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RESEARCH ARTICLE

Cohomology of algebraic varieties over non-archimedean fields

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Abstract
We develop a sheaf cohomology theory of algebraic varieties over an algebraically closed nontrivially valued nonarchimedean field K based on Hrushovski-Loeser's stable completion. In parallel, we develop a sheaf cohomology of definable subsets in o-minimal expansions of the tropical semi-group Γ_∞, where Γ denotes the value group of K. For quasi-projective varieties, both cohomologies are strongly related by a deformation retraction of the stable completion homeomorphic to a definable subset of Γ_∞. In both contexts, we show that the corresponding cohomology theory satisfies the Eilenberg-Steenrod axioms, finiteness and invariance, and we provide natural bounds of cohomological dimension in each case. As an application, we show that there are finitely many isomorphism types of cohomology groups in definable families. Moreover, due to the strong relation between the stable completion of an algebraic variety and its analytification in the sense of V. Berkovich, we recover and extend results on the singular cohomology of the analytification of algebraic varieties concerning finiteness and invariance.

Contents

1. Introduction

Let V be an algebraic variety over a rank 1 nonarchimedean field K. There are two well-behaved cohomology theories that can be attached to the analytification V^{an} of V in the sense of V. Berkovich: the singular cohomology and the étale cohomology. Both have been proven to carry interesting information about V (see [12], [9], [10], to cite just a few).

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When K is a nonarchimedean field of higher rank, E. Hrushovski and F. Loeser [35] introduced a topological space $\widehat{V}(K)$ called the *stable completion of* V, which can be thought as a higher-rank analogue of Berkovich's analytification. If V is a quasi-projective variety, the main theorem in [35] shows a very deep connection between $\widehat{V}(K)$ and the tropical semi-group Γ_∞ associated to the value group Γ of K: there is a deformation retraction from $\widehat{V}(K)$ to a piece-wise semi-linear subset of a finite power of Γ_∞. An analogous result was earlier proved by Berkovich [11] for V^{an} under strong algebraic restrictions on V.

The aim of this article is to introduce a well-behaved cohomology theory of the space $\widehat{V}(K)$ that coincides with the singular cohomology when K has rank 1. In particular, we show our cohomology satisfies finiteness and invariance and has natural bounds on the cohomological dimension. As an application of the above, we extend results of Berkovich on the singular cohomology and obtain a tameness result in families showing there are only finitely many cohomology groups in fibres.

Typically (e.g., when K has rank bigger than one), the underlying topological space of $\widehat{V}(K)$ is not even locally compact (see later Remark 7.5). We bypass such a problem by introducing a suitable site on $\widehat{V}(K)$ (the $\widehat{\text{v+g}}$-site) and using a formalism of sheaves developed by M. Edmundo and L. Prelli ([27]), which builds on a modification of M. Kashiwara and P. Schapira's ([40]). Recent applications such as [2] and [5, 6] are implicitly working within this formalism.

Our approach is reminiscent of J. Tate's initial attempt to study rigid analytic spaces via G-topologies [52]. However, it is worthwhile to mention that, in contrast with cohomology theories defined via the analytification of algebraic varieties, our definition is intrinsically algebraic and makes no use of Tate algebras or affinoid domains. This confirms Hrushovski-Loeser's idea that a 'rigid *algebraic* geometry exists as well' (see [35, Chapter 1]).

2. Main results

2.1. Cohomology theories

Let K be an algebraically closed nonarchimedean field of arbitrary rank, Γ be its value group and $\Gamma_\infty = \Gamma \cup \{\infty\}$ the associated tropical semi-group. In order to fully exploit Hrushovski-Loeser's main theorem, part of the present article consists in developing in parallel a sheaf cohomology of definable (i.e., semi-linear) sets of Γ_∞. More generally, we lay out a sheaf cohomology for definable sets in o-minimal expansions of Γ_∞ over the site of open definable sets, extending theorems from [22, 25, 26, 28] on sheaves on o-minimal expansions of Γ, including the formalism of the six Grothendieck operations (see Remark 6.23). It is worth mentioning that the passage from Γ to Γ_∞ is more subtle than one might expect, as already noted by Hrushovski and Loeser (see the first paragraph in [35, Section 4.1]).

The following is the main theorem concerning our cohomology theories:

Main theorem. *For algebraic varieties (possibly over a subfield of K), their stable completions and definable subsets of* Γ_∞*, the corresponding cohomology theory satisfies:*

(1) *the Eilenberg-Steenrod axioms (Theorem 6.11);*
(2) *finiteness (respectively, Theorems 6.31 and 6.20);*
(3) *invariance (respectively, Theorems 6.32 and 6.22);*
(4) *bounds on cohomological dimensions (respectively, Theorems 6.35, 6.36 and 6.20).*

2.2. Applications

We obtain the following applications:

(I) In rank 1, Hrushovski and Loeser show finiteness of homotopy types in definable families (see [35, Theorems 14.3.1 and 14.4.4]). As they observed (see the beginning of [35, Section 14.3]), such a result is no longer true in higher ranks. However, using invariance of cohomology with definably compact support, we obtain a tameness result showing that given a definable family Z_w of v+g-locally closed subsets of a variety, as w runs through W, there are finitely many isomorphisms types for the $\widehat{\text{v+g}}$-cohomology $H_c^*(\widehat{Z}_w;\mathbb{Q})$ with definably compact supports. (See later Theorem 8.1.)

Results of A. Abbes and T. Saito [1], later generalised by J. Poineau [48], showed that for families parametrised by the norm of an analytic function, there is a finite partition of $\mathbb{R}$ into intervals such that on each piece, π_0 of the fibres are in canonical bijections. For definable families parametrised by Γ_∞, we get analogues of such results for H_c^* and π_0^{def} (both independently of the rank). See Corollaries 8.2 and 8.3.

It is worth pointing out that assuming finiteness and invariance of cohomology without support, similar methods lead to bounds on the v+g-Betti numbers of a uniformly definable family of v+g-locally closed subsets of a variety. Such a result will be an analogue, in all ranks, of a result obtained by S. Basu and D. Patel ([4, Theorem 2]) in the case where the valued field has rank one, using the usual topological (singular) Betti numbers instead. In comparison with [4, Theorem 2], this approach will provide a much simpler proof by taking advantage of the fact that we can work in higher elementary extensions where we can endow the tropical semi-group $\mathbf{\Gamma}_\infty$ with the structure of a real closed field and therefore apply earlier results of Basu himself on bounds in o-minimal expansions of real closed fields ([3, Theorem 2.2]).

(II) We recover results of Berkovich [11] on the singular cohomology of V^{an} concerning finiteness and invariance (see later Theorems 7.12 and 7.13 and Corollaries 7.19 and 7.17).

(III) We extend Berkovich's results in two different ways that we briefly explain. Let F be a nonarchimedean field of rank 1 (not necessarily complete) and V be an algebraic variety over F. Hrushovski and Loeser also introduced a topological space, called the *model-theoretic Berkovich space associated to* V, which they denoted by $B_{\mathbf{F}}(V)$ (see later Section 7 for its formal definition). When F is complete, $B_{\mathbf{F}}(V)$ is homeomorphic to the underlying topological space of V^{an}. Letting $F^{\max}$ be a maximally complete algebraically closed extension of F with value group $\mathbb{R}$, they show the existence of a closed continuous surjection $\pi\colon \widehat{V}(F^{\max}) \to B_{\mathbf{F}}(V)$. Such a map allowed them to transfer results from $\widehat{V}(F^{\max})$ to $B_{\mathbf{F}}(V)$ and draw conclusions about the homotopy type of V^{an} for V quasi-projective. When $F = F^{\max}$, the topological spaces $\widehat{V}(F)$, $B_{\mathbf{F}}(V)$ and V^{an} are all homeomorphic, and we show that, in this particular case, our cohomology coincides with the singular cohomology of V^{an} (Theorems 7.16 and 7.18). Furthermore, the map π allowed us to transfer results from the $\widehat{\text{v+g}}$-cohomology on $\widehat{V}(F^{\max})$ to the singular cohomology of $B_{\mathbf{F}}(V)$ and hence, when F is complete, to the singular cohomology of V^{an}. This is mainly how we recover Berkovich's results. However, note that our results hold more generally for $B_{\mathbf{F}}(V)$, a space that is well-defined without assuming F to be complete.

A second way in which we extend Berkovich's results is that our theorems actually hold for semi-algebraic subsets of $B_{\mathbf{F}}(V)$ (respectively, for semi-algebraic subsets of $\widehat{V}(K)$, for any nontrivially valued algebraically closed field extension of F). Here, a semi-algebraic subset should be understood in the sense of A. Ducros [20] (see later Fact 7.1). Using that $\widehat{V}(K)$ has finitely many definably connected components (Lemma 5.21), we also recover the analogue result due to Ducros [20] for V^{an}. In a similar vein, but using completely different methods, F. Martin [43] extended results of Berkovich concerning the étale cohomology of analytic spaces to the larger category of their semi-algebraic subsets. We would like to explore in subsequent research if the techniques here employed can also be used to develop an analogue of the étale cohomology over the spaces $\widehat{V}(K)$ and $B_{\mathbf{F}}(V)$.

2.3. Layout of the article

Throughout, we will use a formalism of sheaves developed by M. Edmundo and L. Prelli ([27]), which builds on a modification of M. Kashiwara and P. Schapira's ([40]) notion of $\mathcal{T}$-topology (a Grothendieck topology). The semi-algebraic site, the sub-analytic site, the o-minimal site and the $\widehat{\text{v+g}}$-site mentioned above will be examples of such $\mathcal{T}$-topologies. The needed background and properties of such a formalism are presented in Section 3.

The novelty here is the introduction of the notions of $\mathcal{T}$-normality and families of $\mathcal{T}$-normal supports on spaces equipped with a $\mathcal{T}$-topology. When we pass to the $\mathcal{T}$-spectrum of such Grothendieck

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topologies, the corresponding categories of sheaves are isomorphic, and we get normal spectral spaces (respectively, families of normal and constructible supports on such spectral spaces). The latter notion will play the role that paracompactifying families of supports plays in sheaf theory in topological spaces. We study the notion of Φ-soft sheaves when Φ is a normal and constructible family of supports and set up the tools required to obtain the Base change formula, the Vietoris-Begle theorem and, in Section 6, Mayer-Vietoris sequences and bounds on cohomological Φ-dimension. These results were already known in the o-minimal case ([25]) and the semi-algebraic case ([17]), where $\mathcal{T}$-normality and families of $\mathcal{T}$-normal supports correspond to definably normal definable spaces and families of definably normal supports (respectively, regular and paracompact (locally) semi-algebraic spaces and paracompactifying (locally) semi-algebraic families of supports). Here we present a unifying approach to this theory.

In Section 4, we define the o-minimal site on a definable set in an o-minimal expansion $\mathbf{\Gamma}_\infty$ of Γ_∞ and recall the notions of definable connectedness and definable compactness. Moreover, in order to be able to apply the tools of Section 3, we study the notion of definable normality in $\mathbf{\Gamma}_\infty$. Unlike in o-minimal expansions $\mathbf{\Gamma}$ of Γ, in $\mathbf{\Gamma}_\infty$ there are open definable sets that are not definably normal. The main result of Section 4, which ensures that we can later apply the tools of Section 3, is Theorem 4.28, showing that every definably locally closed set in $\mathbf{\Gamma}_\infty$ is the union of finitely many relatively open definable subsets that are definably normal. The proof of this result is rather long and is based in ideas from [28].

The $\widehat{\text{v+g}}$-site on the stable completion $\widehat{V}(K)$ of an algebraic variety V over an algebraically closed nonarchimedean field K (and more generally on the stable completion $\widehat{X}$ of a definable subset $X \subseteq V \times \Gamma_\infty^m$) is defined and studied in Section 5. Here, for the reader's convenience, we recall some background from [35] on stable completions and the notions of definable connectedness and definable compactness. In particular, we include a proof (missing in [35]) of the fact that the stable completion $\widehat{X}$ of a definable subset $X \subseteq V \times \Gamma_\infty^n$ has finitely many definably connected components (Lemma 5.21). The main result of this section is Corollary 5.31 showing that if X is a v+g-locally closed subset, then X is the union of finitely many basic v+g-open subsets that are weakly v+g-normal. When we pass to $\widehat{X}$ equipped with the $\widehat{\text{v+g}}$-site, this gives the required conditions to apply the tools of Section 3.

Section 6 is devoted to showing the Eilenberg-Steinrod axioms, finiteness, invariance and vanishing results for the associated sheaf cohomologies in the algebraic and $\mathbf{\Gamma}_\infty$ cases. The main difficulty concerns the homotopy axiom that, as usual, is proved using the Vietoris-Begle theorem. However, extra work is needed to verify the assumptions of the version of this theorem presented in Section 3. The results concerning finiteness and invariance in $\mathbf{\Gamma}_\infty$ are obtained by adapting the methods used in [7] and [26]. After verifying that the strong deformation retraction given by the main theorem of [35] is a morphism of $\widehat{\text{v+g}}$-sites, we obtain finiteness and invariance for stable completions in the quasi-projective case. The general case is then obtained using Mayer-Vietoris sequences based on the tools of Section 3.

Finally, Sections 7 and 8 gather the above-mentioned applications: the former is devoted to the relation with Berkovich spaces and the latter to the tameness results on cohomological complexity in families.

3. $\mathcal{T}$-spaces and $\mathcal{T}$-sheaves

In this section, we will recall the notions of $\mathcal{T}$-space and $\mathcal{T}$-sheaves as well as some of the basic results obtained in [27]. We then extend the theory of $\mathcal{T}$-sheaves by introducing the notion of families of $\mathcal{T}$-normal supports and proving several results in this context, generalising those already known in the o-minimal or locally semi-algebraic cases.

Notations 3.1. Below, we let A be a commutative ring with unit. If X is a topological space (not necessarily Hausdorff), we denote by $\mathrm{Op}(X)$ the category whose objects are the open subsets of X, and the morphisms are the inclusions. We write A_X for the constant sheaf on X with value A, and we let $\mathrm{Mod}(A_X)$ be the category of sheaves of A_X-modules on X. The category $\mathrm{Mod}(A_X)$ is a Grothendieck category: it admits enough injective objects, it admits a family of generators, filtrant inductive limits are

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exact, and we have on it the classical operations

$$\mathcal{H}om_{A_X}(\bullet,\bullet),\ \bullet\otimes_{A_X}\bullet,\ f_*,\ f^{-1},\ (\bullet)_Z,\ \Gamma_Z(X;\bullet),\ \Gamma(X;\bullet),$$

where $Z\subseteq X$ is a locally closed subset.

Also recall that if $Z\subseteq X$ is a locally closed subset and $i\colon Z\longrightarrow X$ is the inclusion, then we have the operation

$$i_!(\bullet)\colon \mathrm{Mod}(A_Z)\rightarrow \mathrm{Mod}(A_X)$$

of *extension by zero* such that for $\mathcal{F}\in\mathrm{Mod}(A_Z)$, $i_!\mathcal{F}$ is the unique sheaf in $\mathrm{Mod}(A_X)$ inducing $\mathcal{F}$ on Z and zero on $X\setminus Z$. If $\mathcal{G}\in\mathrm{Mod}(A_X)$, then

$$\mathcal{G}_Z\simeq\mathcal{G}\otimes A_Z\simeq i_!\circ i^{-1}\mathcal{G}. \tag{1}$$

If $f\colon X\longrightarrow Y$ is a continuous map, Z is a locally closed subset of Y, and

$$\begin{array}{ccc} f^{-1}(Z) & \overset{j}{\hookrightarrow} & X \\ \downarrow{\scriptstyle f_|} & & \downarrow{\scriptstyle f} \\ Z & \overset{i}{\hookrightarrow} & Y \end{array}$$

is a commutative diagram and $\mathcal{F}\in\mathrm{Mod}(A_Z)$, then

$$f^{-1}\circ i_!\mathcal{F}\simeq j_!\circ(f_|)^{-1}\mathcal{F}. \tag{2}$$

Below, we use these operations freely and refer the reader to [39, Chapter II, Sections 2.1–2.4] or [14, Chapter I, Sections 1–6] (but the notation is different) for details about sheaves on topological spaces and the properties of these basic operations.

3.1. $\mathcal{T}$-sheaves

Here, we recall the definition of $\mathcal{T}$-space given in [27], adapting the construction of Kashiwara and Schapira [40], and we recall some of the results obtained in that paper for the category of sheaves on $X_{\mathcal{T}}$. Those results generalise results from the case of sub-analytic sheaves, semi-algebraic sheaves or o-minimal sheaves. See Examples 3.7 below.

Definition 3.2. Let X be a topological space, and let $\mathcal{T}\subseteq\mathrm{Op}(X)$ be a family of open subsets of X such that

(i) $\mathcal{T}$ is a basis for the topology of X, and $\varnothing\in\mathcal{T}$.
(ii) $\mathcal{T}$ is closed under finite unions and intersections.

Then we say that

- A $\mathcal{T}$-subset is a finite Boolean combination of elements of $\mathcal{T}$.
- A closed (respectively, open) $\mathcal{T}$-subset is a $\mathcal{T}$-subset that is closed (respectively, open) in X.
- A $\mathcal{T}$-connected subset is a $\mathcal{T}$-subset that is not the disjoint union of two proper clopen $\mathcal{T}$-subsets (in the induced topology).

If in addition $\mathcal{T}$ satisfies

(iii) every $U\in\mathcal{T}$ has finitely many $\mathcal{T}$-connected components that are in $\mathcal{T}$,

then we say that X is a $\mathcal{T}$*-space*.

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Let X and $\mathcal{T} \subseteq \mathrm{Op}(X)$ satisfying (i) and (ii) of Definition 3.2. One can endow the category $\mathcal{T}$ with a Grothendieck topology, called the $\mathcal{T}$*-topology*, in the following way: a family $\{U_i\}_i$ in $\mathcal{T}$ is a covering of $U \in \mathcal{T}$ if it admits a finite subcover. We denote by $X_{\mathcal{T}}$ the associated site. For short, we write $A_{\mathcal{T}}$ instead of $A_{X_{\mathcal{T}}}$ for the constant sheaf on $X_{\mathcal{T}}$ with value A, and we let $\mathrm{Mod}(A_{\mathcal{T}})$ be the category of sheaves of $A_{\mathcal{T}}$-modules on $X_{\mathcal{T}}$. If $\rho\colon X \to X_{\mathcal{T}}$ is the natural morphism of sites, then we have functors

$$\mathrm{Mod}(A_X) \underset{\rho^{-1}}{\overset{\rho_*}{\rightleftarrows}} \mathrm{Mod}(A_{\mathcal{T}})$$

with $\rho^{-1} \circ \rho_* \simeq \mathrm{id}$ (equivalently, the functor ρ_* is fully faithful). See [27, Proposition 2.1.6].

By [27, Propositions 2.1.7 and 2.1.8], we have:

Fact 3.3.

