\mathbf{a}}$, $d_H z = d_H(x + iy) = \prod_{v\in\mathbf{a}} y_v^{-2} dx_v dy_v$, and $\mathrm{vol}(D) = \int_D d_H z$. This is independent of the choice of Γ.

We now consider forms of given level and character. Let $\mathfrak{c}$ be an integral ideal in F, and let ϕ be a character of $(\mathfrak{g}/\mathfrak{c})^\times$. We may view ϕ as a character of $\prod_{v|\mathfrak{c}} \mathfrak{g}_v^\times$ by a natural extension $\phi(x) = \phi(a \bmod \mathfrak{c})$ with $a \in \mathfrak{g}$ satisfying $a - x_v \in \mathfrak{c}_v, \forall v \mid \mathfrak{c}$. For an arbitrarily fixed ideal $\mathfrak{b}$ in F, we denote by $\mathcal{M}_k(\mathfrak{b},\mathfrak{c},\phi)$ the set of all f in $\mathcal{M}_k$ such that $f\|_k\gamma = \phi(\alpha_\gamma) f$ for all $\gamma \in \Gamma[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$.

We then put $\mathcal{S}_k(\mathfrak{b},\mathfrak{c},\phi) = \mathcal{S}_k \cap \mathcal{M}_k(\mathfrak{b},\mathfrak{c},\phi)$. In order to get non-trivial spaces of automorphic forms, we shall assume $\phi(-1) = (-1)^{\|k\|}$, where $\|k\| \stackrel{\mathrm{def}}{=} \sum_{v\in\mathbf{a}} k_v$.

There exists $\lambda \in \mathbf{R}^{\mathbf{a}}$ with $\|\lambda\| = 0$ such that we have

$$(1) \qquad \phi(e) = \mathrm{sgn}(e)^k |e|^{-2i\lambda}, \qquad \forall e \in \mathfrak{g}^\times.$$

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This condition, in turn, is equivalent to the existence of a Hecke character Ψ of F such that the conductor of Ψ divides $\mathfrak{c}$, that the restriction of Ψ to $\prod_{v|\mathfrak{c}} \mathfrak{g}_v^\times$ coincides with ϕ, and that

$$(2) \qquad \Psi_{\mathbf{a}}(x) = \operatorname{sgn}(x_{\mathbf{a}})^k \, |x_{\mathbf{a}}|^{2i\lambda}.$$

We can now define adelic automorphic forms on $\widetilde{G}_{\mathbf{A}}$ with level $\mathfrak{c}$ and character Ψ. We shall denote by $\mathcal{S}_k(\mathfrak{c}, \Psi)$ the space of all functions $\mathbf{f} : \widetilde{G}_{\mathbf{A}} \to \mathbb{C}$ such that the following three conditions are satisfied:

(i) $\mathbf{f}(sx) = \Psi(s)\mathbf{f}(x), \qquad \forall s \in F_{\mathbf{A}}^\times,\ \forall x \in \widetilde{G}_{\mathbf{A}},$

(ii) $\mathbf{f}(\alpha x w) = \Psi\big((d_w)_{\mathfrak{c}}\big)\mathbf{f}(x), \qquad \forall \alpha \in \widetilde{G},\ \forall x \in \widetilde{G}_{\mathbf{A}},\ w \in \widetilde{D}[\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{d}]$
with $w_{\mathbf{a}} = 1$, and $d_{\mathfrak{c}} = (d_v)_{v|\mathfrak{c}}$ for $d \in F_{\mathbf{A}}$,

(iii) For every $p \in \widetilde{G}_{\mathbf{h}}$, there is an element f_p in $\mathcal{S}_k$ such that

$$\mathbf{f}(py) = \Psi\big(\det(p)\big) \det(y)^{i\lambda} (f_p \|_k y)(\mathbf{i}), \quad \forall y \in \widetilde{G}_{\mathbf{a}+}.$$

Here the symbol $\mathbf{i}$ stands for $(i, i, \dots, i) \in \mathbb{C}^{\mathbf{a}}$.

Next we explain the relation between the adelic automorphic forms defined on $\widetilde{G}_{\mathbf{A}}$ and the automorphic forms defined on $H^{\mathbf{a}}$. To shorten notations, we write

$$\widetilde{D}_{\mathfrak{c}} = \widetilde{D}[\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{d}], \qquad \widetilde{\Gamma}_{\mathfrak{c}} = \widetilde{\Gamma}[\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{d}], \qquad \text{and} \qquad \widetilde{\Gamma}_p = \widetilde{G} \cap p\widetilde{D}_{\mathfrak{c}}p^{-1},$$

where $\mathfrak{d}$ is the different of F relative to $\mathbb{Q}$ and $p \in \widetilde{G}_{\mathbf{A}}$. We define, for λ satisfying (1),

$$(3) \qquad \mathcal{S}_k\Big(\widetilde{\Gamma}_p, \phi, \lambda\Big) = \Big\{ f \in \mathcal{S}_k \ \big|\ f\|_k \gamma = \phi\big(a(p^{-1}\gamma p)\big) \det(\gamma)^{i\lambda} f, \quad \forall \gamma \in \widetilde{\Gamma}_p \Big\}.$$

Note that this λ is not uniquely defined. In fact we have the following. Let e be an element of $F_{\mathbf{h}}^\times$ such that $e\mathfrak{d} = \mathfrak{b}$, and let $p = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}$. Then $\mathcal{S}_k(\mathfrak{b}, \mathfrak{c}, \phi)$ is a direct sum of several $\mathcal{S}_k\Big(\widetilde{\Gamma}_p, \phi, \lambda_i\Big)$, with each λ_i satisfying (1).