(1) Let $\{\mathcal{F}_i\}_{i\in I}$ be a filtrant inductive system in $\mathrm{Mod}(A_{\mathcal{T}})$, and let $U \in \mathcal{T}$. Then

$$\varinjlim_i \Gamma(U;\mathcal{F}_i) \xrightarrow{\sim} \Gamma(U;\varinjlim_i \mathcal{F}_i).$$

(2) Let $\mathcal{F}$ be a presheaf on $X_{\mathcal{T}}$ and assume that
  (i) $\mathcal{F}(\varnothing) = 0$,
  (ii) For any $U, V \in \mathcal{T}$ the sequence $0 \to \mathcal{F}(U \cup V) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is exact.

  Then $\mathcal{F} \in \mathrm{Mod}(A_{\mathcal{T}})$. □

Recall that any Boolean algebra $\mathcal{A}$ has an associated topological space that we denote by $S(\mathcal{A})$, called its Stone space. The points in $S(\mathcal{A})$ are the ultrafilters α on $\mathcal{A}$. The topology on $S(\mathcal{A})$ is generated by a basis of open and closed sets consisting of all sets of the form

$$\widetilde{C} = \{\alpha \in S(\mathcal{A}) : C \in \alpha\},$$

where $C \in \mathcal{A}$. The space $S(\mathcal{A})$ is a compact[1] totally disconnected space. Moreover, for each $C \in \mathcal{A}$, the subspace $\widetilde{C}$ is compact. See [37] for an introduction to this subject.

Let X and $\mathcal{T} \subseteq \mathrm{Op}(X)$ satisfying (i) and (ii) of Definition 3.2. Then as in [27], we let

$$\mathcal{T}_{loc} = \{U \in \mathrm{Op}(X) : U \cap W \in \mathcal{T} \text{ for every } W \in \mathcal{T}\},$$

and we make the following definitions:

- A subset S of X is a $\mathcal{T}_{loc}$-subset if and only if $S \cap V$ is a $\mathcal{T}$-subset for every $V \in \mathcal{T}$.
- A closed (respectively, open) $\mathcal{T}_{loc}$-subset is a $\mathcal{T}_{loc}$-subset that is closed (respectively, open) in X.
- A $\mathcal{T}_{loc}$-connected subset is a $\mathcal{T}_{loc}$-subset that is not the disjoint union of two proper clopen $\mathcal{T}_{loc}$-subsets (in the induced topology).

Clearly, $\varnothing, X \in \mathcal{T}_{loc}$, $\mathcal{T} \subseteq \mathcal{T}_{loc}$ and $\mathcal{T}_{loc}$ is closed under finite intersections. Moreover, if $\{S_i\}_i$ is a family of $\mathcal{T}_{loc}$-subsets such that $\{i : S_i \cap W \neq \emptyset\}$ is finite for every $W \in \mathcal{T}$, then the union and intersection of the family $\{S_i\}_i$ are $\mathcal{T}_{loc}$-subsets. The complement of a $\mathcal{T}_{loc}$-subset is also a $\mathcal{T}_{loc}$-subset. Therefore, the $\mathcal{T}_{loc}$-subsets form a Boolean algebra.

Definition 3.4. Let X be a topological space with $\mathcal{T} \subseteq \mathrm{Op}(X)$ as above, and let $\mathcal{A}$ be the Boolean algebra of $\mathcal{T}_{loc}$-subsets of X. The topological space $\widetilde{X}_{\mathcal{T}}$ is the data consisting of

- The points α of $S(\mathcal{A})$ such that $U \in \alpha$ for some $U \in \mathcal{T}$;
- A basis for the topology is given by the family of subsets $\{\widetilde{U} : U \in \mathcal{T}\}$.

We call $\widetilde{X}_{\mathcal{T}}$ the $\mathcal{T}$-spectrum of X.

[1] Throughout the paper, for topological spaces, we say compact to mean quasi-compact and Hausdorff.

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With this topology, for $U \in \mathcal{T}$, the set $\widetilde{U}$ is quasi-compact in $\widetilde{X}_{\mathcal{T}}$ since it is quasi-compact in $S(\mathcal{A})$. Hence:

Fact 3.5. $\widetilde{X}_{\mathcal{T}}$ has a basis of quasi-compact open subsets given by $\{\widetilde{U} : U \in \mathcal{T}\}$ closed under finite intersections. Moreover, if $X \in \mathcal{T}$, then $\widetilde{X}_{\mathcal{T}} = \widetilde{X}$ is a spectral topological space. □

Furthermore, by [27, Proposition 2.6.3], we also have:

Fact 3.6. There is an equivalence of categories

$$\mathrm{Mod}(A_{\mathcal{T}}) \simeq \mathrm{Mod}(A_{\widetilde{X}_{\mathcal{T}}}). \qquad \square$$

Note that in the statement of [27, Proposition 2.6.3], X is assumed to be a $\mathcal{T}$-space, but this is not necessary: all that is used is Fact 3.5 and [27, Corollary 1.2.11] for $\widetilde{V}$ with $V \in \mathcal{T}$.

We now recall the main known examples of $\mathcal{T}$-spaces.

Examples 3.7.

(1) Let $R = (R, <, 0, 1, +, \cdot)$ be a real closed field, X be a locally semi-algebraic space and $\mathcal{T} = \{U \in \mathrm{Op}(X) : U \text{ is semi-algebraic}\}$. Then X is a $\mathcal{T}$-space, the associated site $X_{\mathcal{T}}$ is the semi-algebraic site on X of [17, 18] (the $\mathcal{T}$-subsets of X are the semi-algebraic subsets of X (see [13]), and the $\mathcal{T}_{loc}$-subsets of X are the locally semi-algebraic subsets of X (see [18])). When X is semi-algebraic, then:
  - (i) $\widetilde{X}_{\mathcal{T}} = \widetilde{X}$ is the semi-algebraic spectrum of X from [15]. For example, if $V \subseteq R^n$ is an affine real algebraic variety over R, then $\widetilde{V}$ is homeomorphic to Sper $R[V]$, the real spectrum of the coordinate ring $R[V]$ of V (see [13, Chapter 7, Section 7.2] or [17, Chapter I, Example 1.1]).
  - (ii) There is an equivalence of categories $\mathrm{Mod}(A_{\mathcal{T}}) \simeq \mathrm{Mod}(A_{\widetilde{X}})$ (see [17, Chapter 1, Proposition 1.4]).

(2) Let X be a real analytic manifold, and consider $\mathcal{T} = \{U \in \mathrm{Op}(X) : U \text{ is sub-analytic relatively compact}\}$. Then X is a $\mathcal{T}$-space, and the associated site $X_{\mathcal{T}}$ is the sub-analytic site X_{sa} of [40, 49]. In this case, the $\mathcal{T}_{loc}$-subsets are the sub-analytic subsets of X.
Another example in the sub-analytic context is the conic sub-analytic site ([50]).

(3) Let $\mathbf{\Gamma} = (\Gamma, <, \ldots)$ be an arbitrary o-minimal structure without end points, X a definable space and $\mathcal{T} = \{U \in \mathrm{Op}(X) : U \text{ is definable}\}$. For the notion of definable space, see [53, Chapter 10]. Then X is a $\mathcal{T}$-space, and the associated site $X_{\mathcal{T}}$ is the o-minimal site X_{def} of [22]. Also note that the $\mathcal{T}$-subsets of X are exactly the definable subsets of X (see Remark 4.12). Therefore, $\widetilde{X}_{\mathrm{def}} = \widetilde{X}$ is the o-minimal spectrum of X from [46, 22] (i.e., the points are types over Γ concentrated on X). Moreover, there is an equivalence of categories $\mathrm{Mod}(A_{X_{\mathrm{def}}}) \simeq \mathrm{Mod}(A_{\widetilde{X}})$ ([22]).

3.2. *$\mathcal{T}$-normality and families of $\mathcal{T}$-supports*

Let $\mathfrak{T}$ be the category whose objects are pairs $(X, \mathcal{T})$ with X a topological space, $\mathcal{T} \subseteq \mathrm{Op}(X)$ is a family of open subsets of X satisfying conditions (i) and (ii) of Definition 3.2 and such that $X \in \mathcal{T}$, and morphisms $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ are (continuous) maps $f : X \to Y$ such that $f^{-1}(\mathcal{T}') \subseteq \mathcal{T}$.

Note that a morphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ of $\mathfrak{T}$ defines a morphism of sites

$$f : X_{\mathcal{T}} \to Y_{\mathcal{T}'}$$

which extends to a homomorphism from the boolean algebra of $\mathcal{T}'$-subsets of Y to the boolean algebra of $\mathcal{T}$-subsets of X: namely, $f^{-1}(C)$ is a $\mathcal{T}$-subset of X whenever C is a $\mathcal{T}'$-subset of Y. It follows that we have an induced map

$$\widetilde{f} : \widetilde{X}_{\mathcal{T}} \to \widetilde{Y}_{\mathcal{T}'}$$

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between the corresponding spectral topological spaces, where $\widetilde{f}(\alpha)$ is the ultrafilter in $\widetilde{Y}_{\mathcal{T}'}$ given by the collection

$$\{C : C \text{ is a } \mathcal{T}'\text{-subset and } f^{-1}(C) \in \alpha\}.$$

The map $\widetilde{f}\colon \widetilde{X}_{\mathcal{T}} \to \widetilde{Y}_{\mathcal{T}'}$ is continuous since if $V \in \mathcal{T}'$, then clearly $\widetilde{f}^{-1}(\widetilde{V}) = \widetilde{f^{-1}(V)}$.

Later, we will also require the following generalisation of the above constructions:

Remark 3.8. Let $\zeta\colon X_{\mathcal{T}} \to Y_{\mathcal{T}'}$ be a morphism of sites that extends to a homomorphism from the boolean algebra of $\mathcal{T}'$-subsets of Y to the boolean algebra of $\mathcal{T}$-subsets of X. If $\zeta(Y) = X$ and $\zeta(A) \subseteq \zeta(B)$ whenever A and B are $\mathcal{T}'$-subsets such that $A \subseteq B$, then ζ induces a continuous map $\widetilde{\zeta}\colon \widetilde{X}_{\mathcal{T}} \to \widetilde{Y}_{\mathcal{T}'}$, where $\widetilde{\zeta}(\alpha)$ is the ultrafilter of $\mathcal{T}'$-subsets given by the collection

$$\{C : C \text{ is a } \mathcal{T}'\text{-subset and } \zeta(C) \in \alpha\}.$$

If in addition $\zeta\colon X_{\mathcal{T}} \to Y_{\mathcal{T}'}$ is an isomorphism of sites, then $\widetilde{\zeta}$ is a homeomorphism.

Also recall the following facts:

Remark 3.9 (Constructible subsets and topology). Given $(X, \mathcal{T})$ an object of $\mathfrak{T}$, then $\widetilde{X}$ equipped with the constructible topology is a compact totally disconnected space in which each constructible subset is clopen. Recall that by a *constructible subset of* $\widetilde{X}$, we mean a subset of the form $\widetilde{Z}$, where Z is a $\mathcal{T}$-subset of X, and the *constructible topology on* $\widetilde{X}$ is the topology generated by the constructible subsets of $\widetilde{X}$.

Let $\widetilde{\mathfrak{T}}$ be category with objects $\widetilde{X}_{\mathcal{T}}$ for $(X, \mathcal{T})$ an object of $\mathfrak{T}$ and with morphisms $\widetilde{f}\colon \widetilde{X}_{\mathcal{T}} \to \widetilde{Y}_{\mathcal{T}'}$ for $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ a morphism of $\mathfrak{T}$. Below, we denote by

$$\mathfrak{T} \to \widetilde{\mathfrak{T}}$$

the functor sending an object $(X, \mathcal{T})$ of $\mathfrak{T}$ to $\widetilde{X}_{\mathcal{T}}$ and sending a morphism $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ of $\mathfrak{T}$ to $\widetilde{f}\colon \widetilde{X}_{\mathcal{T}} \to \widetilde{Y}_{\mathcal{T}'}$.

Remark 3.10. Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. Then the restriction of the tilde functor $\mathfrak{T} \to \widetilde{\mathfrak{T}}$ induces an isomorphism between the boolean algebra of $\mathcal{T}$-subsets of X and the boolean algebra of constructible subsets of its $\mathcal{T}$-spectrum $\widetilde{X}$ preserving open (respectively, closed), and X is $\mathcal{T}$-connected if and only if its $\mathcal{T}$-spectrum $\widetilde{X}$ is connected. Moreover, if $f\colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a morphism of $\mathfrak{T}$, then:

- If C is a $\mathcal{T}'$-subset of Y, then $\widetilde{f^{-1}(C)} = \widetilde{f}^{-1}(\widetilde{C})$;
- If $\beta \in \widetilde{Y}$, then $\widetilde{f}^{-1}(\beta)$ is a quasi-compact subset of $\widetilde{X}$.

For the latter, observe that $\widetilde{f}^{-1}(\beta) = \bigcap\{\widetilde{f^{-1}(C)} : C \in \beta\}$, so it is compact in the constructible topology.

The goal of this subsection is to extend to $(\mathfrak{T}, \widetilde{\mathfrak{T}})$ some results from [22] and [25] proved for the pair $(\mathrm{Def}, \widetilde{\mathrm{Def}})$, where Def is the subcategory of $\mathfrak{T}$ of Example 3.7 (3): that is, it is the category of definable spaces in some fixed arbitrary o-minimal structure $\mathbf{\Gamma} = (\Gamma, <, \ldots)$ without end points, with morphisms being continuous definable maps between such definable spaces, and $\widetilde{\mathrm{Def}}$ is the image of Def under $\mathfrak{T} \to \widetilde{\mathfrak{T}}$.

For $(\mathfrak{T}, \widetilde{\mathfrak{T}})$, just like in the case of $(\mathrm{Def}, \widetilde{\mathrm{Def}})$, normality will play the role that paracompactness plays in sheaf theory on topological spaces [14, Chapter I, Section 6 and Chapter II, Sections 9–13] or the role that regular and paracompact plays in sheaf theory on locally semi-algebraic spaces [17, Chapter II].

So we introduce and study a notion of normality adapted to $\mathfrak{T}$:

Definition 3.11. Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. We say that X is $\mathcal{T}$*-normal* if and only if one of the following equivalent conditions holds:

(1) For every disjoint, closed $\mathcal{T}$-subsets C and D of X there are disjoint, open $\mathcal{T}$-subsets U and V of X such that $C \subseteq U$ and $D \subseteq V$.

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(2) For every $S \subseteq X$ closed $\mathcal{T}$-subset and $U \subseteq X$ open $\mathcal{T}$-subset such that $S \subseteq U$, there are an open $\mathcal{T}$-subset W of X and a closed $\mathcal{T}$-subset K of X such that $S \subseteq W \subseteq K \subseteq U$.
(3) For every open $\mathcal{T}$-subsets U and V of X such that $X = U \cup V$ there are closed $\mathcal{T}$-subsets $A \subseteq U$ and $B \subseteq V$ of X such that $X = A \cup B$.

We let the reader convince her/himself of the equivalence of (1)–(3).

As in the case of $(\mathrm{Def}, \widetilde{\mathrm{Def}})$ ([22, Theorem 2.13]), we have the following. Here we give a simpler proof.

Proposition 3.12. *Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. Then X is $\mathcal{T}$-normal if and only if its $\mathcal{T}$-spectrum $\widetilde{X}$ is a normal topological space.*

Proof. By Remark 3.10 and quasi-compactness of $\widetilde{X}$, it is immediate that if $\widetilde{X}$ is normal, then X is $\mathcal{T}$-normal. It is a standard fact that a spectral topological space is normal if and only if any two distinct closed points can be separated by disjoint open subsets (see [15, Proposition 2]). So let α and β be two distinct closed points in $\widetilde{X}$. Since α and β are closed points, we have $\{\alpha\} = \bigcap_{i\in I} C_i$ and $\{\beta\} = \bigcap_{j\in J} D_j$, where C_is and D_js are closed constructible subsets of $\widetilde{X}$. Since α and β are distinct, we have

$$\bigcap\{C_i \cap D_j : i \in I, j \in J\} = \emptyset.$$

By quasi-compactness of $\widetilde{X}$, there are $C = C_{i_1} \cap \ldots \cap C_{i_k}$ and $D = D_{j_1} \cap \ldots \cap D_{j_l}$ such that $C \cap D = \emptyset$. Since C and D are disjoint, closed constructible subsets of $\widetilde{X}$, by Remark 3.10, we can use the $\mathcal{T}$-normality of X to find disjoint, open constructible subsets of $\widetilde{X}$ separating C and D and hence α and β. □

As usual, normality implies the shrinking lemma, whose proof is standard; see, for example, [22, Proposition 2.17] or [53, Chapter 6, (3.6)]. In the $\mathcal{T}$-spectra, due to the quasi-compactness of the base, we get a stronger result:

Corollary 3.13 (Shrinking lemma). *Suppose that $(X, \mathcal{T})$ is an object of $\mathfrak{T}$ that is $\mathcal{T}$-normal. If $\{U_i : i = 1, \ldots, n\}$ is a covering of X (respectively, of $\widetilde{X}$) by open $\mathcal{T}$-subsets of X (respectively, open subsets of $\widetilde{X}$), then there are open $\mathcal{T}$-subsets (respectively, open constructible subsets) V_i and closed $\mathcal{T}$-subsets (respectively, closed constructible subsets) K_i of X (respectively, of $\widetilde{X}$) $(1 \le i \le n)$ with $V_i \subseteq K_i \subseteq U_i$ and $X = \cup\{V_i : i = 1, \ldots, n\}$ (respectively, $\widetilde{X} = \cup\{V_i : i = 1, \ldots, n\}$).* □

Later we will require the following weaker notion:

Definition 3.14. Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. We say that X is *weakly $\mathcal{T}$-normal* if and only if one of the following equivalent conditions holds:

(1) For every disjoint, $\mathcal{T}$-closed subsets C and D of X, there are disjoint, $\mathcal{T}$-open subsets U and V of X such that $C \subseteq U$ and $D \subseteq V$.
(2) For every $S \subseteq X$ $\mathcal{T}$-closed subset and $U \subseteq X$ $\mathcal{T}$-open subset such that $S \subseteq U$, there are an $\mathcal{T}$-open subset W of X and a $\mathcal{T}$-closed subset K of X such that $S \subseteq W \subseteq K \subseteq U$.
(3) For every $\mathcal{T}$-open subsets U and V of X such that $X = U \cup V$, there are $\mathcal{T}$-closed subsets $A \subseteq U$ and $B \subseteq V$ of X such that $X = A \cup B$.

We continue with the $\mathfrak{T}$ versions of some definitions from [25] that are needed in the sequel.

Definition 3.15. Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. A *family of $\mathcal{T}$-supports on* $(X, \mathcal{T})$ is a family of closed $\mathcal{T}$-subsets of X such that:

(1) Every closed $\mathcal{T}$-subset contained in a member of Φ is in Φ.
(2) Φ is closed under finite unions.