There exists a finite set $Q \subset \widetilde{G}_{\mathbf{h}}$ such that

$$\widetilde{G}_{\mathbf{A}} = \bigsqcup_{q \in Q} \widetilde{G} q \widetilde{D}_{\mathfrak{c}}.$$

Suppose we are given a character ϕ as in (1) and a Hecke character Ψ as in (2), such that the restriction of Ψ to $\prod_{v|\mathfrak{c}} \mathfrak{g}_v^\times$ is equal to ϕ. Given $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \Psi)$, we observe that the f_q defined by condition (iii) of the definition of $\mathcal{S}_k(\mathfrak{c}, \Psi)$ belongs to $\mathcal{S}_k\Big(\widetilde{\Gamma}_q, \phi, \lambda\Big)$

for every $q \in Q$. In fact, if we let $\mathcal{S}_k(\mathfrak{c}, \phi, \lambda)$ be the space of all $\mathbf{f}$ on $\widetilde{G}_{\mathbf{A}}$ satisfying (ii) and (iii) in that definition, then we have an isomorphism

$$\mathcal{S}_k(\mathfrak{c}, \phi, \lambda) \cong \prod_{q\in Q} \mathcal{S}_k\left(\widetilde{\Gamma}_q, \phi, \lambda\right)$$

given by $\mathbf{f} \mapsto (f_q)_{q\in Q}$. In particular, we have an embedding

$$\mathcal{S}_k(\mathfrak{c}, \Psi) \hookrightarrow \prod_{q\in Q} \mathcal{S}_k\left(\widetilde{\Gamma}_q, \phi, \lambda\right),$$

and we shall from now on identify $\mathbf{f}$ and $(f_q)_{q\in Q}$.

Given $\mathbf{f} = (f_q)_{q\in Q}$ and $\mathbf{g} = (g_q)_{q\in Q}$ in $\mathcal{S}_k(\mathfrak{c}, \Psi)$, we define their Petersson inner product by $\langle \mathbf{f}, \mathbf{g}\rangle = \sum_{q\in Q} \langle f_q, g_q\rangle$. This is of course always convergent. Note also that this definition is independent of the choice of Q. The spaces $\mathcal{M}_k(\mathfrak{c}, \Psi)$, $\mathcal{M}_k(\mathfrak{c}, \phi, \lambda)$, and $\mathcal{M}_k\left(\widetilde{\Gamma}_q, \phi, \lambda\right)$ can be discussed in an entirely similar manner.

We therefore proceed to the consideration of Hecke operators. Let Y be the subset of $\widetilde{G}_{\mathbf{A}}$ consisting of all elements y such that $y_v \in \mathfrak{o}[\mathfrak{d}^{-1}, \mathfrak{c}\mathfrak{d}]_v$ and $a(y_v) \in \mathfrak{g}_v^\times$ for every $v \mid \mathfrak{c}$. Given $y_0 \in Y$, we have a coset decomposition

$$\widetilde{D}_\mathfrak{c} y_0 \widetilde{D}_\mathfrak{c} = \bigsqcup_{w\in W} \widetilde{D}_\mathfrak{c} w,$$

where W is a finite subset of $\widetilde{G}_\mathbf{h}$. More precisely, it is a complete set of representatives of $\left(\widetilde{D}_\mathfrak{c} \cap \widetilde{G}_\mathbf{h}\right)\backslash\left(\widetilde{D}_\mathfrak{c} y_0 \widetilde{D}_\mathfrak{c} \cap \widetilde{G}_\mathbf{h}\right)$.

Let $\mathbf{f} = (f_q)_{q\in Q} \in \mathcal{M}_k(\mathfrak{c}, \phi, \lambda)$. We define

$$\left(\mathbf{f}|\widetilde{D}_\mathfrak{c} y_0 \widetilde{D}_\mathfrak{c}\right)(x) = \sum_{w\in W} \phi(a_w^{-1})\mathbf{f}(xw^*), \quad \forall x \in \widetilde{G}_\mathbf{A}. \tag{4}$$

Here w^* is the main involution of w as usual. Then we have $\mathbf{f}|\widetilde{D}_\mathfrak{c} y_0 \widetilde{D}_\mathfrak{c} \in \mathcal{M}_k(\mathfrak{c}, \phi, \lambda)$. Therefore, we may write $\mathbf{f}|\widetilde{D}_\mathfrak{c} y_0 \widetilde{D}_\mathfrak{c} = (f'_q)_{q\in Q}$, where $f'_q \in \mathcal{M}_k\left(\widetilde{\Gamma}_q, \phi, \lambda\right)$. The f'_q may be related back to the f_q as follows. Given $q \in Q$, there exists a unique $p \in Q$ and an element $\alpha_0 \in \widetilde{G}$ such that $qy_0 \in \alpha_0 p \widetilde{D}_\mathfrak{c}$. Then we may write $\widetilde{\Gamma}_q \alpha_0 \widetilde{\Gamma}_p = \bigsqcup_{\alpha\in A} \widetilde{\Gamma}_q \alpha$ for some finite set A. We then have

$$f'_p = \sum_{\alpha\in A} \phi\left(a\left(q^{-1}\alpha p\right)\right)^{-1} \det(\alpha)^{-i\lambda} f_q\|_k \alpha. \tag{5}$$

Notice that our discussion in the last few paragraphs applies as well to the space $\mathcal{S}_k(\mathfrak{c}, \Psi)$, in which we shall also be interested.

We define the symbols $\mathfrak{T}_v$ and $\mathfrak{S}_v$ for every $v \in \mathbf{h}$ as follows.

$$\mathfrak{T}_v = \widetilde{D}_\mathfrak{c} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} \widetilde{D}_\mathfrak{c}, \ \forall v \in \mathbf{h}; \quad \mathfrak{S}_v = \widetilde{D}_\mathfrak{c} \pi_v \widetilde{D}_\mathfrak{c}, \ \forall v \nmid \mathfrak{c}; \quad \mathfrak{S}_v = 0, \ \forall v \mid \mathfrak{c}.$$

Then we have

(i) $\mathcal{M}_k(\mathfrak{c}, \Psi)$ is stable under all $\widetilde{D}_\mathfrak{c} y_0 \widetilde{D}_\mathfrak{c}$, $y_0 \in Y$;

(ii) $\mathcal{M}_k(\mathfrak{c}, \Psi) = \{\mathbf{f} \in \mathcal{M}_k(\mathfrak{c}, \phi, \lambda) \mid \mathbf{f}|\mathfrak{S}_v = \Psi(\pi_v)\mathbf{f}, \ \forall v \nmid \mathfrak{c}\}$.