Φ is said to be a *family of $\mathcal{T}$-normal supports* if, in addition,

(3) Each element of Φ is $\mathcal{T}$-normal,
(4) For each element S of Φ, if $U \subseteq X$ is an open $\mathcal{T}$-subset such that $S \subseteq U$, then there are an open $\mathcal{T}$-subset W of X and a closed $\mathcal{T}$-subset K of X such that $S \subseteq W \subseteq K \subseteq U$ and $K \in \Phi$.

Example 3.16. Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. Then the family of all closed $\mathcal{T}$-subsets of X is a family of $\mathcal{T}$-supports on $(X, \mathcal{T})$. Moreover, if X is $\mathcal{T}$-normal, then this family is a family of $\mathcal{T}$-normal supports on $(X, \mathcal{T})$. For other examples that play a crucial role in the paper, see Remarks 6.19 and 6.29.

Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. If Z is a $\mathcal{T}$-subset of X and Φ is a family of $\mathcal{T}$-supports on $(X, \mathcal{T})$, then we have families of $\mathcal{T} \cap Z$-supports

$$\Phi \cap Z = \{S \cap Z : S \in \Phi\}$$

and

$$\Phi_{|Z} = \{S \in \Phi : S \subseteq Z\}$$

on $(Z, \mathcal{T} \cap Z)$.

If $f: (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ is a morphism in $\mathfrak{T}$ and Ψ is a family of $\mathcal{T}'$-supports on $(Y, \mathcal{T}')$, then we have a family of $\mathcal{T}$-supports

$$f^{-1}\Psi = \{S \subseteq X : S \text{ is a closed } \mathcal{T}\text{-subset and } \exists B \in \Psi \ (S \subseteq f^{-1}(B)\}$$

on $(X, \mathcal{T})$.

Remark 3.17 (Constructible family of supports). Note that a family of $\mathcal{T}$-supports Φ on an object $(X, \mathcal{T})$ of $\mathfrak{T}$ determines a family of supports

$$\widetilde{\Phi} = \{A \subseteq \widetilde{X} : A \text{ is closed and } \exists B \in \Phi \ (A \subseteq \widetilde{B})\}$$

on the topological space $\widetilde{X} = \widetilde{X_{\mathcal{T}}}$. By Remark 3.10, it follows that

$$\widetilde{\Phi \cap Z} = \widetilde{\Phi} \cap \widetilde{Z}, \ \widetilde{\Phi_{|Z}} = \widetilde{\Phi}_{|\widetilde{Z}} \text{ and } \widetilde{f^{-1}\Psi} = \widetilde{f}^{-1}\widetilde{\Psi}.$$

We will say that the family of supports Ψ on $\widetilde{X}$ is a *constructible family of supports* on $\widetilde{X}$ if $\Psi = \widetilde{\Phi}$ for some family of $\mathcal{T}$-supports on $(X, \mathcal{T})$.

Remark 3.18 (Normal families of supports). Let us call a family Ψ of supports on a topological space Z a *normal family of supports* if:

(1) Each element of Ψ is normal.
(2) Each element S of Ψ has a fundamental system of normal (closed) neighbourhood in Ψ: that is, if $U \subseteq Z$ is an open subset such that $S \subseteq U$, then there are an open subset W of Z and a closed subset K of Z such that $S \subseteq W \subseteq K \subseteq U$ and $K \in \Psi$.

We point out that instead of (2) above, just like for the notion of a paracompactifying family of supports in topology ([14, Chapter I, Section 6 and Chapter II, Sections 9 - 13]) or the notion of a regular and paracompact family of supports in sheaf theory on locally semi-algebraic spaces ([17, Chapter II]), we could have taken the following apparently weaker condition:

(2)* Each element S of Ψ has a (closed) neighbourhood that is in Ψ: that is, there are an open subset W of Z and a closed subset K of Z such that $S \subseteq W \subseteq K$ and $K \in \Psi$.

However, these conditions are equivalent. Assume (2)*. Let $S \in \Psi$, and let $U \subseteq Z$ be an open subset such that $S \subseteq U$. Then there are an open subset W' of Z and a closed subset K' of Z such that $S \subseteq W' \subseteq K'$ and $K' \in \Psi$. Then $S \subseteq W' \cap U \subseteq K'$, S is closed in K' and K' is normal by (1). Therefore, there are

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$W \subseteq K'$ an open subset in K' and $K \subseteq K'$ a closed subset in K' such that $S \subseteq W \subseteq K \subseteq W' \cap U$. It follows that W is open in Z, K is closed in Z, and we get the stronger (2).

Also observe that as in the case of a paracompact family of supports ([14, Chapter I, Section 6, Proposition 6.5]), if Y is a locally closed subset of Z and Ψ is a normal family of supports on Z, then $\Psi_{|Y}$ is also a normal family of supports on Y. Indeed, Y is closed in an open U and $\Psi_{|Y} = (\Psi_{|U})_{|Y} = (\Psi_{|U}) \cap Y$.

By Proposition 3.12, it follows that:

Proposition 3.19. *Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$ and Φ a family of $\mathcal{T}$-supports on $(X, \mathcal{T})$. Then Φ is $\mathcal{T}$-normal if and only if $\widetilde{\Phi}$ is normal.*

Moreover, if Φ is $\mathcal{T}$-normal, then the following holds:

(∗) *Each element S of $\widetilde{\Phi}$ has a fundamental system of normal and constructible (closed) neighbourhoods in $\widetilde{\Phi}$: that is, if $U \subseteq \widetilde{X}$ is an open subset such that $S \subseteq U$, then there are an open constructible subset W of $\widetilde{X}$ and a closed constructible subset K of $\widetilde{X}$ such that $S \subseteq W \subseteq K \subseteq U$ and $K \in \Psi$.*

Proof. Suppose that Φ is $\mathcal{T}$-normal. Let $S \in \widetilde{\Phi}$, and let $U \subseteq \widetilde{X}$ be an open subset such that $S \subseteq U$. Since S is quasi-compact and U is a union of constructible open subsets of $\widetilde{X}$, there is an open constructible subset U' of $\widetilde{X}$ such that $S \subseteq U' \subseteq U$. Since $S \in \widetilde{\Phi}$, there is $S' \in \widetilde{\Phi}$ constructible such that $S \subseteq S'$. Hence $S = \bigcap\{S' \in \widetilde{\Phi} : S' \text{ is constructible and } S \subseteq S'\}$. Since $(\widetilde{X} \setminus U') \cap S = \emptyset$, by quasi-compactness of $\widetilde{X} \setminus U'$, there are $S'_1, \ldots, S'_l \in \widetilde{\Phi}$, constructible with $S \subseteq S'_i$ such that $(\bigcap_{i=1}^{l} S'_i) \cap (\widetilde{X} \setminus U') = \emptyset$. Let $S' = \bigcap_{i=1}^{l} S'_i$. Then $S \subseteq S' \subseteq U' \subseteq U$. Since Φ is $\mathcal{T}$-normal and S' and U' are constructible, there are an open constructible subset W of $\widetilde{X}$ and a closed constructible subset K of $\widetilde{X}$ such that $S \subseteq S' \subseteq W \subseteq K \subseteq U' \subseteq U$. This also shows (∗).

Conversely, suppose $\widetilde{\Phi}$ is normal. If $S \in \widetilde{\Phi}$ is constructible and $U \subseteq \widetilde{X}$ is an open constructible subset such that $S \subseteq U$, then there are an open subset $W' \subseteq \widetilde{X}$ and a closed subset $K' \subseteq \widetilde{X}$ such that $S \subseteq W' \subseteq K' \subseteq U$. By quasi-compactness as above, there are an open constructible subset $W \subseteq \widetilde{X}$ and a closed constructible subset $K \subseteq \widetilde{X}$ such that $S \subseteq W \subseteq W' \subseteq K' \subseteq K \subseteq U$. □

Below, we say that Ψ is a *family of normal and constructible supports* on the spectral topological space $\widetilde{X}$ if $\Psi = \widetilde{\Phi}$ for some family of $\mathcal{T}$-normal supports on $(X, \mathcal{T})$.

Example 3.20. The main example of a family of normal and constructible supports on a spectral topological space $\widetilde{X}$ is the family of all closed subsets of $\widetilde{X}$ when X is $\mathcal{T}$-normal. See also Remarks 6.19 and 6.29.

3.3. Sheaf cohomology with $\mathcal{T}$-supports

Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. Due to the isomorphism

$$\mathrm{Mod}(A_{X_{\mathcal{T}}}) \simeq \mathrm{Mod}(A_{\widetilde{X}}),$$

in analogy to what happened in the case $(\mathrm{Def}, \widetilde{\mathrm{Def}})$ ([22] for o-minimal sheaf cohomology without supports and in [25] in the presence of families of definable supports), in this subsection, we will develop sheaf cohomology on $X_{\mathcal{T}}$ via this tilde isomorphism.

Definition 3.21. Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$, Φ a family of $\mathcal{T}$-supports in $(X, \mathcal{T})$ and $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathcal{T}}})$. We define the $\mathfrak{T}$*-sheaf cohomology groups with $\mathcal{T}$-supports in* Φ via the tilde isomorphism by

$$H^*_{\Phi}(X; \mathcal{F}) = H^*_{\widetilde{\Phi}}(\widetilde{X}; \widetilde{\mathcal{F}}),$$

where $\widetilde{\mathcal{F}}$ is the image of $\mathcal{F}$ via the isomorphism $\mathrm{Mod}(A_{X_{\mathcal{T}}}) \simeq \mathrm{Mod}(A_{\widetilde{X}})$.

If $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ is a morphism in $\mathfrak{T}$, we define the induced homomorphism

$$f^* : H^*_{\Phi}(Y; \mathcal{F}) \longrightarrow H^*_{f^{-1}\Phi}(X; f^{-1}\mathcal{F})$$

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in cohomology to be the same as the homomorphism

$$\widetilde{f}^*: H^*_{\widetilde{\Phi}}(\widetilde{Y};\widetilde{\mathcal{F}}) \longrightarrow H^*_{\widetilde{f}^{-1}\widetilde{\Phi}}(\widetilde{X};\widetilde{f}^{-1}\widetilde{\mathcal{F}})$$

in cohomology induced by the continuous map $\widetilde{f}: \widetilde{X} \longrightarrow \widetilde{Y}$ of topological spaces.

Below, we shall use a few facts about sheaf cohomology in topological spaces that can be found, for example, in [14, Chapter II, Sections 1–8]. We will also need the following $\mathfrak{T}$ versions of [14, Chapter II, 9.5] and [14, Chapter II, 9.21], respectively.

Lemma 3.22. *Let X be an object of $\widetilde{\mathfrak{T}}$. Let Z be a subspace of X and Y a quasi-compact subset of Z having a fundamental system of normal and constructible locally closed neighbourhoods in X. Then for every $\mathcal{G} \in \mathrm{Mod}(A_Z)$, the canonical morphism*

$$\varinjlim_{Y\subseteq U} \Gamma(U\cap Z;\mathcal{G}) \longrightarrow \Gamma(Y;\mathcal{G}_{|Y})$$

where U ranges through the family of open constructible subsets of X containing Y, is an isomorphism.

Proof. Since Y is quasi-compact, the family of open neighbourhoods of Y in Z of the form $V\cap Z$, where V is an open constructible subset of X, is a fundamental system of neighbourhoods of Y in Z. Hence, the morphism of the lemma is certainly injective.

To prove that it is surjective, consider a section $s \in \Gamma(Y;\mathcal{G}_{|Y})$. There is a covering $\{U_j : j \in J\}$ of Y by open constructible subsets of X and sections $s_j \in \Gamma(U_j\cap Z;\mathcal{G}_{|U_j\cap Y})$, $j \in J$, such that $s_{j|U_j\cap Y} = s_{|U_j\cap Y}$. Since Y is quasi-compact, we can assume that J is finite, so $\cup\{U_j : j \in J\}$ is an open constructible subset of X. Therefore, there are an open constructible subset O' in X and a normal, constructible locally closed subset X' in X such that $Y \subseteq O' \subseteq X' \subseteq \cup\{U_j : j \in J\}$. For each $j \in J$, let $U'_j = U_j \cap X'$. Since X' is normal and constructible, by the shrinking lemma, there are open constructible subsets $\{V'_j : j \in J\}$ of X' and closed constructible subsets $\{K'_j : j \in J\}$ of X' such that $V'_j \subseteq K'_j \subseteq U'_j$ for every $j \in J$ and $X' = \cup\{V'_j : j \in J\}$. Since X' is a constructible locally closed subset of X, it is a constructible closed subset of an open subset of X, which by quasi-compactness we may assume is also constructible. Replacing X by that constructible open subset, we may assume that X' is a constructible closed subset of X. For each $j \in J$, let $V_j = V'_j \cap O'$ and $K_j = K'_j$. Then for each $j \in J$, V_j is an open constructible subset of X and K_j is a closed constructible subset of X with $V_j \subseteq K_j \subseteq U_j$ and $Y \subseteq \cup\{V_j : j \in J\}$.

For $x \in Z \cap O'$, set $J(x) = \{j \in J : x \in K_j\}$, and let

$$W_x = \Big(\bigcap_{x\in V_l} V_l \cap \bigcap_{j\in J(x)} U_j\Big) \setminus \bigcup_{k\notin J(x)} K_k.$$

Then W_x is a constructible neighbourhood of x in X such that if $y \in W_x$, then $J(y) \subseteq J(x)$. Indeed, suppose that $y \in W_x$, and let $i \in J(y)$. Then $y \in K_i$ and either $x \in K_i$, in which case $i \in J(x)$, or $x \notin K_i$, in which case $i \notin J(x)$, so $y \in W_x \cap K_i = \emptyset$.

Observe that for all $i, j \in J(x)$, we have that W_x is an open subset of both U_i and U_j. Hence, for every $i, j \in J(x)$, we have $s_{i|W_x\cap Y} = s_{|W_x\cap Y} = s_{j|W_x\cap Y}$. So, for $y \in W_x \cap Y$, we have $(s_i)_y = (s_j)_y$ for any $i, j \in J(x)$. This implies that the set

$$W = \{z \in \Big(\bigcup_{j\in J} V_j\Big) \cap Z : (s_i)_z = (s_j)_z \text{ for any } i, j \in J(z)\}$$

contains Y (clearly $Y \subseteq \bigcup_{x\in Z} W_x \cap Y \subseteq (\bigcup_{j\in J} V_j) \cap Z$). On the other hand, if $z \in W$, then for any $i, j \in J(z)$, we have $(s_i)_z = (s_j)_z$, so $s_i = s_j$ in an open neighbourhood of z. Since $J(z)$ is finite, z has an open neighbourhood in Z on which $s_i = s_j$ for any $i, j \in J(z)$. Thus W is an open neighbourhood of Y in Z. Since Y is quasi-compact, we may assume that W is of the form $U \cap Z$ for some open constructible

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subset U of X. Since $s_{i|W\cap V_i\cap V_j} = s_{j|W\cap V_i\cap V_j}$, there exists $t \in \Gamma(W;\mathcal{G})$ such that $t_{|W\cap V_j} = s_{j|W\cap V_j}$. This proves that the morphism is surjective. □

A general form of Lemma 3.22 is:

Lemma 3.23. *Assume that X is an object of $\widetilde{\mathfrak{T}}$. Let Z be a subspace of X, Φ a normal and constructible family of supports on X and Y a subset of Z such that for every constructible $D \in \Phi$,, $D \cap Y$ is a quasi-compact subset of Z having a fundamental system of normal and constructible locally closed neighbourhoods in X. Then for every $\mathcal{G} \in \mathrm{Mod}(A_Z)$, the canonical morphism*

$$\varinjlim_{Y\subseteq U} \Gamma_{\Phi\cap U\cap Z}(U \cap Z;\mathcal{G}) \longrightarrow \Gamma_{\Phi\cap Y}(Y;\mathcal{G}_{|Y}),$$

where U ranges through the family of open constructible subsets of X containing Y, is an isomorphism.

Proof. Let us prove injectivity. Let $s \in \Gamma_{D\cap U\cap Z}(U\cap Z;\mathcal{G})$, with $D \in \Phi$ and $U \supset Y$ open constructible subset of X and such that $s_{|D\cap Y} = 0$. By quasi-compactness of $D\cap Y$, there exists an open constructible neighbourhood V of $D\cap Y$ in X such that $s_{|V\cap D\cap Z} = 0$. Replacing V with its intersection with U if necessary, we may assume that $V \subseteq U$. Set $W = V \cup (U \setminus D)$. Then W is open constructible in X, $Y \subseteq W \subseteq U$ and $s_{|W\cap Z} = 0$.

Let us prove that the morphism is surjective. Let $s \in \Gamma_{\Phi\cap Y}(Y;\mathcal{G}_{|Y})$, and consider a closed, normal and constructible set $C \in \Phi$ such that the support of s is contained in $C \cap Y$. Since Φ is normal and constructible, there exist a closed, normal and constructible set $D \in \Phi$ and an open constructible set $O \subseteq X$ such that $C \subseteq O \subseteq D$. We shall find $\widetilde{t} \in \Gamma_D(U\cap Z;\mathcal{G})$ such that $\widetilde{t}_{|Y} = s$. After applying Lemma 3.22 above to X, $D\cap Z$ and $D\cap Y$, we see that there exist an open, constructible neighbourhood V of $D\cap Y$ in X and a section $t \in \Gamma(V\cap D\cap Z;\mathcal{G})$ such that $t_{|D\cap Y} = s_{|D\cap Y}$. Since $t_{|(D\setminus O)\cap Y} = 0$, then each point x of $(D\setminus O)\cap Y$ has an open constructible neighbourhood $W_x \subseteq V$ such that $t_{|W_x\cap D\cap Z} = 0$. Using quasi-compactness of $(D\setminus O)\cap Y$ (it is closed on the quasi-compact set $D\cap Y$), there exists a finite number of points $x_1,\dots,x_n$ such that $(D\setminus O)\cap Y \subseteq \bigcup_{i=1}^n W_{x_i}$. Then $W := \bigcup_{i=1}^n W_{x_i}$ is an open constructible subset and $t_{|W\cap D\cap Z} = 0$. Let $U_1 = (V\cap O)\cup W$. Then U_1 is open constructible, and $D\cap Y \subseteq U_1 \subseteq V$. Define $t' \in \Gamma(U_1\cap Z;\mathcal{G})$ by: $t'_{|V\cap O\cap Z} = t_{|V\cap O\cap Z}$ and $t'_{|W\cap Z} = 0$. This is well defined since $(V\cap O\cap Z)\cap(W\cap Z) \subseteq W\cap D\cap Z$ and $t_{|W\cap D\cap Z} = 0$. Also observe that $t'_{|U_1\cap D\cap Z} = t$. Let $U_2 = X\setminus D$. Then $U = U_1 \cup U_2$ is open constructible, $Y \subseteq U$, $U_1\cap U_2 \subseteq W$, and we can define $\widetilde{t} \in \Gamma(U\cap Z;\mathcal{G})$ in the following way: $\widetilde{t}_{|U_1\cap Z} = t'_{|U_1\cap Z}$, $\widetilde{t}_{|U_2\cap Z} = 0$. It is well defined since $U_1\cap U_2 \subseteq W$ and $t'_{|W\cap Z} = 0$. Moreover $\mathrm{supp}\,\widetilde{t} \subseteq D$ and $\widetilde{t}_{|Y} = s$ as required. □

For the locally semi-algebraic analogues of Lemmas 3.22 and 3.23, see [17, Chapter II, Lemma 3.1 and Proposition 3.2]. Note that in the locally semi-algebraic case, one only requires that Y has a paracompact and regular locally semi-algebraic neighbourhood in X, instead of a fundamental system of such neighbourhoods, since under this assumption, the family of all open locally semi-algebraic neighbourhoods of Y in X is a fundamental system of neighbourhoods of Y in X ([17, Chapter I, Lemma 5.5]). Moreover, these are paracompact and regular: that is, they are normal, and one can apply the shrinking lemma ([17, Chapter I, Proposition 5.1]). The assumption for Φ implies that the same holds for each $D\cap Y$.