For the remainder of this section, we turn our attention to G and cusp forms with respect to G. Roughly speaking, our discussion here corresponds to the case $p = 1$ in [**7**]. Let us simplify notation one step further by writing

$$\Gamma = G \cap \widetilde{\Gamma}_\mathfrak{c}, \qquad \text{and} \qquad D = G_\mathbf{A} \cap \widetilde{D}_\mathfrak{c}.$$

Then, given $f \in \mathcal{S}_k(\Gamma, \phi, \lambda)$, we can define a function $f_\mathbf{A}$ on $G_\mathbf{A}$ by

$$f_\mathbf{A}(\alpha w) = \phi(a_w)^{-1}(f\|_k w)(\mathbf{i}), \qquad \forall \alpha \in G, \ \forall w \in D. \tag{6}$$

It is then straightforward that we have

$$f_\mathbf{A}(\alpha x w) = \phi(a_w)^{-1} j(w, \mathbf{i})^{-k} f_\mathbf{A}(x), \qquad \forall \alpha \in G, \ \forall x \in G_\mathbf{A}, \ \forall w \in D. \tag{7}$$

In fact, the mapping $f \mapsto f_\mathbf{A}$ is an injection of $\mathcal{S}_k(\Gamma, \phi, \lambda)$ into the space of all functions g on $G_\mathbf{A}$ satisfying the equation (7) with $f_\mathbf{A}$ replaced by g.

Suppose $\mathbf{f} \in \mathcal{S}_k(\mathfrak{c}, \Psi)$. Consider the form $f_1 \in \mathcal{S}_k\left(\widetilde{\Gamma}, \phi, \lambda\right)$ as promised by condition (iii) of the definition of $\mathcal{S}_k(\mathfrak{c}, \Psi)$. Then the restriction of $\mathbf{f}$ to $G_\mathbf{A}$ coincides with $(f_1)_\mathbf{A}$ as defined by (6):

$$\mathbf{f}|_{G_\mathbf{A}} = (f_1)_\mathbf{A}.$$

Let $y \in G_\mathbf{A} \cap F_\mathbf{A}^\times Y$. Then we can find a finite subset $W \subset G_\mathbf{h}$ such that the decompositions

$$\widetilde{D}_\mathfrak{c} y \widetilde{D}_\mathfrak{c} = \bigsqcup_{w \in W} \widetilde{D}_\mathfrak{c} w, \qquad \text{and} \qquad DyD = \bigsqcup_{w \in W} Dw$$

hold simultaneously. Furthermore, there exists an element α_0 and a finite subset I of G such that

$$DyD = D\alpha_0 D = D\alpha_0\Gamma = \bigsqcup_{\iota \in I} D\iota, \qquad \text{and} \qquad \Gamma\alpha_0\Gamma = \bigsqcup_{\iota \in I} \Gamma\iota$$

hold simultaneously.

Now let g be a mapping on $G_{\mathbf{A}}$ such that g satisfies (7) with the symbol $f_{\mathbf{A}}$ replaced by g. Then, given $y \in G_{\mathbf{A}} \cap F_{\mathbf{A}}^{\times}Y$ and W as above, we define

$$(8) \qquad (g|DyD)(x) = \sum_{w \in W} \phi(a_w)^{-1} g(xw^{-1}), \qquad \forall x \in G_{\mathbf{A}}.$$

Because of the simultaneous decompositions shown above, if g is the restriction of some $\mathbf{g} \in \mathcal{S}_k(\mathfrak{c}, \Psi)$ to $G_{\mathbf{A}}$, then our definition here is consistent with (4).

Let $f \in \mathcal{S}_k(\Gamma, \phi, \lambda)$. Then, with the α_0 and I given above, we define another element $f|\Gamma\alpha_0\Gamma$ of $\mathcal{S}_k(\Gamma, \phi, \lambda)$ by

$$(9) \qquad f|\Gamma\alpha_0\Gamma = \sum_{\iota \in I} \phi(a_\iota)^{-1} f\|_k \iota.$$

We then have

$$(10) \qquad (f|\Gamma\alpha_0\Gamma)_{\mathbf{A}} = f_{\mathbf{A}}|DyD.$$

Because of this, we shall from now on use the same symbol DyD to denote the double coset operation on f as well as on $f_{\mathbf{A}}$. That is, we shall write $f|DyD$ for $f|\Gamma\alpha_0\Gamma$.

2. Automorphic forms on $\mathcal{G}'_{\mathbf{A}}$ and $\mathcal{G}_{\mathbf{A}}$

In this section we consider (matrix valued) adelic automorphic forms defined with respect to a quaternion algebra over a totally real quadratic extension E of F. Hecke operators are also considered in the same setting. Many aspects of our development here parallel that in the previous section. We use the symbols $\mathfrak{g}^E$ and $\mathfrak{d}^E$ to denote the ring of integers and the different of E (over $\mathbb{Q}$). Let B_E be a quaternion algebra over E, and define $B = \{x \in B_E \mid x^\tau = x\}$, where τ is an involution of B_E but its restriction to E is not the identity mapping. Then B is a quaternion algebra over F, and we have $B_E = B \otimes_F E$.

Denote by $\mathfrak{d}_B$ the product of all finite primes of F which are ramified in B, and by $\mathfrak{d}_B^E$ the product of all finite primes of E which are ramified in B_E. It is not difficult to check that if $v \in F$ is inertial in E, then the factor of v in E is unramified in B_E. Consequently, a prime u of E divides $\mathfrak{d}_B^E$ if and only if it divides $\mathfrak{d}_B$ and splits over F.