The o-minimal versions are [25, Lemmas 3.2 and 3.3]. Unfortunately, in the statement of these lemmas and also in [25, Proposition 3.7], the assumption that Y (respectively, $D\cap Y$) has a fundamental system of normal and constructible locally closed neighbourhoods in X is missing. However, note that this does not affect the main results of that paper since in those results, $Y = X$ and each D satisfies the hypothesis as Φ is normal and constructible. The only results affected are the Vietoris-Begle theorem and the homotopy axiom ([25, Theorems 2.13 and 2.14]) that we fix below.

Recall that a sheaf $\mathcal{F}$ on a topological space X with a family of supports Φ is Φ-*soft* if and only if the restriction $\Gamma(X;\mathcal{F}) \longrightarrow \Gamma(S;\mathcal{F}_{|S})$ is surjective for every $S \in \Phi$. If Φ consists of all closed subsets of X, then $\mathcal{F}$ is simply called *soft*.

The topological analogue of the next result is [14, Chapter II, 9.3].

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Proposition 3.24. *Let X be a topological space and $\mathcal{F}$ be a sheaf in* $\mathrm{Mod}(A_X)$. *Suppose that Φ is a family of supports on X such that every $C \in \Phi$ has a neighbourhood D in X with $D \in \Phi$. Then the following are equivalent:*

(1) *$\mathcal{F}$ is Φ-soft;*
(2) *$\mathcal{F}_{|S}$ is soft for every $S \in \Phi$;*
(3) *$\Gamma_\Phi(X;\mathcal{F}) \longrightarrow \Gamma_{\Phi_{|S}}(S;\mathcal{F}_{|S})$ is surjective for every closed subset S of X;*
If in addition X is an object of $\widetilde{\mathfrak{T}}$ and Φ is a constructible family of supports on X, then the above are also equivalent to:
(4) *$\mathcal{F}_{|Z}$ is soft for every constructible subset Z of X that is in Φ;*
If moreover Φ is a normal and constructible family of supports on X, then the above are also equivalent to:
(5) *$\Gamma(X;\mathcal{F}) \longrightarrow \Gamma(Z;\mathcal{F}_{|Z})$ is surjective for every constructible subset Z of X that is in Φ.*

Proof. By our hypothesis on Φ, the arguments in [14, Chapter II, 9.3] show the equivalence of (1), (2) and (3). The equivalence of (2) and (4) is obvious since every $S \in \Phi$ is contained in some constructible subset Z of X that is in Φ. Also, (1) implies that (5) is obvious.

Assume (5), and let $S \in \Phi$ and $s \in \Gamma(S;\mathcal{F}_{|S})$. Since Φ is normal and constructible, S has a fundamental system of normal and constructible closed neighbourhoods in Φ. In particular, there is an open constructible subset V of X and a closed constructible subset D of X such that $S \subseteq V \subseteq D$ and $D \in \Phi$. By Lemma 3.22, s can be extended to a section $t \in \Gamma(W;\mathcal{F})$ of $\mathcal{F}$ over an open constructible subset W of X such that $S \subseteq W \subseteq V$. Applying the shrinking lemma in D, we find a closed constructible neighbourhood Z of S in W. Since $D \in \Phi$, we have $Z \in \Phi$. So $t_{|Z} \in \Gamma(Z;\mathcal{F}_{|Z})$, $(t_{|Z})_{|S} = s$, and $t_{|Z}$ can be extended to X by (5). □

The locally semi-algebraic analogue of Proposition 3.24 is [17, Chapter II, Propositions 4.1 and 4.2], and the o-minimal version is [25, Proposition 3.4].

The following topological result is often useful. It is an immediate consequence of Proposition 3.24, and we omit the proof and refer the reader to [25, Proposition 3.6] (compare also with [14, Chapter II, Propositions 9.2 and 9.12 and Corollary 9.13]).

Fact 3.25. Let X be a topological space and Φ be a family of supports on X such that every $C \in \Phi$ has a neighbourhood D in X with $D \in \Phi$. Let W be a locally closed subset of X. The following hold:

(i) If $\mathcal{F} \in \mathrm{Mod}(A_X)$ is Φ-soft, then $\mathcal{F}_{|W}$ is $\Phi_{|W}$-soft.
(ii) $\mathcal{G}$ in $\mathrm{Mod}(A_W)$ is $\Phi_{|W}$-soft if and only if $i_{W!}\mathcal{G}$ is Φ-soft.
(iii) If $\mathcal{F} \in \mathrm{Mod}(A_X)$ is Φ-soft, then $\mathcal{F}_W$ is Φ-soft.

We now come to the main result here. Its topological analogue is [14, Chapter II, 9.6, 9.9 and 9.10], and the locally semi-algebraic analogue is [17, Chapter II, Propositions 4.8, 4.12 and Corollary 4.13]. The o-minimal version is [25, Proposition 3.7] (as mentioned above, the assumption that $D \cap Y$ has a fundamental system of normal and constructible locally closed neighbourhoods in X is missing).

Proposition 3.26. *Assume that X is an object in $\widetilde{\mathfrak{T}}$. Let Z be a subspace of X, Φ be a normal and constructible family of supports on X and Y be a subspace of Z such that for every constructible $D \in \Phi$, $D \cap Y$ is a quasi-compact subset of Z having a fundamental system of normal and constructible locally closed neighbourhoods in X. Then the full additive subcategory of* $\mathrm{Mod}(A_Y)$ *of $\Phi \cap Y$-soft sheaves is $\Gamma_{\Phi\cap Y}(Y;\bullet)$-injective: that is,*

(1) *For every $\mathcal{F} \in \mathrm{Mod}(A_Y)$, there exists a $\Phi \cap Y$-soft $\mathcal{F}' \in \mathrm{Mod}(A_Y)$ and an exact sequence $0 \to \mathcal{F} \to \mathcal{F}'$.*
(2) *If $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$ is an exact sequence in $\mathrm{Mod}(A_Y)$ and $\mathcal{F}'$ is $\Phi \cap Y$-soft, then*

$$0 \longrightarrow \Gamma_{\Phi\cap Y}(Y;\mathcal{F}') \xrightarrow{\alpha} \Gamma_{\Phi\cap Y}(Y;\mathcal{F}) \xrightarrow{\beta} \Gamma_{\Phi\cap Y}(Y;\mathcal{F}'') \longrightarrow 0$$

is an exact sequence.

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(3) *If* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is an exact sequence in* $\mathrm{Mod}(A_Y)$ *and* $\mathcal{F}'$ *and* $\mathcal{F}$ *are* $\Phi \cap Y$*-soft, then* $\mathcal{F}''$ *is* $\Phi \cap Y$*-soft.*

Proof. Point (1) follows since on any topological space Y, the full additive subcategory of $\mathrm{Mod}(A_Y)$ of injective (and flabby) A-sheaves is co-generating (see, for example, [39, Proposition 2.4.3]) and injective A-sheaves are flabby, so $\Phi \cap Y$-soft by Lemma 3.23. On the other hand, (3) follows as usual from (2) by a simple diagram chase using Proposition 3.24 (3).

We now prove (2). Let $s'' \in \Gamma_{\Phi\cap Y}(Y, \mathcal{F}'')$. We have to find $s \in \Gamma_{\Phi\cap Y}(Y, \mathcal{F})$ such that $\beta(s) = s''$. Since Φ is normal and constructible, there exist an open constructible subset U of X and a closed constructible subset C of X such that $\mathrm{supp}\, s'' \subseteq U \subseteq C$ and $C \in \Phi$.

Suppose there is $t \in \Gamma(C \cap Y; \mathcal{F})$ such that $\beta(t) = s''_{|C\cap Y}$. Then $\beta(t_{|(C\setminus U)\cap Y}) = 0$, so $t_{|(C\setminus U)\cap Y} = \alpha(s')$ for some $s' \in \Gamma((C \setminus U) \cap Y; \mathcal{F}')$. Since $\mathcal{F}'$ is $\Phi \cap Y$-soft, s' can be extended to a section $s' \in \Gamma(C \cap Y; \mathcal{F}')$. Then $(t - \alpha(s'))_{|(C\setminus U)\cap Y} = 0$ and so can be extended, by 0, to a section $s \in \Gamma(Y; \mathcal{F})$ with $\mathrm{supp}\, s \subseteq C \cap Y \in \Phi \cap Y$ and $\beta(s) = s''$.

Now, considering the exact sequence

$$0 \to \mathcal{F}'_{C\cap Y} \xrightarrow{\alpha} \mathcal{F}_{C\cap Y} \xrightarrow{\beta} \mathcal{F}''_{C\cap Y} \to 0,$$

since by Fact 3.25 (iii) $\mathcal{F}'_{C\cap Y}$ is still $\Phi \cap Y$-soft, replacing $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ with $\mathcal{F}'_{C\cap Y}$, $\mathcal{F}_{C\cap Y}$ and $\mathcal{F}''_{C\cap Y}$, respectively, we are reduced to proving that the sequence

$$0 \to \Gamma(C \cap Y; \mathcal{F}') \xrightarrow{\alpha} \Gamma(C \cap Y; \mathcal{F}) \xrightarrow{\beta} \Gamma(C \cap Y; \mathcal{F}'') \to 0$$

is exact.

Let $s'' \in \Gamma(C \cap Y; \mathcal{F}'')$. There is a covering $\{U_j : j \in J\}$ of $C \cap Y$ by open constructible subsets of X and sections $s_j \in \Gamma(U_j \cap (C \cap Y); \mathcal{F})$, $j \in J$, such that $\beta(s_j) = s''_{|U_j\cap(C\cap Y)}$. By the assumptions on $C \cap Y$, we can assume that J is finite, so $\cup\{U_j : j \in J\}$ is an open constructible subset of X, and furthermore, there are an open constructible subset O' in X and a normal, constructible locally closed subset X' in X such that $C \cap Y \subseteq O' \subseteq X' \subseteq \cup\{U_j : j \in J\}$. For each $j \in J$, let $U'_j = U_j \cap X'$. Since X' is normal and constructible, by the shrinking lemma, there are open constructible subsets $\{V'_j : j \in J\}$ of X' and closed constructible subsets $\{K'_j : j \in J\}$ of X' such that $V'_j \subseteq K'_j \subseteq U'_j$ for every $j \in J$ and $X' = \cup\{V'_j : j \in J\}$. Now we may replace X by a constructible open subset and assume that X' is a constructible closed subset of X. For each $j \in J$, let $V_j = V'_j \cap O'$ and $K_j = K'_j$. Then for each $j \in J$, V_j is an open constructible subset of X and K_j is a closed constructible subset of X with $V_j \subseteq K_j \subseteq U_j$ and $C \cap Y \subseteq \cup\{V_j : j \in J\}$.

We now proceed by induction on $\#J$. Suppose that $J = I \cup \{j\}$, and let $V_I = \cup\{V_i : i \in I\}$, $K_I = \cup\{K_i : i \in I\}$ and $U_I = \cup\{U_i : i \in I\}$. By induction hypothesis, let $s_I \in \Gamma(K_I \cap (C \cap Y); \mathcal{F})$ be such that $\beta(s_I) = s''_{|K_I\cap(C\cap Y)}$. Since $\beta((s_I - s_j)_{|K_I\cap K_j}) = 0$, there is $s' \in \Gamma(K_I \cap K_j \cap (C \cap Y); \mathcal{F}')$ such that $\alpha(s') = (s_I - s_j)_{|K_I\cap K_j}$ that extends to $s' \in \Gamma(C \cap Y; \mathcal{F}')$ since $\mathcal{F}'$ is $\Phi \cap Y$-soft. Replacing s_I with $s_I - \alpha(s'_{|K_I\cap(C\cap Y)})$, we may suppose that $s_I = s_j$ on $K_I \cap K_j \cap (C \cap Y)$. Then since $(K_I \cup K_j) \cap (C \cap Y) = C \cap Y$, there exists $s \in \Gamma(C \cap Y; \mathcal{F})$ such that $s_{|K_I\cap(C\cap Y)} = s_I$ and $s_{|K_j\cap(C\cap Y)} = s_j$. Thus the induction proceeds. □

We now include several corollaries that will be useful later.

Corollary 3.27. *Assume that X is an object in $\widetilde{\mathfrak{T}}$. Suppose either that Φ is a normal and constructible family of supports on X and Z is a (constructible) locally closed subset of X or that Φ is any family of supports on X and Z is a closed subset of X. If $\mathcal{F} \in \mathrm{Mod}(A_Z)$, then*

$$H^*_\Phi(X; i_{Z!}\mathcal{F}) = H^*_{\Phi|Z}(Z; \mathcal{F}).$$

Proof. The second case is covered by [14, Chapter II, 10.1]. If Z is closed in an open subset U of X, then $\Phi_{|U}$ is a normal and constructible family of supports on U and $\Phi_{|Z} = \Phi_{|U} \cap Z$. In particular, if $D \in \Phi_{|U}$,

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then $D \cap Z \in \Phi_{|U}$, so $D \cap Z$ is a quasi-compact subset of Z having a fundamental system of normal and constructible locally closed neighbourhoods in U. Now the result follows from Proposition 3.26, Fact 3.25 (ii) and the fact that $\Gamma_\Phi(X; i_{Z!}\mathcal{F}) \simeq \Gamma_{\Phi|Z}(Z;\mathcal{F})$ ([14, Chapter I, Proposition 6.6]). □

It also follows that if X, Z, Y and Φ are as in Proposition 3.26, then $H^q_{\Phi\cap Y}(Y;\mathcal{G})$ is the qth cohomology of the cochain complex

$$0 \to \Gamma_{\Phi\cap Y}(Y;\mathcal{I}^0) \to \Gamma_{\Phi\cap Y}(Y;\mathcal{I}^1) \to \dots$$

where

$$0 \to \mathcal{G} \to \mathcal{I}^0_{|Y} \to \mathcal{I}^1_{|Y} \to \dots$$

is a resolution of $\mathcal{G}$ by $\Phi \cap Y$-soft sheaves in $\mathrm{Mod}(A_Y)$.

Moreover, if $\mathcal{G}$ is any flabby sheaf in $\mathrm{Mod}(A_X)$, then the restriction $\Gamma_\Phi(X;\mathcal{G}) \to \Gamma_{\Phi\cap Y}(Y;\mathcal{G}_{|Y})$ is surjective (by Lemma 3.23) and $H^p_{\Phi\cap Y}(Y;\mathcal{G}_{|Y}) = 0$ for all $p > 0$ (since $\mathcal{G}_{|Y}$ is $\Phi\cap Y$-soft by Lemma 3.22). This means Y is Φ-taut in X (see [14, Chapter II, Definition 10.5]).

Since on any topological space X an open subset is Ψ-taut in X for any family of supports Ψ on X, it follows from [14, Chapter II, Theorem 10.6] that:

Corollary 3.28. *Assume that X is an object in $\widetilde{\mathfrak{T}}$. Let Z be a subspace of X, Φ a normal and constructible family of supports on X and Y a subspace of Z such that for every constructible $D \in \Phi$, $D \cap Y$ is a quasi-compact subset of Z having a fundamental system of normal and constructible locally closed neighbourhoods in X. Then for every $\mathcal{G} \in \mathrm{Mod}(A_Z)$, the canonical homomorphism*

$$\varinjlim_{Y\subseteq U} H^q_{\Phi\cap U\cap Z}(U\cap Z;\mathcal{G}) \longrightarrow H^q_{\Phi\cap Y}(Y;\mathcal{G}_{|Y}),$$

where U ranges through the family of open constructible subsets of X containing Y, is an isomorphism for every $q \geq 0$.

For the locally semi-algebraic analogue of the above, compare with [17, Chapter II, Theorem 5.2].

Since fibres of morphisms in $\widetilde{\mathfrak{T}}$ are quasi-compact (Remark 3.10), applying Corollary 3.28, we obtain the following result (compare with [17, Chapter II, Theorem 7.1]).

Theorem 3.29 (Base change formula). *Let $f\colon X \to Y$ be a morphism in $\widetilde{\mathfrak{T}}$. Assume that f maps constructible closed subsets of X to closed subsets of Y and that every $\alpha \in Y$ has an open constructible neighbourhood W in Y such that α is closed in W and $f^{-1}(W)$ is a normal and constructible subset of X. Let $\mathcal{F} \in \mathrm{Mod}(A_X)$. Then for every $\alpha \in Y$, the canonical homomorphism*

$$(R^q f_*\mathcal{F})_\alpha \longrightarrow H^q(f^{-1}(\alpha);\mathcal{F}_{|f^{-1}(\alpha)})$$

is an isomorphism for every $q \geq 0$.

Proof. Fix $\alpha \in Y$. First, notice that

$$\{f^{-1}(V) : V \subseteq W \text{ an open constructible neighbourhood of } \alpha\}$$

is a fundamental system of open neighbourhoods of $f^{-1}(\alpha)$. Indeed, let U be an open neighbourhood of $f^{-1}(\alpha)$. Since $f^{-1}(\alpha)$ is quasi-compact, we may assume that U is an open constructible neighbourhood of $f^{-1}(\alpha)$. Then by assumption, $f(X \setminus U)$ is a closed subset of Y not containing α. Let V be an open constructible neighbourhood of α in W contained in $Y \setminus f(X \setminus U)$. Then $f^{-1}(V) \subseteq U$.

Also notice that $R^q f_*\mathcal{F}$ is the sheaf associated to the presheaf sending V to $H^q(f^{-1}(V);\mathcal{F})$. So

$$(R^q f_*\mathcal{F})_\alpha = \varinjlim_{\alpha\subseteq V\subseteq W} H^q(f^{-1}(V);\mathcal{F}) = \varinjlim_{f^{-1}(\alpha)\subseteq U} H^q(U;\mathcal{F}),$$

where V (respectively, U) ranges through the family of open constructible subsets of Y (respectively, X).