Proposition 2.1. *There exists a maximal order $\mathfrak{o}$ in B_E which contains a maximal order of B, such that we can find, for every $v \in \mathbf{h}$ prime to $\mathfrak{d}_B^E$, an E_v-linear isomorphism $\mu_v : (B_E)_v \to M_2(E_v)$ with the property $\mu_v(\mathfrak{o}_v) = \mathrm{M}_2\left(\mathfrak{g}_v^E\right)$. Moreover, if $v \nmid \mathfrak{d}_B$, then $\mu_v(x^\tau) = \mu_v(x)^\tau$, where we understand that τ acts entry-wise on $\mathrm{M}_2\left(E_v\right)$.*

Proof: Let $\mathfrak{r}$ be a maximal order in B. Then we may take a maximal order $\mathfrak{o}$ in B_E such that $\mathfrak{g}^E\mathfrak{r} \subset \mathfrak{o}$. For $v \in \mathbf{h}$, $v \nmid \mathfrak{d}_B^E$, there exists $\mu_v : \left(B_E\right)_v \to \mathrm{M}_2\left(E_v\right)$

such that $\mu_v(\mathfrak{o}_v) = \mathrm{M}_2\left(\mathfrak{g}_v^E\right)$. Moreover, if $v \nmid \mathfrak{d}_B$, we already have an isomorphism $\lambda_v : B_v \to \mathrm{M}_2(F_v)$ such that $\lambda_v(\mathfrak{r}_v) = \mathrm{M}_2(\mathfrak{g}_v)$. Now $(B_E)_v = B_v \otimes_{F_v} E_v$. Thus λ_v can be extended E_v-linearly to $\mu_v : (B_E)_v \to \mathrm{M}_2(E_v)$. Then $\mu_v(x^\tau) = \mu_v(x)^\tau$ and $\mu_v\Big(\big(\mathfrak{g}^E\mathfrak{r}\big)_v\Big) = \mathrm{M}_2\left(\mathfrak{g}_v^E\right)$. Certainly for those $v \in \mathbf{h}$ such that $v \nmid \mathfrak{d}_B^E$ and $v \mid \mathfrak{d}_B$, we still have isomorphisms with $\mu_v(\mathfrak{o}_v) = \mathrm{M}_2\left(\mathfrak{g}_v^E\right)$. This completes the proof. ∎

We shall from now on fix $\mathfrak{o}$ and μ_v as in the above proposition. To define adelic automorphic forms in this setting, it is necessary to recall some relevant facts from [**1**]. Let δ denote the set of primes $v \in \mathbf{a}$ which are unramified in B, and $\delta' = \mathbf{a} - \delta$ the set of those ramified in B. The subsets of $J(E)$ consisting of extensions of primes in δ and δ' are then denoted by ζ and ζ', respectively. Here $J(E) \overset{\mathrm{def}}{=} \{\text{archimedean primes of } E\}$. For each $v \in \mathbf{a}$, we fix, once and for all, an extension of u of v in $J(E)$. The collection of these primes is then written ι. Further, we denote by η and η', respectively, the subsets of ι corresponding to δ and δ'. Throughout this paper, we assume $\zeta \neq \emptyset$.

For every $m \geq 0$, there is an $\mathbb{R}$-rational irreducible polynomial representation $\sigma_m : \mathbb{H}^\times \longrightarrow \mathrm{GL}_{m+1}(\mathbb{C})$ of degree m, which is unique up to equivalence. By fixing suitable isomorphisms for $B_\mathbf{a}$ and $(B_E)_\mathbf{a}$, we may assume that σ_m respects the $\overline{\mathbb{Q}}$-structure. If $k \in \mathbb{Z}^\iota$ is a weight such that $k_u \geq 1$ for $u \in \eta$ and $k_u \geq 2$ for $u \in \eta'$, then we define a representation on $\left(B_E^\times\right)_\mathbf{A}$ by

$$\sigma(\alpha) = \bigotimes_{u \in \eta'} \sigma_{k_u - 2}(\alpha_u), \qquad \forall \alpha \in \left(B_E^\times\right)_\mathbf{A}. \tag{11}$$

The representation space for σ will be denoted by $\mathcal{X}$. It may be identified with $\bigotimes_{u \in \eta'} \mathbb{C}^{k_u - 1}$.

To define the factor of automorphy in the definition of automorphic forms, we need some more notation. Let

$$\mathcal{G}' = (B_E)^\times, \qquad \mathcal{G}'_{\mathbf{a}+} = \left\{x \in \mathcal{G}'_\mathbf{a} \;\middle|\; N(x) \gg 0\right\},$$
$$\mathcal{G}'_{\mathbf{A}+} = \{x \in \mathcal{G}'_\mathbf{A} \mid x_\mathbf{a} \in \mathcal{G}'_{\mathbf{a}+}\}, \qquad \mathcal{G}'_1 = \left\{x \in \mathcal{G}' \;\middle|\; \mathrm{N}(x) = 1\right\}.$$

Thus the representation in (11) is $\sigma : \mathcal{G}'_\mathbf{A} \to \mathrm{GL}(\mathcal{X})$. For $\alpha \in \mathcal{G}'_\mathbf{A}$ and $w \in H^\zeta$, we put

$$\alpha w = \alpha(w) = (\alpha_u w_u)_{u \in \zeta} = \left(\frac{a_u w_u + b_u}{c_u w_u + d_u}\right)_{u \in \zeta}$$

and

$$j(\alpha, w) = \big(j(\alpha_u, w_u)\big)_{u \in \zeta} = \Big(\big|\det(\alpha_u)\big|^{-1/2}(c_u w_u + d_u)\Big)_{u \in \zeta},$$

where a_u, b_u, c_u and d_u are the entries of α_u in the standard order.