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Now, by assumption, $\alpha \in W$ is closed in W and $f^{-1}(W)$ is an open normal and constructible subset of X containing $f^{-1}(\alpha)$. So the family of all closed subsets of $f^{-1}(W)$ is a normal and constructible family of supports on $f^{-1}(W)$ (Example 3.20); and since $f^{-1}(\alpha)$ is closed in $f^{-1}(W)$, for every constructible closed subset $D \subseteq f^{-1}(W)$, $D \cap f^{-1}(\alpha)$ is a quasi-compact subset of $f^{-1}(W)$ having a fundamental system of normal and constructible locally closed neighbourhoods in $f^{-1}(W)$.

Therefore the result follows from Corollary 3.28 applied to $X = Z = f^{-1}(W)$, Φ the family of all closed subsets of $f^{-1}(W)$ and $Y = f^{-1}(\alpha)$. □

Using classical arguments, the Base change formula implies the following form of the Vietoris-Begle theorem:

Theorem 3.30 (Vietoris-Begle theorem). *Let $f\colon X \longrightarrow Y$ be a surjective morphism in $\mathfrak{T}$. Assume that f maps constructible closed subsets of X onto closed subsets of Y and that every $\alpha \in Y$ has an open constructible neighbourhood W in Y such that α is closed in W and $f^{-1}(W)$ is a normal and constructible subset of X. Let $\mathcal{F} \in \mathrm{Mod}(A_Y)$, and suppose that $f^{-1}(\alpha)$ is connected and $H^q(f^{-1}(\alpha); f^{-1}\mathcal{F}_{|f^{-1}(\alpha)}) = 0$ for $q > 0$ and all $\alpha \in Y$. Then for any constructible family of supports Ψ on Y, the induced map*

$$f^*\colon H^*_\Psi(Y;\mathcal{F}) \longrightarrow H^*_{f^{-1}\Psi}(X; f^{-1}\mathcal{F})$$

is an isomorphism.

Proof. The homomorphism $f^*\colon H^*_\Psi(Y;\mathcal{F}) \longrightarrow H^*_{f^{-1}\Psi}(X; f^{-1}\mathcal{F})$ is the composition $\epsilon \circ \eta$, where

$$\epsilon\colon H^*_\Psi(Y; f_*(f^{-1}\mathcal{F})) \to H^*_{f^{-1}\Psi}(X, f^{-1}\mathcal{F})$$

is the canonical edge homomorphism $E_2^{*,0} \to E^*$ in the Leray spectral sequence

$$H^p_\Psi(Y; R^q f_*(f^{-1}\mathcal{F})) \Rightarrow H^{p+q}_{f^{-1}\Psi}(X; f^{-1}\mathcal{F})$$

of $f^{-1}\mathcal{F}$ with respect to f and $\eta\colon H^*_\Psi(Y;\mathcal{F}) \to H^*_{f^{-1}\Psi}(X; f_*(f^{-1}\mathcal{F}))$ is induced by the canonical adjunction homomorphism

$$\mathcal{F} \to f_* f^{-1}\mathcal{F}.$$

By Theorem 3.29 and the hypothesis, $R^q f_*(f^{-1}\mathcal{F}) = 0$ for all $q > 0$, so the Leray sequence splits and ϵ is an isomorphism. On the other hand, by Theorem 3.29 and since $f^{-1}(\alpha)$ is connected, we have

$$\begin{aligned}(f_* f^{-1}\mathcal{F})_\alpha &= (R^0 f_* f^{-1}\mathcal{F})_\alpha\\ &\simeq H^0(f^{-1}(\alpha); (f^{-1}\mathcal{F})_{|f^{-1}(\alpha)})\\ &\simeq H^0(f^{-1}(\alpha); \mathcal{F}_\alpha)\\ &\simeq \mathcal{F}_\alpha.\end{aligned}$$

Hence adjunction homomorphism $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is an isomorphism and η is also an isomorphism. □

We end this section with the following, which follows quickly from previous results exactly the same way as its topological analogue ([14, Chapter II, 16.1]):

Proposition 3.31. *Assume that X is an object in $\widetilde{\mathfrak{T}}$, Φ is a normal and constructible family of supports on X and $\mathcal{F}$ is a sheaf in* $\mathrm{Mod}(A_X)$. *Then the following are equivalent:*

(1) *$\mathcal{F}$ is Φ-soft.*
(2) *$\mathcal{F}_U$ is Γ_Φ-acyclic for all open and constructible $U \subseteq X$.*

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(3) $H^1_{\Phi}(X; \mathcal{F}_U) = 0$ *for all open and constructible* $U \subseteq X$.
(4) $H^1_{\Phi|U}(U, \mathcal{F}_{|U}) = 0$ *for all open and constructible* $U \subseteq X$.

Proof. (1) ⇒ (2) follows from Proposition 3.26 and Fact 3.25 (iii). (2) ⇒ (3) is trivial. (3) is equivalent to (4) by Corollary 3.27. For (4) ⇒ (1), consider a constructible closed set C in Φ and the exact sequence $0 \longrightarrow \mathcal{F}_{X\setminus C} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_C \longrightarrow 0$. The associated long exact cohomology sequence

$$\cdots \to \Gamma_{\Phi}(X; \mathcal{F}) \to \Gamma_{\Phi|C}(X; \mathcal{F}_{|C}) \to H^1_{\Phi|X\setminus C}(X \setminus C; \mathcal{F}_{|X\setminus C}) \to \dots$$

shows that $\Gamma_{\Phi}(X; \mathcal{F}) \longrightarrow \Gamma_{\Phi|C}(X; \mathcal{F}_{|C})$ is surjective. Hence $\mathcal{F}$ is Φ-soft by Proposition 3.24 (5). □

4. A site on definable sets in Γ_∞

We will assume some familiarity with o-minimality. We refer the reader to classical texts like [53]. Note that we allow o-minimal structures to have end points, which is a subtle difference from [53]. Most results from [53] still hold in this context.

Let $\mathbf{\Gamma} = (\Gamma, <, \dots)$ be an o-minimal structure without end points. Let ∞ be a new symbol, and set $\Gamma_\infty = \Gamma \cup \{\infty\}$ with $x < \infty$ for all $x \in \Gamma$. The example that we have in mind is the case where $\mathbf{\Gamma} = (\Gamma, <, +)$ is the value group of algebraically closed field and ∞ is the valuation of 0.

When $(\Gamma, <)$ is nonarchimedean as a linear order (i.e., does not embed into the reals), then infinite definable sets in the structure $\mathbf{\Gamma}_\infty$ induced by $\mathbf{\Gamma}$ on Γ_∞ with their natural topology are in general totally disconnected and not locally compact. The goals in this section are (i) to introduce an appropriate site that will replace the topology just mentioned and show that definable sets with this site are $\mathcal{T}$-spaces (Subsection 4.1); and (ii) to show that for the associated notion of $\mathcal{T}$-normality, when $\mathbf{\Gamma}$ is an o-minimal expansion of an ordered group, $\mathcal{T}$-locally closed subsets are finite unions of $\mathcal{T}$-open subsets that are $\mathcal{T}$-normal (Subsection 4.2). We point out that normality in $\mathbf{\Gamma}_\infty$ is more complicated than in $\mathbf{\Gamma}$ (see Example 4.15 below), which justifies the extra work that will be done below.

4.1. The o-minimal site on $\mathbf{\Gamma}_\infty$-definable sets

Here we introduce the natural first-order logic structure $\mathbf{\Gamma}_\infty$ induced by $\mathbf{\Gamma}$ on Γ_∞ and show that $\mathbf{\Gamma}_\infty$-definable sets are equipped with a site making them $\mathcal{T}$-spaces.

For $x = (x_1, \dots, x_m) \in \Gamma^m_\infty$, the *support of* x, denoted $\operatorname{supp} x$, is defined by

$$\operatorname{supp} x = \{i \in \{1, \dots, m\} : x_i \neq \infty\}.$$

For $L \subseteq \{1, \dots, m\}$, let

$$(\Gamma^m_\infty)_L = \{x \in \Gamma^m_\infty : \operatorname{supp} x = L\}.$$

Then $\Gamma^m_\infty = \bigsqcup_{L\subseteq\{1,\dots,m\}} (\Gamma^m_\infty)_L$, $(\Gamma^m_\infty)_{\{1,\dots,m\}} = \Gamma^m$ and, if $\tau_L : \Gamma^m_\infty \to \Gamma^{|L|}_\infty$ is the projection onto the $|L|$ coordinates in L, then the restriction $\tau_{L|} : (\Gamma^m_\infty)_L \to \Gamma^{|L|}$ is a bijection and $\tau_{\{1,\dots,m\}|} : \Gamma^m \to \Gamma^m$ is the identity.

If $\pi : \Gamma^m_\infty \to \Gamma^k_\infty$ is the projection onto the first k coordinates, $\pi' : \Gamma^m_\infty \to \Gamma^{m-k}_\infty$ is the projection onto the last $m - k$ coordinates, and for $L \subseteq \{1, \dots, m\}$, $\pi(L) = \{1, \dots, k\} \cap L$ and $\pi'(L) = -k + \{k + 1, \dots, m\} \cap L$, then

(∗)

$$\operatorname{supp} x = \operatorname{supp} \pi(x) \sqcup (k + \operatorname{supp} \pi'(x))$$

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and moreover,

$$\begin{array}{ccccc} (\Gamma_\infty^m)_L & \hookrightarrow & \Gamma_\infty^m & \xrightarrow{\tau_L} & \Gamma_\infty^{|L|} \\ \downarrow{\scriptstyle \pi_|} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_k^L} \\ (\Gamma_\infty^k)_{\pi(L)} & \hookrightarrow & \Gamma_\infty^k & \xrightarrow{\tau_{\pi(L)}} & \Gamma_\infty^{|\pi(L)|} \end{array}$$

where π_k^L is a projection onto the first $\#\pi(L)$ coordinates and

$$\begin{array}{ccccc} (\Gamma_\infty^m)_L & \hookrightarrow & \Gamma_\infty^m & \xrightarrow{\tau_L} & \Gamma_\infty^{|L|} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ (\Gamma_\infty^k)_{\pi(L)} \times (\Gamma_\infty^{m-k})_{\pi'(L)} & \hookrightarrow & \Gamma_\infty^k \times \Gamma_\infty^{m-k} & \xrightarrow{\tau_{\pi(L)} \times \tau_{\pi'(L)}} & \Gamma_\infty^{|\pi(L)|} \times \Gamma_\infty^{|\pi'(L)|} \end{array}$$

are commutative diagrams.

If, for $X \subseteq \Gamma_\infty^m$ and for $L \subseteq \{1, \ldots, m\}$, we set

$$X_L = X \cap (\Gamma_\infty^m)_L,$$

then $X = \bigsqcup_{L \subseteq \{1,\ldots,m\}} X_L$ and $X_{\{1,\ldots,m\}} = X \cap \Gamma^m$. Furthermore, the restriction $\tau_{L|} \colon X_L \longrightarrow \tau_L(X_L)$ is a bijection, $\tau_L(X_L) \subseteq \Gamma^{|L|}$ and $\tau_{\{1,\ldots,m\}|} \colon X \cap \Gamma^m \longrightarrow X \cap \Gamma^m$ is the identity.

For each m, let

$$\mathfrak{G}_m = \{X \subseteq \Gamma_\infty^m : \tau_L(X_L) \subseteq \Gamma^{|L|} \text{ is } \mathbf{\Gamma}\text{-definable for every } L \subseteq \{1, \ldots, m\}\}.$$

Recall that an o-minimal structure (possibly with end points) has definable Skolem functions if, given a definable family $\{Y_t\}_{t \in T}$, there is a definable function $f \colon T \longrightarrow \bigcup_{t \in T} Y_t$ such that $f(t) \in Y_t$ for each $t \in T$. By the (observations before the) proof of [53, Chapter 6, (1.2)] (see also Comment (1.3) there), the o-minimal structure has definable Skolem functions if and only if every nonempty definable set X has a definable element $e(X) \in X$.

The following Proposition is left to the reader.

Proposition 4.1. $\mathbf{\Gamma}_\infty = (\Gamma_\infty, (\mathfrak{G}_m)_{m \in \mathbb{N}})$ *is an o-minimal structure with right end point* ∞*. Moreover, if* $\mathbf{\Gamma}$ *has definable Skolem functions, then* $\mathbf{\Gamma}_\infty$ *has definable Skolem functions.* □

Results in o-minimality usually are stated and proved for o-minimal structures without end points. Nearly all of those results can be checked to hold with exactly the same proof when there is an end point. Nevertheless, for convenience, we will now introduce a new structure without end points that contains as a substructure a copy of $\mathbf{\Gamma}_\infty$.

Let $\Sigma = \{0\} \times \Gamma_\infty \cup \{1\} \times \Gamma$ be equipped with the natural order extending $<$ on Γ and on Γ_∞ such that $(0, x) < (1, y)$ for all $x \in \Gamma_\infty$ and all $y \in \Gamma$.

Set $\Sigma_0 = \{0\} \times \Gamma_\infty$, $\Sigma_1 = \{1\} \times \Gamma$ and, for $\alpha \in 2^m$, which below we identify with a sequence of 0s and 1s of length m, let

$$\Sigma_\alpha = \Pi_{i=1}^m \Sigma_{\alpha(i)}.$$

Then $\Sigma^m = \bigsqcup_{\alpha \in 2^m} \Sigma_\alpha$, $\Sigma_{\bar{0}} = \Sigma_0^m$ and $\Sigma_{\bar{1}} = \Sigma_1^m$. If

$$\sigma_m \colon \Sigma^m \longrightarrow \Gamma_\infty^m \colon ((\alpha(1), x_1), \ldots, (\alpha(m), x_m)) \mapsto (x_1, \ldots, x_m)$$

is the natural projection, then the restriction $\sigma_{m|} \colon \Sigma_\alpha \longrightarrow \Gamma_\infty^m$ is injective, $\sigma_{m|} \colon \Sigma_{\bar{0}} \longrightarrow \Gamma_\infty^m$ is a bijection with inverse the natural inclusion $\iota_m \colon \Gamma_\infty^m \longrightarrow \Sigma_0^m \colon (x_1, \ldots, x_m) \mapsto ((0, x_1), \ldots, (0, x_m))$, and

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$\sigma_{m|}\colon \Sigma_{\bar{1}} \longrightarrow \Gamma^m$ is a bijection with inverse the natural inclusion $\iota_m\colon \Gamma^m \longrightarrow \Sigma_1^m\colon (x_1,\ldots,x_m) \mapsto ((1,x_1),\ldots,(1,x_m))$

If $\pi\colon \Sigma^m \longrightarrow \Sigma^k$ is the projection onto the first k pairs of coordinates, $\pi'\colon \Sigma^m \longrightarrow \Sigma^{m-k}$ is the projection onto the last $m-k$ pairs of coordinates, for $\alpha \in 2^m$, $\pi(\alpha) \in 2^k$ is given by $\pi(\alpha) = \alpha_{|\{1,\ldots,k\}}$ and $\pi'(\alpha) \in 2^{m-k}$ is given by $\pi'(\alpha)(s) = \alpha(s+k)$, then:

(∗∗)

$$\alpha = \pi(\alpha) * \pi'(\alpha),$$

where $*$ is a concatenation of sequences, and moreover,

$$\begin{array}{ccccc} \Sigma_\alpha & \hookrightarrow & \Sigma^m & \xrightarrow{\sigma_m} & \Gamma^m_\infty \\ \downarrow{\scriptstyle \pi_|} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Sigma_{\pi(\alpha)} & \hookrightarrow & \Sigma^k & \xrightarrow{\sigma_k} & \Gamma^k_\infty \end{array}$$

and

$$\begin{array}{ccccc} \Sigma_\alpha & \hookrightarrow & \Sigma^m & \xrightarrow{\sigma_m} & \Gamma^m_\infty \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \Sigma_{\pi(\alpha)} \times \Sigma_{\pi'(\alpha)} & \hookrightarrow & \Sigma^k \times \Sigma^{m-k} & \xrightarrow{\sigma_k \times \sigma_{m-k}} & \Gamma^k_\infty \times \Gamma^{m-k}_\infty \end{array}$$

are commutative diagrams.

If, for $X \subseteq \Sigma^m$ and for $\alpha \in 2^m$, we set

$$X_\alpha = X \cap \Sigma_\alpha,$$

then $X = \bigsqcup_{\alpha\in 2^m} X_\alpha$. Furthermore, the restriction $\sigma_{m|}\colon X_\alpha \longrightarrow \sigma_m(X_\alpha)$ is a bijection, $\sigma_m(X_\alpha) \subseteq \Gamma^m_\infty$.

For each m, let

$$\mathfrak{S}_m = \{X \subseteq \Sigma^m : \sigma_m(X_\alpha) \subseteq \Gamma^m_\infty \text{ is } \mathbf{\Gamma}_\infty\text{-definable for every } \sigma \in 2^m\}.$$

Similarly to Proposition 4.1, we have:

Proposition 4.2. $\mathbf{\Sigma} = (\Sigma, (\mathfrak{S}_m)_{m\in\mathbb{N}})$ *is an o-minimal structure. Moreover, if* $\mathbf{\Gamma}$ *has definable Skolem functions, then* $\mathbf{\Sigma}$ *has definable Skolem functions.* □

The following remarks are obvious from the constructions above:

Remark 4.3 (Natural embeddings). We have:

(a) The structure $\mathbf{\Gamma}$ and the substructure $(\Gamma, (\mathfrak{G}_m \cap \Gamma^m)_{m\in\mathbb{N}})$ of $\mathbf{\Gamma}_\infty$ have the same definable sets.
(b) Under the inclusion $\iota_1\colon \Gamma_\infty \longrightarrow \Sigma_0$, the structure $\mathbf{\Gamma}_\infty$ and the substructure $(\Sigma_0, (\mathfrak{S}_m \cap \Sigma_0^m)_{m\in\mathbb{N}})$ of $\mathbf{\Sigma}$ have the same definable sets.

So when convenient, below we will often identify $\mathbf{\Gamma}_\infty$ with its copy in $\mathbf{\Sigma}$.

We now make a couple of observations comparing $\mathbf{\Gamma}$, $\mathbf{\Gamma}_\infty$ and $\mathbf{\Sigma}$.

Remark 4.4 (The topologies). Let $\mathbf{\Gamma} = (\Gamma, <, \ldots)$ be an arbitrary o-minimal structure without end points. Then the topology of Γ is generated by the open intervals $(-\infty, a), (a,b)$ and (b,∞) with $a,b \in \Gamma$ and the topology of Γ^n is generated by the products of n such open intervals.

In $\mathbf{\Gamma}_\infty$ the topology of Γ_∞ is generated by the open intervals $(-\infty, a)$, (a, b) and $(b, \infty]$ with $a, b \in \Gamma$ and the topology of Γ_∞^n is generated by the products of n such open intervals.

It follows that:

- Γ^m is open in Γ_∞^m, and the topology on Γ^m is the induced topology from Γ_∞^m.
- Γ_∞^m is closed in Σ^m, and the topology on Γ_∞^m is the induced topology from Σ^m.