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We now define $\mathrm{End}(\mathcal{X})$-valued holomorphic automorphic forms of weight $k+\tau k$ on H^ς, where $k \in \mathbb{Z}^\iota$ as above. Given a mapping $f : H^\varsigma \to \mathrm{End}(\mathcal{X})$, we define another mapping of the same kind, which we denote by $f\|_{k+\tau k}\alpha$, by the following formula:

$$(f\|_{k+\tau k}\alpha)(w) = j(\alpha, w)^{-k\eta-\tau(k\eta)}\sigma\Big(\mathrm{N}(\alpha)^{1/2}\alpha^{-1}\Big)f(\alpha w)\sigma\Big(\mathrm{N}(\alpha)^{-\tau/2}\alpha^{\tau}\Big).$$

Let Γ be a congruence subgroup of $\mathcal{G}'_1$, then the space of holomorphic automorphic forms of weight $k+\tau k$ with respect to Γ is the set of all holomorphic mappings $f :$ $H^\varsigma \to \mathrm{End}(\mathcal{X})$ such that $f\|\alpha = f$ for all $\alpha \in \Gamma$, and also the usual cusp condition in the case where $B_E = \mathrm{M}_2(E)$. This space is denoted by $\mathcal{S}_{k+\tau k}(\Gamma)$. The union of such spaces over all congruence subgroups is denoted by $\mathcal{S}_{k+\tau k}(B_E)$. Denoting by $d_H^\varsigma w$ the Haar measure on H^ς, we define an inner product of two $\mathbb{C}^\infty$-mappings f and g of H^ς into $\mathrm{End}(\mathcal{X})$, such that $f\|\alpha = f$ and $g\|\alpha = g$ for all $\alpha \in \Gamma$ for some congruence subgroup Γ, by the formula

$$\langle f, g\rangle = \mathrm{vol}(D)^{-1}\int_D \mathrm{Tr}\left(\overline{{}^t f(w)}g(w)\right)\mathrm{Im}(w)^{k\eta+\tau(k\eta)}d_H^\varsigma w.$$

Here $D \stackrel{\mathrm{def}}{=} \Gamma\backslash H^\varsigma$, and $\mathrm{vol}(D) \stackrel{\mathrm{def}}{=} \int_D d_H^\varsigma w$. This definition is independent of the choice of Γ.

We resume the development of the adelic forms. Let $\mathfrak{m}$ be an integral ideal in E. We define an order of level $\mathfrak{m}$ to be the $\mathfrak{g}^E$-lattice $\mathfrak{o}_1 \subset B_E$ given by

(i) $\mathfrak{o}_{1v} = \mathfrak{g}_v^E + \mathfrak{m}\mathfrak{o}_v$, if $v|\mathfrak{d}_B^E$;

(ii) $\mathfrak{o}_{1v} = \mu_v^{-1}\Big(\big\{x \in \mathrm{M}_2(E_v) \;\big|\; a_x \in \mathfrak{g}_v^E,\ b_x \in \mathfrak{d}^{-1}\mathfrak{g}_v^E,\ c_x \in \mathfrak{d}\mathfrak{m}_v,$
$d_x \in \mathfrak{g}_v^E\big\}\Big)$, if $v \nmid \mathfrak{d}_B^E$.

We further define a subgroup $W'_\mathfrak{m} \subset \mathcal{G}'_\mathbf{A}$ by

$$W'_\mathfrak{m} = \mathcal{G}'_{\mathbf{a}+}\prod_{v\in\mathbf{h}}\mathfrak{o}_{1v}^\times,$$

where $\mathcal{G}'_{\mathbf{a}+} \stackrel{\mathrm{def}}{=} \big\{x \in \mathcal{G}'_\mathbf{a} \,\big|\, \mathrm{N}(x) \gg 0\big\}$. When $\mathfrak{m}$ is understood we sometimes simply write W'. There exists a finite subset $Q' \subset \mathcal{G}'_\mathbf{h}$ such that we have a coset decomposition

$$\mathcal{G}'_\mathbf{A} = \bigsqcup_{q\in Q'}\mathcal{G}'qW'_\mathfrak{m}. \tag{12}$$

Finally, let Φ be a Hecke character of E such that the conductor of Φ, cond(Φ), is prime to $\mathfrak{d}_B^E$, is a divisor of $\mathfrak{m}$, and is such that

$$\Phi_\mathbf{a}(x) = \mathrm{sgn}(x_\mathbf{a})^{k+\tau k}|x_\mathbf{a}|^{2i\mu}, \tag{13}$$

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where $\mu \in E_{\mathbf{a}} = \mathbb{R}^{J(E)}$, and $\|\mu\| = 0$.

The space of adelic automorphic forms $\mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi; B_E)$ is now defined to be the set of all mappings $\mathbf{g} : \mathcal{G}'_{\mathbf{A}} \to \mathrm{End}\,(\mathcal{X})$ satisfying the following three conditions:

(a) $\mathbf{g}(sx) = \Phi(s)\mathbf{g}(x), \qquad \forall s \in E^{\times}_{\mathbf{A}},\ \forall x \in \mathcal{G}'_{\mathbf{A}}$.

(b) $\mathbf{g}(\beta x w) = \Phi_{\mathfrak{m}}(d_w)\mathbf{g}(x), \qquad \forall \beta \in \mathcal{G}',\ \forall w \in W'$ with $w_{\mathbf{a}} = 1,\ \forall x \in \mathcal{G}'_{\mathbf{A}}$.

(c) For every $p \in \mathcal{G}'_{\mathbf{h}}$, there is an element g_p of $\mathcal{S}_{k+\tau k}(B_E)$ such that

$$\mathbf{g}(py) = \Phi\big(\mathrm{N}\,(p)\big)\,\mathrm{N}\,(y)^{i\mu}(g_p\|_{k+\tau k}y)(\mathbf{i}), \qquad \forall y \in \mathcal{G}'_{\mathbf{a}+}.$$

As usual, we have here $\mathbf{i} = (i, i, \ldots, i) \in H^{\zeta}, \Phi_{\mathfrak{m}} = \prod_{u|\mathfrak{m}} \Phi_u$, while $d_w \in E^{\times}_{\mathbf{h}}$ is defined as follows. The v-component of d_w is 1 for all v except when $v \mid \mathrm{N}\,(\mathfrak{m})$ and v is prime to $\mathfrak{d}^E_B$, in which case it is defined to be the d-entry of $\mu(w)$.

Let Φ and $\mathfrak{m}$ be given as above, and choose μ such that (13) holds. For each $p \in \mathcal{G}'_{\mathbf{h}}$, we put

$$\Delta'_p = pW'p^{-1} \cap \mathcal{G}'.$$

We then define the space $C\big(\Delta'_p, \Phi_{\mathfrak{m}}, \mu\big)$ for any given p to be the set of all C^{∞} mappings $h : H^{\zeta} \to \mathrm{End}\,(\mathcal{X})$ such that

$$h\|_{k+\tau k}\gamma = \Phi_{\mathfrak{m}}\Big(a\big(p^{-1}\gamma p\big)\Big)\,\mathrm{N}\,(\gamma)^{i\mu}h, \quad \forall \gamma \in \Delta'_p. \tag{14}$$

As in Section 1, the symbol $a\big(p^{-1}\gamma p\big)$ stands for $a_{(p^{-1}\gamma p)}$, and is defined in the same way as d_w in (b) of the above definition.

The development in this section has so far parallelled that of Section 1. The space $C\big(\Delta'_p, \Phi_{\mathfrak{m}}, \mu\big)$ above parallels the space $\mathcal{S}_k\Big(\widetilde{\Gamma}_p, \phi, \lambda\Big)$ there. With respect to the coset decomposition (12) we have here also an embedding

$$\mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi, B_E) \hookrightarrow \prod_{q \in Q'} C\big(\Delta'_q, \Phi_{\mathfrak{m}}, \mu\big). \tag{15}$$

For a given form $\mathbf{g} \in \mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi, B_E)$, the embedding above is defined by condition (c) of the definition of the adelic forms, with the p there replaced by the various $q \in Q'$. It is straightforward to check that $g_q \in C\big(\Delta'_q, \Phi_{\mathfrak{m}}, \mu\big)$ for every $q \in Q'$. We shall identify from now on the image of $\mathbf{g}$ under (15) with $\mathbf{g}$ itself, and write $\mathbf{g} = (g_q)_{q \in Q'}$.

If $\mathbf{f} = (f_q)_{q \in Q'}$ and $\mathbf{g} = (g_q)_{q \in Q'}$ are two elements of $\mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi, B_E)$, then their inner product is defined by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \|Q'\|^{-1} \sum_{q \in Q'} \langle f_q, g_q \rangle.$$

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Next we consider Hecke operators. We define a subset $\mathcal{Y}$ of $\mathcal{G}'_{\mathbf{A}+}$ as follows:

$$\mathcal{Y} = \Big\{ y \in \mathcal{G}'_{\mathbf{A}+} \;\Big|\; y_v \in \mathfrak{o}_{1v} \ (\forall v \in \mathbf{h}),\ y_u \in \mathfrak{o}_{iu}^{\times} (u \mid \mathfrak{d}_B^E + \mathfrak{m}),$$
$$\mu_v(y_v) = \begin{pmatrix} a & * \\ * & * \end{pmatrix},\ a \in \left(\mathfrak{g}_u^E\right)^{\times},\ \left(u \mid \mathfrak{m}, u \nmid \mathfrak{d}_B^E\right) \Big\}.$$

For $y_0 \in \mathcal{Y}$ and $\mathbf{g} \in \mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi, B_E)$, we have a coset decomposition

$$W' y_0 W' = \bigsqcup_{y \in J} W' y$$

for some finite set J. More precisely, J is a complete set of representatives for the quotient space $(W' \cap \mathcal{G}'_{\mathbf{h}}) \backslash (W' y_0 W' \cap \mathcal{G}'_{\mathbf{h}})$. We then define $\mathbf{g} | W' y_0 W'$ by

$$(16) \qquad \left(\mathbf{g} | W' y_0 W'\right)(x) = \sum_{y \in J} \Phi_{\mathfrak{m}}(a_y)^{-1} \mathbf{g}(x y^*), \quad \forall x \in \mathcal{G}'_{\mathbf{A}}.$$

If $y_0 \in \mathcal{Y}$ and $q \in Q'$ are given, there exist an element $\alpha_0 \in \mathcal{G}'$ and a unique $p \in Q'$, such that $q y_0 \in \alpha_0 p W'$. We can then take a coset decomposition

$$(17) \qquad \Delta'_q \alpha_0 \Delta'_p = \bigsqcup_{\alpha \in A} \Delta'_q \alpha$$

for some finite set A. This fact can be verified by essentially repeating verbatim the proof of [**6**, Lemma 2.2] and so will be omitted here.

Keeping the notation as above, we let $\mathbf{g} = (g_q)_{q \in Q'} \in \mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi, B_E)$, and suppose $\mathbf{g} \mid W' y_0 W' = (\widetilde{g}_q)_{q \in Q'}$. Then

$$(18) \qquad \widetilde{g}_p = \sum_{\alpha \in A} \Phi_{\mathfrak{m}}\left(a(q^{-1} y p)\right)^{-1} \mathrm{N}(\alpha)^{-i\mu} g_q \| \alpha.$$

This follows from a direct computation. Because of this result, given $(f_q)_{q \in Q'} \in \prod_{q \in Q'} C(\Delta'_q, \Phi_{\mathfrak{m}}, \mu)$, we may define

$$(f_q) | W' y_0 W' = \left(\widetilde{f}_q\right)_{q \in Q'},$$

where the $\widetilde{f}_q$ are determined by the above formula, with g replaced by f.

PROPOSITION 2.2. *Let* $y_0 \in \mathcal{Y} \cap \mathcal{G}'_{\mathbf{h}}$. *If* $\mathrm{N}(y_0)\mathfrak{g}^E$ *is prime to* $\mathfrak{m}$, *then* $W' y_0 W' = W' y_0^* W'$.