Recall the notion of *cell in* Γ_∞^n defined inductively as follows:

- A cell in Γ_∞ is either a singleton $\{a\}$ or $\{\infty\}$ or an open interval of the form $(-\infty, a)$ or (a, b) for $a, b \in \Gamma$ with $a < b$.
- A cell in Γ_∞^{n+1} is a set of the form

$$\Gamma(h) = \{(x, y) \in \Gamma_\infty^{n+1} : x \in X \text{ and } y = h(x)\},$$

or

$$(f, g)_X = \{(x, y) \in \Gamma_\infty^{n+1} : x \in X \text{ and } f(x) < y < g(x)\},$$

or

$$(-\infty, g)_X = \{(x, y) \in \Gamma_\infty^{n+1} : x \in X \text{ and } y < g(x)\},$$

for some $\mathbf{\Gamma}_\infty$-definable and continuous maps $f, g, h\colon X \to \Gamma_\infty$ with $f < g$, where X is a cell in Γ_∞^n.

In either case, X is called *the domain* of the defined cell.

Remark 4.5 (Cell decompositions). As in [53, Chapter 3, (2.11)] (the proof also works in the case with end points), we have:

- Given any $\mathbf{\Gamma}_\infty$-definable sets $A_1, \ldots, A_k \subseteq \Gamma_\infty^n$, there is a decomposition $\mathcal{C}$ of Γ_∞^n that partitions each A_i.
- Given a $\mathbf{\Gamma}_\infty$-definable map $f\colon A \to \Gamma_\infty$, there is a decomposition $\mathcal{C}$ of Γ_∞^n that partitions A such that the restriction $f_{|B}$ to each $B \in \mathcal{C}$ with $B \subseteq A$ is continuous.

Here (see [53, Chapter 3, (2.10)]), a decomposition of Γ_∞ is a partition of Γ_∞ into cells and a decomposition of Γ_∞^{k+1} is a partition of Γ_∞^{k+1} into cells such that its projection to Γ_∞^k is a decomposition of Γ_∞^k.

By the inductive construction of cells, it is easy to see that:

(a) If $\mathcal{C}$ is a cell decomposition of Γ_∞^m, then (i) $\mathcal{C}$ partitions the sets in $\{(\Gamma_\infty^m)_L : L \subseteq \{1, \ldots, m\}\}$; in particular, $\mathcal{C}_{|\Gamma^m}$ is a cell decomposition of Γ^m. (ii) The cells in $\mathcal{C}$ are built using $\mathbf{\Gamma}_\infty$-definable continuous maps $f\colon X \to \Gamma_\infty$ with $X \subseteq (\Gamma_\infty^m)_L$ a cell and such that f is constant with value ∞ or

$$f = f_L \circ \tau_L,$$

where $f_L\colon \tau_L(X) \to \Gamma$ is a $\mathbf{\Gamma}$-definable continuous map.

(b) If $\mathcal{B}$ is a cell decomposition of Σ^m that partitions the sets in $\{(\Gamma_\infty^m)_L : L \subseteq \{1, \ldots, m\}\}$, then $\mathcal{B}_{|\Gamma_\infty^m}$ is a cell decomposition of Γ_∞^m.

In an arbitrary o-minimal structure, a definable set X is *definably connected* if and only if the only clopen definable subsets of X are $\emptyset$ and X. We say that a $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m is $\mathbf{\Gamma}_\infty$*-definably locally closed* if and only if it is the intersection of an open $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m and a closed $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m or equivalently if and only if it is open in its closure in Γ_∞^m. A similar definition applies in $\mathbf{\Gamma}$.

Remark 4.6 (Definable connectedness). From the above remark about the topologies, a $\mathbf{\Gamma}$-definable subset of Γ^m is $\mathbf{\Gamma}$-definably connected if and only if it is $\mathbf{\Gamma}_\infty$-definably connected, and a $\mathbf{\Gamma}_\infty$-definable

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subset of Γ_∞^m is $\mathbf{\Gamma}_\infty$-definably connected if and only if it is $\mathbf{\Sigma}$-definably connected. Hence, just like in $\mathbf{\Gamma}$, by [53, Chapter 3, Proposition 2.18]:

- Every $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m has finitely many $\mathbf{\Gamma}_\infty$-definably connected components that are clopen and partition the $\mathbf{\Gamma}_\infty$-definable set.

Remark 4.7 (Definably locally closed subsets). Since cells in Γ^m are $\mathbf{\Gamma}$-definably locally closed ([53, page 51]), by cell decomposition theorem, every $\mathbf{\Gamma}$-definable subset of Γ^m is a finite union of $\mathbf{\Gamma}$-definably locally closed sets. Working in $\mathbf{\Sigma}$, it follows, by Remark 4.4, that:

- Every $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m is a finite union of $\mathbf{\Gamma}_\infty$-definably locally closed sets.

Just like in $\mathbf{\Gamma}$, if $(\Gamma, <)$ is nonarchimedean as a linear order, then infinite $\mathbf{\Gamma}_\infty$-definable spaces with the topology generated by open definable subsets are in general totally disconnected and not locally compact. So, as in Example 3.7 (3), to develop cohomology, one studies $\mathbf{\Gamma}_\infty$-definable spaces X equipped with the o-minimal site:

Definition 4.8 (O-minimal site on $\mathbf{\Gamma}_\infty$-definable subsets). Let $\mathbf{\Gamma} = (\Gamma, <, \ldots)$ be an arbitrary o-minimal structure without end points. If X is a $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m, then the o-minimal site X_{def} on X is the category $\mathrm{Op}(X_{\mathrm{def}})$ whose objects are open (in the topology of X mentioned above) $\mathbf{\Gamma}_\infty$-definable subsets of X, the morphisms are the inclusions, and the admissible covers $\mathrm{Cov}(U)$ of $U \in \mathrm{Op}(X_{\mathrm{def}})$ are covers by open $\mathbf{\Gamma}_\infty$-definable subsets of X with finite subcovers.

Proposition 4.9. *Let $\mathbf{\Gamma} = (\Gamma, <, \ldots)$ be an arbitrary o-minimal structure without end points. If X is a $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m, let*

$$\mathcal{T} = \{U \in \mathrm{Op}(X) : U \text{ is } \mathbf{\Gamma}_\infty\text{-definable}\}.$$

Then X is a $\mathcal{T}$-space, $X_{\mathcal{T}} = X_{\mathrm{def}}$, and $\widetilde{X}_{\mathcal{T}}$ is the o-minimal spectrum $\widetilde{X}$ of X: that is, its points are types over Γ_∞ concentrated on X. Furthermore, there is an equivalence of categories

$$\mathrm{Mod}(A_{X_{\mathrm{def}}}) \simeq \mathrm{Mod}(A_{\widetilde{X}}).$$

Proof. By Remark 4.4, $\mathcal{T}$ is a basis for a topology of X, and $\emptyset \in \mathcal{T}$; it is clear that $\mathcal{T}$ is closed under finite unions and intersections; by Remark 4.7, the $\mathcal{T}$-subsets of X are exactly the $\mathbf{\Gamma}_\infty$-definable subsets of X. Therefore, from Remark 4.6, it follows that every $U \in \mathcal{T}$ has finitely many $\mathcal{T}$-connected components. The rest follows from the definitions and results in Subsection 3.1. □

The following remark will allow us to work in $\mathbf{\Gamma}_\infty$ instead of $\mathbf{\Gamma}$ or in $\mathbf{\Sigma}$ instead of in $\mathbf{\Gamma}_\infty$ when convenient:

Remark 4.10. Let $\mathbf{\Gamma} = (\Gamma, <, \ldots)$ be an arbitrary o-minimal structure without end points. By Remark 4.4, we have:

- If $X \subseteq \Gamma^m$ is a $\mathbf{\Gamma}$-definable subset, then the o-minimal site of X in $\mathbf{\Gamma}$ is the same as the o-minimal site of X in $\mathbf{\Gamma}_\infty$, so the o-minimal sheaf and cohomology theories of X in $\mathbf{\Gamma}$ are the same as the o-minimal sheaf and cohomology theories of X in $\mathbf{\Gamma}_\infty$.
- If $X \subseteq \Gamma_\infty^m$ is a $\mathbf{\Gamma}_\infty$-definable subset, then the o-minimal site of X in $\mathbf{\Gamma}_\infty$ is the same as the o-minimal site of X in $\mathbf{\Sigma}$, so the o-minimal sheaf and cohomology theories of X in $\mathbf{\Gamma}_\infty$ are the same as the o-minimal sheaf and cohomology theories of X in $\mathbf{\Sigma}$.

4.2. Definable compactness and definable normality

In this subsection, we recall the notion of definable compactness for definable sets in $\mathbf{\Gamma}_\infty$ (defined in analogy to the case $\mathbf{\Gamma}$) and make a couple of remarks about locally definably compact and definable completions that will be used later. We introduce the notion of definable normality (which corresponds

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to $\mathcal{T}$-normality for the associated $\mathcal{T}$-topology) and prove that when $\mathbf{\Gamma}$ is an o-minimal expansion of an ordered group, every definably locally closed subset of Γ_∞^n is the union of finitely many open definable subsets that are definably normal (Theorem 4.28).

As in [45], we say that a $\mathbf{\Gamma}_\infty$-definable subset $X \subseteq \Gamma_\infty^m$ is *$\mathbf{\Gamma}_\infty$-definably compact* if and only if for every $\mathbf{\Gamma}_\infty$-definable and continuous map from an open interval in Γ_∞ into X, the limits at the end points of the interval exist in X. Recall that X is *locally $\mathbf{\Gamma}_\infty$-definably compact* if and only if every point in X has a $\mathbf{\Gamma}_\infty$-definably compact neighbourhood.

Remark 4.11 (Definable compactness). Note that in the above definition it is enough to consider open intervals of the form $(-\infty, a)$, (a, b) or (b, ∞), where $a, b \in \Gamma$ and $a < b$. Therefore:

- A $\mathbf{\Gamma}$-definable subset $X \subseteq \Gamma^m$ is $\mathbf{\Gamma}$-definably compact if and only if it is $\mathbf{\Gamma}_\infty$-definably compact.

On the other hand, since for any interval I in $\Sigma = \Sigma_0 \sqcup \Sigma_1$, we have that $I \cap \Sigma_0$ and $I \cap \Sigma_1$, if non-empty, are intervals in Γ_∞, it follows that:

- A $\mathbf{\Gamma}_\infty$-definable subset $X \subseteq \Gamma_\infty^m$ is $\mathbf{\Gamma}_\infty$-definably compact if and only if it is Σ-definably compact.

In particular, by [45, Theorem 2.1], we have:

- A $\mathbf{\Gamma}_\infty$-definable subset $X \subseteq \Gamma_\infty^m$ is $\mathbf{\Gamma}_\infty$-definably compact if and only if it is closed and *bounded in* Γ_∞^m: that is, $X \subseteq \Pi_{i=1}^m [c_i, \infty]$ for some $c_i \in \Gamma$ $(i = 1, \ldots, m)$.

Recall that a type β on X (i.e., an ultrafilter of definable subsets of X) is a *definable type on* X if and only if for every uniformly definable family $\{Y_t\}_{t \in T}$ of definable subsets of X, the set $\{t \in T : Y_t \in \beta\}$ is a definable set. And we say a definable type β on X has limit $a \in X$ if for any definable neighbourhood $a \in U$, we have β concentrates on U. It is worth pointing out that the definition in [35] of definable compactness of X is that any definable type on X has a limit in X. At first glance, this seems to be a stronger criterion than defined above. However, both of them turn out to be the same as being closed and bounded. The main advantage of the definable type definition is that one cannot use a definable path on X to describe the topological closure of X, yet the set of limits of definable types on X is the topological closure of X. This failure is also noted in [45, Theorem 2.3]. Nonetheless, this does not affect the characterisation of definable compactness in $\mathbf{\Gamma}_\infty$.

Remark 4.12 (Locally definably compact). As in topology, locally definably compact is equivalent to definably locally closed. Suppose that X is locally definably compact. Let $a \in X$ and K be a definably compact definable neighbourhood of a in X. Let U be an open definable neighbourhood of a such that $U \cap X \subseteq K$. Taking closures in $\overline{X}$ and noting that $\overline{U \cap \overline{X}} = \overline{U \cap X}$, we obtain

$$U \cap \overline{X} \subseteq \overline{U \cap X} \subseteq \overline{K} = K \subseteq X,$$

which shows that X is open in $\overline{X}$. The other implication is also easy.

Remark 4.13 (Definable completions). Let $X \subseteq \Gamma_\infty^m$ be a $\mathbf{\Gamma}_\infty$-definable set that is bounded in $\mathbf{\Gamma}_\infty$ and is locally $\mathbf{\Gamma}_\infty$-definably compact. Then the inclusion $i \colon X \hookrightarrow \overline{X}$ into the closure of X in Γ_∞^m is a $\mathbf{\Gamma}_\infty$-definable completion of X: that is,

(i) $\overline{X}$ is $\mathbf{\Gamma}_\infty$-definably compact.
(ii) $i \colon X \hookrightarrow \overline{X}$ is a $\mathbf{\Gamma}_\infty$-definable open immersion (i.e., $i(X)$ is open in $\overline{X}$ and $i \colon X \to i(X)$ is a $\mathbf{\Gamma}_\infty$-definable homeomorphism).
(iii) The image $i(X)$ is dense in $\overline{X}$.

Everything here is clear, and $i(X)$ is open in $\overline{X}$ by Remark 4.12.

It follows that:

- If $\mathbf{\Gamma} = (\Gamma, <, +, \ldots)$ is an o-minimal expansion of an ordered group $(\Gamma, <, +)$, then every $\mathbf{\Gamma}_\infty$-definably locally closed subset of Γ_∞^m is $\mathbf{\Gamma}_\infty$-definably completable.

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Indeed, let $p\colon \Gamma_\infty \to \Gamma_\infty \times \Gamma_\infty$ be the $\mathbf{\Gamma}_\infty$-definable map given by

$$p(x) = \begin{cases} (-x, 0) & \text{if } x < 0 \\ (0, x) & \text{otherwise.} \end{cases}$$

Then $p(\Gamma_\infty) \subseteq [0,\infty] \times [0,\infty]$ and $p\colon \Gamma_\infty \to p(\Gamma_\infty)$ is a $\mathbf{\Gamma}_\infty$-definable homeomorphism. So every $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^m is $\mathbf{\Gamma}_\infty$-definably homeomorphic to a $\mathbf{\Gamma}_\infty$-definable subset of $[0,\infty]^{2m}$ (which is in particular bounded in $\mathbf{\Gamma}_\infty$).

However, note that $\mathbf{\Gamma}$-definable completions do not exist in general for locally $\mathbf{\Gamma}$-definably compact spaces. Suppose that $\mathbf{\Gamma} = (\Gamma, <, +, \ldots)$ is a semi-bounded o-minimal expansion of an ordered group $(\Gamma, <, +)$ (see [21, Definition 1.5]). Let $X = \Gamma$, and suppose that there is a $\mathbf{\Gamma}$-definable completion $\iota\colon X \hookrightarrow P$ of X. Since P is $\mathbf{\Gamma}$-definably compact, the limit $\lim_{t\to+\infty} \iota(t)$ exists in P: call it a. Let (P_i, θ_i) be a $\mathbf{\Gamma}$-definable chart of P such that $a \in P_i$. Let $\Pi_{j=1}^{n_i}(c_j^-, c_j^+)$ be an open and bounded box in Γ^{n_i} containing $\theta_i(a)$. By continuity, there exists $b \in \Gamma$ be such that $\theta_i \circ \iota((b,+\infty)) \subseteq \Pi_{j=1}^{n_i}(c_j^-, c_j^+)$. Hence, $\theta_i \circ \iota_{|}\colon (b,+\infty) \to \Gamma^{n_i}$ has bounded image; so, by [21, Fact 1.6 and Proposition 3.1 (1)], this map must be eventually constant. This contradicts the fact that the map is injective.

Recall that, in an arbitrary o-minimal structure, a definable subset X with the induced topology is *definably normal* if and only if every two disjoint closed definable subsets of X can be separated by disjoint open definable subsets of X. This corresponds to X being $\mathcal{T}$-normal (Definition 3.11) for the $\mathcal{T}$-topology given by the o-minimal site on X.

Remark 4.14 (Definable normality). By Remark 4.4, definable normality of a definable set does not change when moving between $\mathbf{\Gamma}$, $\mathbf{\Gamma}_\infty$ and $\mathbf{\Sigma}$. Therefore, by Remark 4.11 and [23, Theorem 2.11], we have:

- If $\mathbf{\Gamma}$ has definable Skolem functions, then every $\mathbf{\Gamma}_\infty$-definably compact set is $\mathbf{\Gamma}_\infty$-definably normal.

On the other hand, if $\mathbf{\Gamma} = (\Gamma, <, +, \ldots)$ is an o-minimal expansion of an ordered group $(\Gamma, <, +)$, then by [53, Chapter 6 (3.5)], every $\mathbf{\Gamma}$-definable set is $\mathbf{\Gamma}$-definably normal. However, this is not true in $\mathbf{\Gamma}_\infty$ (and so it is also not true in $\mathbf{\Sigma}$), as shown in the following example.

Example 4.15. Let $a \in \Gamma$, and let $U = \Gamma_\infty^2 \setminus \{(\infty, a)\}$. Let $C = \Gamma \times \{a\}$, and let $D = \{\infty\} \times (\Gamma_\infty \setminus \{a\})$. Then U is an open $\mathbf{\Gamma}_\infty$-definable subset of Γ_∞^2 and C and D are closed and disjoint $\mathbf{\Gamma}_\infty$-definable subsets of U. We claim that there are no disjoint open $\mathbf{\Gamma}_\infty$-definable subsets V and W of U such that $C \subseteq V$ and $D \subseteq W$.

For a contradiction, suppose that such V and W exist. Let $b \in \Gamma$ be such that $b < a$. Let $f\colon \Gamma \to [b,a)$ be the $\mathbf{\Gamma}$-definable function given by $f(t) = \inf\{x \in (b,a) : \{t\} \times (x,a) \subseteq V \cap \Gamma^2\}$. By [21, Fact 1.6 and Proposition 3.1 (1)], there is $s \in \Gamma$ and $c \in [b,a)$ such that $f_{|(s,\infty)}$ is constant and equal to c. So $(s,\infty) \times (c,a) \subseteq V$. Let $(\infty, u) \in \{\infty\} \times (c,a) \subseteq D$ be any point. Since W is an open $\mathbf{\Gamma}_\infty$-definable neighbourhood of D in U, it is an open $\mathbf{\Gamma}_\infty$-definable neighbourhood of (∞, u) in U. So there are $c < u_1 < u < u_2 < a$ and $v < \infty$ such that $(v,\infty] \times (u_1,u_2) \subseteq W$. Let $w = \max\{s, v\}$. Then $(w,\infty) \times (u_1,u_2) \subseteq V \cap W$, contradicting the fact that V and W are disjoint.