PROOF: This identity may be checked by considering the various cases.

Let us give one example. Suppose $u \nmid \mathfrak{d}_B^E$, and $u \nmid \mathfrak{m}$. In this case W'_u consists of non-singular matrices whose entries in the usual order belong to $\mathfrak{g}_u^E, \mathfrak{d}_u^{-1}, \mathfrak{d}_u$, and $\mathfrak{g}_u^E$, respectively. Thus we may write $W'_u = \varepsilon_v \mathrm{M}_2\left(\mathfrak{g}_v^E\right)^{\times}\varepsilon_v^{-1}$. From this we easily deduce that $(W'y_0W')_v = (W'y_0^*W')_v$. Therefore the assertion holds in this case. The verification for the other cases is omitted. The condition on $\mathrm{N}(y_0)$ is needed for the case $u \nmid \mathfrak{d}_B^E$ and $u \mid \mathfrak{m}$. ∎

We now establish the precise formula regarding Hecke operators.

Theorem 2.3. *Let $(f_q)_{q\in Q'} \in \prod_{q\in Q'} C(\Delta'_q, \Phi_\mathfrak{m}, \mu)$, and define $\left(\widetilde{f}_q\right)_{q\in Q'}$ by (18) with g there replaced by f. Let $\mathrm{g} = (g_q)_{q\in Q'} \in \mathcal{S}_{k+\tau k}(\mathfrak{m}, \Phi, B_E)$, and write $\mathrm{g}|W'y_0W' = (\widetilde{g}_q)_{q\in Q'}$. Suppose $y_0 \in \mathcal{Y}$ and $\mathrm{N}(y_0)\mathfrak{g}^E$ is prime to $\mathfrak{m}$. Then*

$$\Phi^*\left(\mathrm{N}(y_0)\mathfrak{g}^E\right)\langle \widetilde{f}_p, g_p\rangle = \langle f_q, \widetilde{g}_q\rangle,$$

so long as the inner products are convergent.

Proof: We may assume that $y_0 \in \mathcal{G}'_\mathrm{h}$. Observe that $\langle f\|\alpha, g\rangle = \langle f, g\|\alpha^*\rangle$. Therefore we may write

$$\langle \widetilde{f}_p, g_p\rangle = \left\langle f_q, \sum_\alpha \Phi_\mathfrak{m}\left(a\left(q^{-1}\alpha p\right)\right)\mathrm{N}(\alpha)^{i\mu} g_p\|\alpha^*\right\rangle. \tag{19}$$

Recall from [7, Lemma 4.3] that the set $\Delta'_p \cap L^\times$ and $\mathrm{vol}\left(\Delta'_p\backslash H^\varsigma\right)$ are independent of p. Thus the quotient spaces $\Delta'_q\alpha_0\Delta'_p/\Delta'_p$ and $\Delta'_q\backslash\Delta'_q\alpha_0\Delta'_p$ have the same number of cosets. For this reason, we can find a finite set A such that $\Delta'_q\alpha_0\Delta'_p = \bigsqcup_{\alpha\in A}\Delta'_q\alpha = \bigsqcup_{\alpha\in A}\alpha\Delta'_p$. Since $W'y_0W' = W'y_0^*W'$ by Proposition 2.2, we have $W'y_0W' = W'\mathrm{N}(y_0)y_0^{-1}W' = W'\mathrm{N}(y_0)p^{-1}\alpha_0^{-1}qW'$. Now $\Delta'_q\alpha_0^{-1}\Delta'_p = \bigsqcup_{\alpha\in A}\Delta'_p\alpha^{-1}$, so we have, by (17),

$$W'p^{-1}\alpha_0^{-1}qW' = \bigsqcup_{\alpha\in A} W'p^{-1}\alpha^{-1}q.$$

We now set y in the defining equation (16) to be the $\left(\mathrm{N}(y_0)p^{-1}\alpha^{-1}q\right)_\mathrm{h}$. That is, we now have

$$(\mathrm{g}|W'y_0W')(x) = \sum_{\alpha\in A}\Phi_\mathfrak{m}\left(a\left(\mathrm{N}(y_0)p^{-1}\alpha^{-1}q\right)\right)^{-1}\mathrm{g}\left(x\,\mathrm{N}(y_0)\left(p^{-1}\alpha^{-1}q\right)_\mathrm{h}^*\right).$$

Combining this equation with the g_p in the definition of the adelic form, we compute, for $u \in \mathcal{G}'_{\mathbf{a}+}$:

$$
\begin{aligned}
&(\widetilde{g}_q \| u)(\mathbf{i}) \\
&\quad = \Phi\big(\mathrm{N}(q)\big)^{-1} \mathrm{N}(u)^{-i\mu} \sum_{\alpha} \Phi_{\mathrm{m}}\big(a(\mathrm{N}(y_0)p^{-1}\alpha^{-1}q)\big)^{-1} \mathbf{g}\Big(qu\,\mathrm{N}(y_0)\big(p^{-1}\alpha^{-1}q\big)^*_{\mathbf{h}}\Big) \\
&\quad = \Phi^*\big(\mathrm{N}(y_0)\mathfrak{g}^E\big)\Phi\big(\mathrm{N}(p)\big)^{-1}\mathrm{N}(u)^{-i\mu} \sum_{\alpha} \Phi_{\mathrm{m}}\big(a(q^{-1}\alpha p)\big)\mathbf{g}\big(\alpha^{-*}p\alpha^*_{\mathbf{a}}u\big) \\
&\quad = \Phi^*\big(\mathrm{N}(y_0)\mathfrak{g}^E\big) \sum_{\alpha} \Phi_{\mathrm{m}}\big(a(q^{-1}\alpha p)\big)\,\mathrm{N}(\alpha)^{i\mu}(g_p \| \alpha^* u)(\mathbf{i}).
\end{aligned}
$$