Let us now recall the following result from [28] that will be true in $\mathbf{\Gamma}_\infty$ by working in $\mathbf{\Sigma}$. After a couple of lemmas, we will prove a result (Theorem 4.22 below) generalising this fact.

Fact 4.16 [28, Theorem 2.20]. Let Z and K be definable spaces in an o-minimal structure with definable Skolem functions with Z definably normal and K Hausdorff and definably compact. Then $Z \times K$ is definably normal. □

So from now on, let $\mathbf{M} = (M, <, \ldots)$ be an arbitrary o-minimal structure without end points and with definable Skolem functions.

Lemma 4.17. *If $I \subseteq M$ is an interval, then I is definably normal. In particular, every open definable subset of M has a finite cover by open, definable subsets that are definably normal.* [2]

Proof. Let C and D be disjoint, closed, definable subsets of I. Then C is a finite union of intervals. The result follows by induction on the number of such intervals. Suppose that C is an interval. Let c be the left end point of C and c' the right end point. Let $d = \sup\{x \in D : x < c\}$, and let $d' = \inf\{x \in D : c' < x\}$. Since C and D are closed in I and disjoint, we have $d < c \leq c' < d'$. Now take $d < u < v < c$ and $c' < v' < u' < d'$. Then $U = ((-\infty, u) \cup (u', \infty)) \cap I$ and $V = (v, v')$ are disjoint, open, definable subsets of I such that $D \subseteq U$ and $C \subseteq V$.

If C is a union of $n + 1$ intervals, let C' be the union of the leftmost n intervals, and let C'' be the rightmost interval. By induction, there are open definable subsets U', V' and U'', V'' of I such that $D \subseteq U', C' \subseteq V', D \subseteq U''$, $C'' \subseteq V'', U' \cap V' = \emptyset$ and $U'' \cap V'' = \emptyset$. Let $U = U' \cap U''$ and $V = V' \cup V''$. Then U and V are disjoint, open, definable subsets of I such that $D \subseteq U$ and $C \subseteq V$. □

Below, we use the following notation: for each $i \in \{1, \dots, m\}$, let

$$\pi_i \colon M^m \to M^{m-1}$$

be the projection omitting the ith coordinate, and let

$$\pi_i' \colon M^m \to M$$

be the projection onto the ith coordinate. If $Z \subseteq M^{m-1}$ is a definable subset and $f, g \colon Z \to M$ are continuous definable maps with $f < g$, then for each $i \in \{1, \dots, m\}$, we let

$$[f, g]_Z^i = \{x \in M^m : \pi_i(x) \in Z \text{ and } f \circ \pi_i(x) \leq x_i \leq g \circ \pi_i(x)\}$$

and we define

$$(f, g)_Z^i, \quad (f, g]_Z^i \text{ and } [f, g)_Z^i$$

in a similar way.

Lemma 4.18. *The restriction*

$$\pi_{i|} \colon [f, g]_Z^i \to Z$$

is a continuous, closed, definable map.

Proof. For $z \in Z$, let

$$D(z) = \{((d_1^-, d_1^+), \dots, (d_{m-1}^-, d_{m-1}^+) \in M^{2(m-1)} : z \in \Pi_{i=1}^{m-1}(d_i^-, d_i^+) \cap Z\}$$

and for $d \in D(z)$, let

$$U(z, d) = \Pi_{i=1}^{m-1}(d_i^-, d_i^+) \cap Z.$$

Then $\{U(z, d)\}_{d \in D(z)}$ is a uniformly definable system of fundamental open neighbourhoods of z in Z. Moreover, since the relation $d \preceq d'$ on $D(z)$ given by $U(z, d) \subseteq U(z, d')$ is a definable downwards directed order on $D(z)$, by [35, Lemma 4.2.18] (or [34, Lemma 2.19]), there is a definable type β on $D(z)$ such that for every $d \in D(z)$, we have $\{d' \in D(z) : d' \preceq d\} \in \beta$.

Let $S \subseteq [f, g]_Z^i$ be a closed definable subset. Suppose that $\pi_i(S)$ is not closed in Z. Then there is $z \in Z \setminus \pi_i(S)$ such that for all $d \in D(z)$, we have $U(z, d) \cap \pi_i(S) \neq \emptyset$. Then by definable Skolem

[2]Here, the existence of definable Skolem functions is not needed.

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functions, there is a definable map

$$h\colon D(z) \to S \subseteq [f,g]^i_Z$$

such that for every $d \in D(z)$, we have $\pi_i(h(d)) \in U(z,d) \cap \pi_i(S)$.

Let α be the definable type on S determined by the collection $\{A \subseteq S : h^{-1}(A) \in \beta\}$. Let α_1 be the definable type on $\pi_i(S)$ determined by the collection $\{A \subseteq \pi_i(S) : (\pi_i)^{-1}(A) \in \alpha\}$, and let α_2 be the definable type on M determined by the collection $\{A \subseteq M : (\pi'_i)^{-1}(A) \in \alpha\}$.

Claim 4.19. *The limit of α exists, it is of the form (z,c) and belongs to $[f,g]^i_Z$.*

Proof. We have that z is the limit of α_1: that is, for every open definable subset V of Z such that $z \in V$, we have $V \in \alpha_1$. Indeed, given any such V, there is d' such that $U(z,d') \subseteq V$ and

$$\begin{aligned} h^{-1}((\pi_i)^{-1}(V)) &\supseteq h^{-1}((\pi_i)^{-1}(\pi_i(S) \cap U(z,d'))) \\ &\supseteq h^{-1}((S \cap (\pi_i)^{-1}(U(z,d')))) \\ &\supseteq \{d'' \in D(z) : d'' \preceq d'\}. \end{aligned}$$

Since $[f,g]^i_Z \in \alpha$, it follows that for every open definable subset V of Z such that $z \in V$, we have $[f,g]^i_V = (\pi_i)^{-1}(V) \cap [f,g]^i_Z \in \alpha$. On the other hand, since α_2 is a definable type on M, by [41, Lemma 2.3], α_2 is not a cut, so α_2 is determined by either (i) $\{b < x : b \in M\}$, (ii) $x = a$, (iii) $\{b < x < a : b \in M,\ b < a\}$, (iv) $\{a < x < b : b \in M,\ a < b\}$ or (v) $\{x < b : b \in M\}$, where $a \in M$. In case (i), the limit of α_2 is $+\infty$; in cases (ii), (iii) and (iv), a is the limit of α_2; and in case (v), $-\infty$ is the limit of α_2. Let c be the limit of α_2.

We show that $f(z) \leq c \leq g(z)$. If $g(z) < c$, let l be such that $g(z) < l < c$. Then since g is continuous, there is an open definable subset V of Z such that $z \in V$ and $g(v) < l$ for all $v \in V$. Case (v) does not occur since we cannot have $g(z) < -\infty$; in the remaining cases, we would have $\emptyset = [f,g]^i_V \cap (\pi'_i)^{-1}((l,+\infty)) \in \alpha$, which is absurd. If $c < f(z)$ let l be such that $c < l < f(z)$. Then since f is continuous, there is an open definable subset V of Z such that $z \in V$ and $c < l < f(v)$ for all $v \in V$. Case (i) does not happen; in the remaining cases, we would have $\emptyset = [f,g]^i_V \cap (\pi'_i)^{-1}((-\infty,l)) \in \alpha$, which is absurd. □

It follows that $(z,c) \in [f,g]^i_Z$ is the limit of α. Since S is closed in $[f,g]^i_Z$ and α is a definable type on S, its limit (z,c) is in S. But then $z \in \pi_i(S)$, which is a contradiction. □

The previous result allows us to obtain a slight generalisation of [28, Lemma 2.23]. In that Lemma, instead of $[f,g]^i_Z$, we have $Z \times [a,b]$. The proof of this new version is exactly the same using Lemma 4.18 instead of the fact that the projection $\pi\colon Z \times [a,b] \to Z$ is a continuous, closed definable map. For the reader's convenience, we include the details.

Lemma 4.20. *Let $Z \subseteq M^{m-1}$ be a definably normal definable subset. Let $S \subseteq [f,g]^i_Z$ be a closed definable subset and $W \subseteq [f,g]^i_Z$ an open definable subset. Then for every closed definable subset $F \subseteq \pi_i(S)$ such that $S \cap (\pi_i)^{-1}(F) \subseteq W$, there is an open definable neighbourhood O of F in Z such that $O \subseteq \overline{O} \cap Z \subseteq \pi_i(W)$ and $S \cap (\pi_i)^{-1}(\overline{O} \cap Z) \subseteq W$.*

Proof. Let $W^c = ([f,g]^i_Z) \setminus W$. If $S \subseteq W$, then since $\pi_i(S) \subseteq \pi_i(W)$ is closed in Z (by Lemma 4.18) and Z is definably normal, there is an open definable neighbourhood O of $\pi_i(S) \supseteq F$ in Z such that $O \subseteq \overline{O} \cap Z \subseteq \pi_i(W)$, so $S \cap (\pi_i)^{-1}(\overline{O} \cap Z) = S \subseteq W$. So we may suppose that $S \cap W^c \neq \emptyset$.

For $z \in Z$, let $\{U(z,d)\}_{d \in D(z)}$ be the uniformly definable system of fundamental open neighbourhoods of z in Z given above. Recall that by [35, Lemma 4.2.18] (or [34, Lemma 2.19]), there is a definable type β on $D(z)$ such that for every $d \in D(z)$, we have $\{d' \in D(z) : d' \preceq d\} \in \beta$.

Suppose that $z \in F$ and for all $d \in D(z)$, we have $(S \cap (\pi_i)^{-1}(U(z,d))) \cap W^c \neq \emptyset$. Then by definable Skolem functions, there is a definable map

$$h\colon D(z) \to S \cap W^c \subseteq [f,g]^i_Z$$

such that for every $d \in D(z)$, we have $h(d) \in (S \cap (\pi_i)^{-1}(U(z,d))) \cap W^c$.

Let α be the definable type on $S \cap W^c$ determined by the collection $\{A \subseteq S \cap W^c : h^{-1}(A) \in \beta\}$. We are in the setup of Claim 4.19, so the limit of α exists; it is of the form $(z,c) \in [f,g]^i_Z$. Since $S \cap W^c$ is closed and α is a definable type on $S \cap W^c$, its limit (z,c) is in $S \cap W^c$. But then $(z,c) \in S \cap (\pi_i)^{-1}(z) \subseteq W^c$, which contradicts the assumption on F.

So for each $z \in F$, there is $d \in D(z)$ such that $S \cap (\pi_i)^{-1}(U(z,d)) \subseteq W$. By definable Skolem functions, there is a definable map $\epsilon\colon F \to M^{2(m-1)}$ such that for each $z \in F$, we have $\epsilon(z) \in D(z)$ and $S \cap (\pi_i)^{-1}(U(z,\epsilon(z))) \subseteq W$. Then

$$U(F,\epsilon) = \bigcup_{z \in F} U(z,\epsilon(z))$$

is an open definable neighbourhood of F in Z such that

$$S \cap (\pi_i)^{-1}(U(F,\epsilon)) = \bigcup_{z \in F} S \cap (\pi_i)^{-1}(U(z,\epsilon(z))) \subseteq W.$$

Since $U(F,\epsilon) \cap \pi_i(W)$ is an open definable neighbourhood of F in Z, F is closed in Z and Z is definably normal, there is an open definable neighbourhood O of F in Z such that $O \subseteq \overline{O} \cap Z \subseteq U(F,\epsilon) \cap \pi_i(W) \subseteq \pi_i(W)$ and $S \cap (\pi_i)^{-1}(\overline{O} \cap Z) \subseteq W$. □

We now obtain the following generalisation of the fact that if $Z \subseteq M^{m-1}$ is definably normal, then $Z \times [a,b]$ is definably normal. We omit the proof since it is exactly the same as in [28, Lemma 2.24, proof of Proposition 2.21] using Lemma 4.20 and basic o-minimality from [53].

Proposition 4.21. *If $Z \subseteq M^{m-1}$ is definably normal, then $[f,g]^i_Z$ is also definably normal.* □

We now extend this result a bit further. First we introduce some notation. Given $I \subseteq \{1,\dots,m\}$, we let $I' = \{1,\dots,m\} \setminus I$, we let

$$\pi_I : M \to M^{m-|I|}$$

be the projection omitting the coordinates in I, and we let

$$\pi'_I : M \to M^{|I|}$$

be the projection onto the coordinates in I (so $\pi'_I = \pi_{I'}$). Given $Z \subseteq M^{|I|}$ a definable set and families $\{f^l : Z \to M\}_{l \in I'}$ and $\{g^l : Z \to M\}_{l \in I'}$ of continuous definable functions, we set

$$[\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z = \{x \in M^m : \pi_{I'}(x) \in Z \text{ and } f^l \circ \pi_{I'}(x) \leq x_l \leq g^l \circ \pi_{I'}(x),\ \forall l \in I'\}.$$

Similarly, we define

$$(\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'})^I_Z,\quad (\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z \text{ and } [\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'})^I_Z.$$

If $g^l = g$ for all $l \in I'$, we write

$$[\{f^l\}_{l\in I'}, g]^I_Z, (\{f^l\}_{l\in I'}, g)^I_Z,\quad (\{f^l\}_{l\in I'}, g]^I_Z \text{ and } [\{f^l\}_{l\in I'}, g)^I_Z$$

instead and similarly for the case $f^l = f$ for all $l \in I'$ (although we will not need this other case).

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Note that if $I = \{1, \ldots, m\}$, then

$$[\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z = (\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z = \ldots = Z.$$

Theorem 4.22. *If $Z \subseteq M^{|I|}$ is definably normal, then $[\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z$ is also definably normal.*

Proof. The proof is by induction on m. The case $m = 0$ is clear, so assume $m > 0$ and the result holds for $m - 1$. If $|I'| = 0$, then $I = \{1, \ldots, m\}$, so $[\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z = Z$. So suppose that $|I'| > 0$ and choose $i \in I'$ and set $J = I$ and $J' = I' \setminus \{i\}$. Then $X = [\{f^l\}_{l\in J'}, \{g^l\}_{l\in J'}]^J_Z \subseteq M^{m-1}$ is definably normal by the induction hypothesis.

Now let $F : X \to M$ be given by $F(u) = f^i \circ \pi_{J'}(u)$ and $G : X \to M$ be given by $G(u) = g^i \circ \pi_{J'}(u)$. Then F and G are continuous definable functions, $F \circ \pi_i(x) = f^i \circ \pi_{J'}(\pi_i(x)) = f^i \circ \pi_{I'}(x)$ and $G \circ \pi_i(x) = g^i \circ \pi_{I'}(x)$. Therefore,

$$[\{f^l\}_{l\in I'}, \{g^l\}_{l\in I'}]^I_Z = [F, G]^i_X$$

and so, by Proposition 4.21, this set is definably normal. □

To proceed, we need to recall the following results:

Fact 4.23 [24, Theorem 2.2]. Let U be an open definable subset of M^m. Then U is a finite union of open definable sets definably homeomorphic, after possibly reordering of coordinates, to open cells. □

Let $\pi : M^m \to M^{m-1}$ be the projection onto the first $m - 1$ coordinates. Clearly, the following also holds with π replaced by each $\pi_i : M^m \to M^{m-1}$ as defined above.

Fact 4.24 [19, Theorem 3.4]. Let U be an open definable subset of M^m. Then there is a finite cover $\{U_j : j = 1, \ldots, l\}$ of $\pi(U)$ by open definable subsets such that for each i there is a continuous definable section $s_j : U_j \to U$ of π (i.e., $\pi \circ s_j = \mathrm{id}_{U_j}$). □

We now go back to the setting $\mathbf{\Gamma}, \mathbf{\Gamma}_\infty$ and $\mathbf{\Sigma}$. Note that by the several remarks made previously regarding definability and definable topological notions (including connectedness, compactness, normality), we may safely omit the prefix $\mathbf{\Gamma}$ (respectively, $\mathbf{\Gamma}_\infty$ and $\mathbf{\Sigma}$) and simply say definable (respectively, definably connected, definably compact, definably normal) when talking about subsets of Γ^m or Γ^m_∞ and Σ^m.

Recall that given $I \subseteq \{1, \ldots, m\}$, $I' = \{1, \ldots, m\} \setminus I$,

$$\pi_I : \Sigma^m \to \Sigma^{m-|I|}$$

is the projection omitting the coordinates in I and

$$\pi'_I : \Sigma^m \to \Sigma^{|I|}$$

is the projection onto the coordinates in I (so $\pi'_I = \pi_{I'}$). If $Z \subseteq \Sigma^{|I|}$ is a definable set and $\{f^l : Z \to \Sigma\}_{l\in I'}$ a family of continuous definable functions, we set

$$(\{f^l\}_{l\in I'}, \infty]^I_Z = \{x \in \Sigma^m : \pi_{I'}(x) \in Z \text{ and } f^l \circ \pi_{I'}(x) < x_l \leq \infty \text{ for all } l \in I'\}.$$

Also recall that, if $I = \{1, \ldots, m\}$, then $(\{f^l\}_{l\in I'}, \infty]^I_Z = Z$.

Proposition 4.25. *Let $O \subseteq \Gamma^m_\infty$ be an open definable subset. Then O is a finite union of open definable subsets of the form $(\{f^l_V\}_{l\in I'}, \infty]^I_V$ with $I \subseteq \{1, \ldots, m\}$ and V an open definable subset of $\Gamma^{|I|}$.*

Proof. Let $I \subsetneq \{1, \ldots, m\}$ be such that $O_I \neq \emptyset$. For $A \subseteq \tau_I(O_I)$ and $B \subseteq \Gamma^{|I'|}_\infty$, we let $A \star_I B$ denote the set defined by

$$(a_1, \ldots, a_m) \in A \star_I B \Leftrightarrow \begin{cases} a_i \in B & \text{if } i \in I' \\ a_i \in \pi'_i(A) & \text{if } i \in I, \end{cases}$$

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where $\pi_i' : \tau_I(O_I) \to \Gamma$ is the restriction of the projection onto the ith coordinate. Consider the definable set

$$U_I := \left\{(a,b) \in \tau_I(O_I) \times \Gamma : \begin{array}{l}\text{there is an open neighbourhood } U \text{ of } a \text{ in } \tau_I(O_I) \\ \text{and } c \in \Gamma \text{ such that } c < b \text{ and } U \star_I (c,\infty]^{|I'|} \subseteq O\end{array}\right\}.$$

Let $\pi : U_I \to \tau_I(O_I) \subseteq \Gamma^{|I|}$ be the projection to the first $|I|$-coordinates.

Claim 4.26. $\pi(U_I) = \tau_I(O_I)$.

The left-to-right inclusion is trivial. From right to left, let a be an element of $\tau_I(O_I)$. Then there is $a' \in O$ such that $\tau_I(a') = a$. Since O is open, let $V = \prod_{i=1}^m J_i \subseteq O$ be a product of basic open sets containing a'. Then $U := \tau_I(V)$ is a neighbourhood of a in $\tau_I(O_I)$. Moreover, for $i \in I'$, we may suppose $J_i = (b_i, \infty]$ for some $b_i \in \Gamma$. Let $c = \max\{b_i : i \in I'\}$. Then $U \star_I (c,\infty]^{|I'|} \subseteq O$. In particular, $(a,b) \in U_I$ for every element $b \in (c,\infty)$, which shows that $a \in \pi(U_I)$. This proves the claim.