Therefore, we conclude that

$$
\widetilde{g}_q = \Phi^*\big(\mathrm{N}(y_0)\mathfrak{g}^E\big) \sum_{\alpha \in A} \Phi_{\mathrm{m}}\Big(a\big(q^{-1}\alpha p\big)\Big)\,\mathrm{N}(\alpha)^{i\mu}(g_p \| \alpha^*). \tag{20}
$$

The equations (19) and (20) together prove our assertion. ∎

As the final topic of this article, we shall consider automorphic forms with respect to a subgroup $\mathcal{G}$ of $\mathcal{G}'$ defined by

$$
\mathcal{G} = \big\{x \in \mathcal{G}' \bigm| \mathrm{N}(x) \in F\big\}.
$$

The significance of so doing is that the vector space $V \overset{\mathrm{def}}{=} \{x \in B_E \mid x^\tau = -x^*\}$, which is a four-dimensional vector space over F, is stable under the mapping $x \mapsto \alpha x \alpha^{-\tau}$ for every $\alpha \in \mathcal{G}$. (V is stable under $x \mapsto \alpha x \alpha^{*\tau}$ for all $x \in \mathcal{G}'$.)

Put

$$
W = W_{\mathrm{m}} = W'_{\mathrm{m}} \cap \mathcal{G}_{\mathbf{A}}, \qquad \text{and} \qquad \mathcal{G}_+ = \mathcal{G} \cap \mathcal{G}'_+.
$$

Then there is a finite set $Q \subset \mathcal{G}_{\mathrm{h}}$ such that we have the following coset decomposition:

$$
\mathcal{G}_{\mathbf{A}} = \bigsqcup_{q \in Q} \mathcal{G} q W_{\mathrm{m}}. \tag{21}
$$

A theory similar to the one we have developed for W'_{m} exists with respect to W_{m} and can be developed following the same outline. Therefore we shall leave most of the development of this theory to the reader. However, we do wish to point out the precise connection between the operators $W'yW'$ and WyW for $y \in G_{\mathbf{A}}$.

Suppose, therefore, that we are given $y \in G_{\mathbf{A}}$. Consider a coset decomposition

$$
WyW = \bigsqcup_{r} Wr, \tag{22}
$$

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where r runs through a finite subset of $\mathcal{G}_h$. Also, given $q \in Q$, there is (as usual) a unique $p \in Q$ and some $\alpha_0 \in \mathcal{G}$ such that $qy \in \alpha_0 pW$. Letting

$$\Delta_q = \Delta'_q \cap \mathcal{G},$$

we can take a coset decomposition

$$\Delta_q \alpha_0 \Delta_p = \bigsqcup_{\alpha \in A} \Delta_q \alpha. \tag{23}$$

We then have the following technical result.

PROPOSITION 2.4. *With notation as above, we have*

$$W'yW' = W'yW,$$

and, more generally,

$$\Delta'_q \alpha_0 \Delta'_p = \Delta'_q \alpha_0 \Delta_p = \bigsqcup_{\alpha \in A} \Delta'_q \alpha.$$

PROOF: Observe that we have $\mathrm{N}\left(W' \cap yW'y^{-1}\right) = \mathrm{N}(W')$. Applying [**4**, Lemma 1.1], we have

$$W'yW' \subset W'y\Big(W' \cap \{x \in \mathcal{G}' \mid \mathrm{N}(x) = 1\}\Big) \subset W'yW.$$

This yields the first assertion of the proposition. More generally, we have

$$qW'q^{-1}zpW'p^{-1} = qW'q^{-1}z\Delta_{1p},$$

where we have written $\Delta_{1p} = pW'p^{-1} \cap \{x \in \mathcal{G}' \mid \mathrm{N}(x) = 1\}$. Therefore

$$\Delta'_q \alpha_0 \Delta'_p \subset qW'q^{-1}\alpha_0 \Delta_{1p} \cap \mathcal{G}' = \Delta'_q \alpha_0 \Delta_{1p}.$$

It follows that $\Delta'_q \alpha_0 \Delta'_p = \Delta'_q \alpha_0 \Delta_p$. The proof is complete. ∎

In view of this proposition, we have

$$\left(\mathrm{g} | W'y_0W'\right)(x) = \sum_r \Phi_{\mathrm{m}}(a_r)^{-1} \mathrm{g}(xr^*)$$

with the same r as those in the formula (22). Moreover, again because of the proposition, we see that the formula (18) is valid in this new setting—that is, with respect to the A in (23)—as well.

References

[1] Z.-L. Dou, 'On a theta correspondence with respect to a quadratic extension', *J. Number Theory* **79** (1999), 1–18.

[2] Z.-L. Dou, 'Theta correspondence and Hecke operators relative to a quadratic extension', *Acta Arith.* **93** (2000), 237–255.

[3] Z.-L. Dou, 'Arithmeticity and theta correspondence on an orthogonal group', in *Proc. New York Number Theory Seminar*, (Chudnovsky, Chudnovsky, and Nathanson, Editors) (Springer-Verlag, Berlin, Heidelberg, New York) (to appear).

[4] G. Shimura, 'On certain zeta functions attached to two Hilbert modular forms, II', *Ann. Math.* **114** (1981), 569–607.

[5] G. Shimura, 'On the critical values of certain Dirichlet series and the periods of automorphic forms', *Invent. Math.* **94** (1988), 245–305.

[6] G. Shimura, 'On the fundamental periods of automorphic forms of arithmetic type', *Invent. Math.* **102** (1990), 399–428.

[7] G. Shimura, 'The critical values of certain Dirichlet series attached to Hilbert modular forms', *Duke Math. J.* **63** (1991), 557–613.

[8] H. Yoshida, 'On a conjecture of Shimura concerning periods of Hilbert modular forms', *Amer. J. Math.* **117** (1995), 1019–1038.

Mathematics Department
P.O. Box 298900
Texas Christian University
Fort Worth, TX 76129
United States of America
z.dou@tcu.edu