Claim 4.27. *The set U_I is open.*

Let (a,b) be an element in U_I. By definition, let U be a neighbourhood of a in $\tau_I(O_I)$ and $c \in \Gamma$ such that $c < b$ and $U \star_I (c,\infty]^{|I'|} \subseteq O$. We let the reader verify that $U \times (c,\infty) \subseteq U_I$, which shows the claim.

By Fact 4.24 applied to U_I and the projection $\pi : U_I \to \Gamma^{|I|}$, there exists a finite cover $\mathcal{V}_I$ of $\pi(U_I) = \tau_I(O_I)$ by open definable subsets, and for each $V \in \mathcal{V}_I$, there is a continuous definable section $s_V : V \to U_I$ of π. We let the reader verify that the previous claims imply that the family

$$\{(\{s_V\}_{i\in I'}, \infty]_V^I\}_{V\in\mathcal{V}_I} \tag{$*$}$$

is a finite family of open definable subsets of O covering O_I.

The union of $O_{\{1\ldots,m\}}$ and all families as in equation $(*)$ for all $I \subsetneq \{1,\ldots,m\}$ provide the required family of open sets. □

We are finally ready to prove the main goal of this subsection:

Theorem 4.28. *Suppose that* $\mathbf{\Gamma} = (\Gamma, <, +, \ldots)$ *is an o-minimal expansion of an ordered group* $(\Gamma, <, +)$. *Let Z be a definably locally closed subset of Γ_∞^m. Then Z is the union of finitely many relatively open definable subsets that are definably normal.*

Proof. Since a definably locally closed set is of the form $O \cap S$ with O open definable and S closed definable, it is a closed definable subset of an open definable subset. Therefore it is enough to prove the result for an open definable subset O of Γ_∞^m.

By Proposition 4.25, it is enough to show that open definable subsets of O of the form $(\{f_V^l\}_{l\in I'}, \infty]_V^I$ with $I \subseteq \{1,\ldots,m\}$ and V an open definable subset of $\Gamma^{|I|}$ are definably normal. So fix $I \subseteq \{1,\ldots,m\}$ and V an open definable subset of $\Gamma^{|I|}$. If $I' = \emptyset$, the $(\{f_V^l\}_{l\in I'}, \infty]_V^I = V \subseteq \Gamma^m$ is definably normal (Remark 4.14). Suppose otherwise, let $P = (\{f_V^l\}_{l\in I'}, \infty]_V^I$, and consider

$$X = (\{f_V^l\}_{l\in I'}, \{2f_V^l\}_{l\in I'})_V^I, \quad Y = (\{\tfrac{3}{2}f_V^l\}_{l\in I'}, \{\tfrac{5}{2}f_V^l\}_{l\in I'})_V^I$$

and

$$Z = [\{2f_V^l\}_{l\in I'}, \infty]_V^I.$$

Since $X, Y \subseteq \Gamma^m$, by Remark 4.14, they are both definably normal, and by Theorem 4.22, Z is also definably normal.

Let $C, D \subseteq P$ be closed, disjoint definable subsets. Since $C \cap Y, D \cap Y \subseteq Y$ are closed, disjoint definable subsets and Y is definably normal, there are $U_Y, V_Y \subseteq Y$ open, disjoint definable subsets of Y

such that $C \cap Y \subseteq U_Y$ and $D \cap Y \subseteq V_Y$. Similarly, there are $U_Z, V_Z \subseteq Z$ open, disjoint definable subsets of Z such that $C \cap Z \subseteq U_Z$ and $D \cap Z \subseteq V_Z$, and there are $U_X, V_X \subseteq X$ open, disjoint definable subsets of X such that $C \cap X \subseteq U_X$ and $D \cap X \subseteq V_X$. Again, by definable normality, let $U'_X \subseteq X$ be an open definable subset of X such that $C \cap X \subseteq U'_X \subseteq \overline{U'_X} \subseteq U_X$, and similarly let $U'_Z \subseteq Z$ be an open definable subset of Z such that $C \cap Z \subseteq U'_Z \subseteq \overline{U'_Z} \subseteq U_Z$ and let $U'_Y \subseteq Y$ be an open definable subset of Y such that $C \cap Y \subseteq U'_Y \subseteq \overline{U'_Y} \subseteq U_Y$.

Let $V'_Y = V_Y \setminus (\overline{U'_Z} \cup \overline{U'_X})$, $V'_X = V_X \setminus (\overline{U'_Y} \cup \overline{U'_Z})$ and $V'_Z = V_Z \setminus (\overline{U'_Y} \cup \overline{U'_X})$. Let

$$Q = \{x \in \Gamma^m : \pi_{I'}(x) \in V \text{ and } x_l = (2f^l_V) \circ \pi_{I'}(x) \text{ for all } l \in I'\}.$$

We have that Q is a closed definable subset of P and of Y (the f^l_Vs are continuous). Let $U''_Z = U'_Z \setminus Q$, and let $V''_Z = V'_Z \setminus Q$.

Let $U = U'_Y \cup U''_Z \cup U'_X$, and let $V = V'_Y \cup V''_Z \cup V'_X$. Since $C \cap Q \subseteq U'_Y$, we clearly have $C \subseteq U$. On the other hand, $D \cap \overline{U'_X} = D \cap \overline{U'_Z} = D \cap \overline{U'_Y} = \emptyset$; otherwise, $D \cap U_X \neq \emptyset$ and $U_X \cap V_X \neq \emptyset$ or, similarly, $U_Z \cap V_Z \neq \emptyset$ or $U_Y \cap V_Y \neq \emptyset$. Thus, since $D \cap Q \subseteq V'_Y$, we also have $D \subseteq V$. By construction, we have $U \cap V = \emptyset$. Now U'_Y is open in Y and Y is open in P, U'_X is open in X and X is open in P, and U''_Z is the interior of U'_Z in P, so U is open in P. Similarly, V is open in P. □

5. Sites on definable sets and in their stable completions in ACVF

In this section, we will introduce the v+g-site on definable sets of algebraically closed nontrivially valued fields and the $\widehat{\text{v+g}}$-site on their corresponding stable completion. We will further show that the latter forms a $\mathcal{T}$-space in the sense of Section 3. Definable compactness and normality will be discussed in Subsection 5.4.

We begin by recalling the needed model-theoretic background on ACVF. Some familiarity with valued fields and their model-theory will be assumed, however. For further references, we refer the reader to [51] or [33].

5.1. Preliminaries on ACVF

Let $\mathcal{L}_{k,\Gamma}$ be the three sorted languages of valued fields: in both the valued field sort **VF** and the residue field sort k, we put the language of rings $\mathcal{L}_{\text{ring}}$, and in the value group sort Γ_∞, the language $\mathcal{L}^\infty_{\text{og}}$ of ordered groups with an additional constant symbol for ∞, together with symbols for the valuation val and the map Res: $\mathbf{VF}^2 \to k$ sending (x, y) to $\text{res}(xy^{-1})$ if $\text{val}(x) \geq \text{val}(y)$ and $y \neq 0$ and to 0 otherwise.

Let $\mathcal{L}_\mathcal{G}$ be the geometric language extending $\mathcal{L}_{k,\Gamma}$ with new sorts described in [32, Section 3.1]. By [33, Theorem 2.1.1], the theory ACVF of algebraically closed fields with a nontrivial valuation has quantifier elimination in the language $\mathcal{L}_{k,\Gamma}$; and by [33, Theorem 7.3], it has quantifier elimination and elimination of imaginaries in the language $\mathcal{L}_\mathcal{G}$.

Below, we fix a monster model $\mathbb{U}$ of ACVF in the language $\mathcal{L}_\mathcal{G}$ and assume that all sets of parameters we consider are small substructures C of $\mathbb{U}$ (i.e., subsets C of $\mathbb{U}$ such that $\text{dcl}(C) = C$ and $|C| < |\mathbb{U}|$) and all models K of ACVF considered are elementary substructures of $\mathbb{U}$, again of smaller cardinality. By $\mathcal{R}$, $\mathcal{M}$ and k, we mean the valuation ring, its maximal ideal and the residue field in $\mathbb{U}$.

It is worth noting that over an elementary small substructure K of $\mathbb{U}$, the residue field and the value group are *stably embedded*: that is, every definable subset of $\Gamma^n_\infty(K)$ (respectively, $k(K)^n$) is already definable in $(\Gamma_\infty(K), \mathcal{L}^\infty_{\text{og}})$ (respectively, $(k(K), \mathcal{L}_{\text{ring}})$). See [32, Proposition 2.1.3].

Let C be a small substructure of $\mathbb{U}$. By an algebraic variety V over C, we mean the set of $\mathbb{U}$-closed points of a separated reduced scheme of finite type over the valued field sort of C (e.g., $\mathbb{A}^n$ stands for $\mathbb{U}^n$). By an algebraic variety, we mean an algebraic variety V over some small substructure C. Note that the category of separated reduced schemes of finite type over $\mathbb{U}$ and the category of their $\mathbb{U}$-closed points are equivalent (see [31, Proposition 2.6]).

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Given an algebraic variety V over C, there are finitely many charts $(V_i, U_i, f_i)_{i\le m}$ such that (i) $V = V_1 \cup \ldots \cup V_m$; (ii) each $f_i : V_i \to U_i$ is a isomorphism with $U_i \subseteq \mathbb{A}^{n_i}$ an affine variety over C; (iii) $U_{ij} = f_i(V_i \cap V_j)$ is a Zariski open subset of U_i; and (iv) $f_j \circ f_i^{-1}$ is an isomorphism between the quasi-affine varieties U_{ij} and U_{ji} over C. If W is another variety over C, then $f : V \to W$ is a morphism if and only if, for some charts $(W_j, U'_j, g_j)_{j\le k}$ for W, each $(g_j \circ f \circ f_i^{-1})_| : f_i(f^{-1}(W_j) \cap U_i) \to g_j(U'_j)$ is a morphism of quasi-affine varieties over C. As in [47, Section 3], by a *C-definable subset of* V, we mean a subset S such that for each i, $f_i(S \cap V_i)$ is a C-definable subset of U_i. Note that this notion is independent of the choice of the charts.

By a C-definable set, we mean a C-definable subset of some product of sorts and of varieties over C in $\mathbb{U}$. By a definable set, we mean a C-definable set for some small substructure C. We denote by

$$\mathrm{Def}_C$$

the category whose objects are C-definable sets and whose morphisms are C-definable maps between C-definable sets (i.e., maps between C-definable sets whose graphs are C-definable sets). Given a substructure F containing C, we have an inclusion functor $\mathrm{Def}_C \to \mathrm{Def}_F$ since every C-definable set is also F-definable.

Given an algebraic variety V with charts $(V_i, U_i, f_i)_{i\le m}$, we have that the Zariski topology on V: $U \subseteq V$ is open if and only if each $f_i(U \cap V_i)$ is a Zariski open subset of U_i. Besides the Zariski topology, we can also equip V with the *valuation topology*: $U \subseteq V$ is open if and only if each $f_i(U \cap V_i)$ is an open subset of $U_i \subseteq \mathbb{A}^{n_i}$ in the valuation topology. Recall that the valuation topology on $\mathbb{A}^1$ is generated by the open balls

$$B^\circ(a, \gamma) = \{x \in \mathbb{A}^1 : \mathrm{val}(x - a) > \gamma\}$$

centred at $a \in \mathbb{A}^1$ and with radius $\gamma \in \Gamma_\infty$, and the valuation topology on $\mathbb{A}^n$ is the product topology. Note that these topologies are independent of the choice of the charts.

The valuation topology, however, is not suitable for developing cohomology as it is totally disconnected (the open balls are also closed sets) and not necessarily locally compact. In Subsection 5.2, we will introduce an appropriate site on a definable set that will replace the valuation topology. However, with this site, definable sets do not form a $\mathcal{T}$-space. To obtain results for the cohomology theory on such sites, we have to go through another category associated to Def_C, the category of the stable completions of objects of Def_C with morphisms the C-pro-definable morphisms between such objects. We then introduce another site, isomorphic to the previous one, in the stable completion of a definable set. In Subsection 5.3, we show, using Hrushovski and Loeser's main theorem, that the open sets of the site make the stable completion of a definable set into a $\mathcal{T}$-space. Finally, in Subsection 5.4, we show that in the stable completion of a definable set, for the associated notion of $\mathcal{T}$-normality, $\mathcal{T}$-locally closed subsets are finite unions of $\mathcal{T}$-open subsets that are $\mathcal{T}$-normal.

In what follows, for simplicity, we will adopt the following conventions:

Remark 5.1. Let K be a small elementary substructure of $\mathbb{U}$ and X a K-definable set (e.g., a variety V over K). We write X for the set $X(\mathbb{U})$ of $\mathbb{U}$-points of X, and we write $X(K)$ for the set of K-points of X. Note that even though in what follows all definitions (respectively, results) are stated for $\mathbb{U}$-points, they make sense (respectively, they hold) for K-points unless otherwise stated. In fact, a K-definable set X can also be seen as a functor from the category of all elementary extensions of K to the category of sets. The category of such functors is actually equivalent to the category of definable sets of $\mathbb{U}$ of the form $X(\mathbb{U})$. Both points of view will be used indistinguishably in what follows. This justifies why we write X instead of $X(\mathbb{U})$. In case a statement involves two small elementary substructures K and K' of $\mathbb{U}$, we will use the notation $X(K)$ and $X(K')$ to disambiguate.

We treat Γ_∞, $\mathcal{R}$, $\mathcal{M}$ and k as $\emptyset$-definable sets. Thus, following our convention, the set $\Gamma_\infty(K)$ (respectively, $\mathcal{R}(K)$, $\mathcal{M}(K)$ and $k(K)$) corresponds to the value group of (K, val) (respectively, valuation ring, maximal ideal and residue field of (K, val)).

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Given an algebraic variety V, we will consider the sheaf $\mathcal{O}_V$ (for the Zariski topology) of regular functions on V. Since, given charts $(V_i, U_i, f_i)_{i\leq m}$ for V, the open subsets $U \subseteq V_i$ for some i are a basis for the Zariski topology, $\mathcal{O}_V$ is completely determined by the sheaves of regular functions $\mathcal{O}_{U_i}$ on the quasi-affine varieties U_i.

5.2. *The* v+g*-sites*

We start with the definition of the site replacing the valuation topology. As we will explain below, this site is based on the work of Hrushovski and Loeser [35]. We then recall Hrushovski and Loeser's category of the stable completions of objects of Def_C and the topology on such spaces. In general, this topology is also not suitable for cohomology (it is not locally compact), and we will replace it by a site as well.

Definition 5.2 (The v+g-site). Let V be an algebraic variety. A subset of V is a *basic* v+g-*open subset* if it is of the form

$$\bigcap_{j\in J}\{u \in U_j : \mathrm{val}(f_j(u)) < \mathrm{val}(g_j(u))\},$$

where J is a finite set and $f_j, g_j \in \mathcal{O}_V(U_j)$ are regular functions on a Zariski open subset $U_j \subseteq V$. A subset of V is a v+g-*open subset* if it is a finite union of basic v+g-open subsets; it is v+g-*closed* if it is the complement of a v+g-open subset.

A subset of $V \times \Gamma_\infty^m$ is a *basic* v+g-*open subset* if and only if its pullback under $\mathrm{id}\times \mathrm{val}\colon V\times \mathbb{A}^m \to V\times\Gamma_\infty^m$ is a basic v+g-open subset of $V\times\mathbb{A}^m$; v+g-open and v+g-closed subsets of $V\times\Gamma_\infty^m$ are defined analogously.

If $X \subseteq V\times\Gamma_\infty^m$ is a definable subset, we say that a subset of X is v+g-open (respectively, v+g-closed) if and only if it is of the form $X\cap O$ (respectively, $X\cap D$), where O (respectively, D) is a v+g-open (respectively, v+g-closed) subset of $V\times\Gamma_\infty^m$. Similarly, a *basic* v+g-*open subset of* X is a set of the form $X \cap O$, where O is basic v+g-subset of $V\times\Gamma_\infty^m$.

The v+g-*site on* X, denoted $X_{\text{v+g}}$, is the category $\mathrm{Op}(X_{\text{v+g}})$ whose objects are the v+g-open subsets of X, the morphisms are the inclusions, and the admissible covers $\mathrm{Cov}(U)$ of $U\in\mathrm{Op}(X_{\text{v+g}})$ are covers by v+g-open subsets of X with finite subcovers.

Remark 5.3 (v-open and g-open subsets). Note that if V is an algebraic variety, then:

- Since regular functions are continuous for the valuation topology, a v+g-open subset is v-*open*: that is, it is a definable subset that is open in the valuation topology.
- Since the constant function zero is a regular function and $\mathrm{val}(f(u)) < \infty$ is the same as $f(u)\neq 0$, it follows that a v+g-open subset is g-*open*: that is, it is a positive finite Boolean combination of Zariski closed sets, Zariski open sets and sets of the form $\{u\in U : \mathrm{val}(f(u)) < \mathrm{val}(g(u))\}$ with $f, g \in \mathcal{O}_V(U)$.

Thus it follows from the characterisation of subsets that are both v-closed and g-closed when V is affine or projective ([35, Proposition 3.7.3]; see also below) that the v+g-open subsets as described above are exactly the subsets that are both v-open and g-open. The sets that are both v-open and g-open (respectively, v-closed and g-closed) are called in [35] v+g-open subsets (respectively, v+g-closed subsets).

The following will be useful later:

Fact 5.4 [35, Proposition 3.7.3]. Let V be an affine (respectively, projective) algebraic variety. Then a subset of V is v+g-closed if and only if it is of the form

$$\bigcup_{i\in I}\bigcap_{j\in J}\{x\in V : h_{ij}(x) = 0,\ \mathrm{val}(f_{ij}(x)) \leq \mathrm{val}(g_{ij}(x))\},$$

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where I, J are finite and $h_{ij}, f_{ij}, g_{ij} \in \mathcal{O}_V(V)$ are regular functions on V (respectively, homogeneous polynomials on V). □

Remark 5.5 (v+g-opens and valuation topology). The v+g-open subsets of an algebraic variety are a basis for the valuation topology. However, not every definable open subset is a v+g-open. For example, in $\mathbb{A}^1$, the valuation ring $\mathcal{R} = \bigcup_{a \in \mathcal{R}} (a + \mathcal{M})$ is definable, open in the valuation topology but it is not v+g-open.

In Γ_∞^m, by Fact 5.4, the v+g-open subsets coincide with the open definable subsets of Γ_∞^m for the product of the order topology on Γ_∞ (Remark 4.4). So the v+g-site $X_{\text{v+g}}$ on a definable subset $X \subseteq \Gamma_\infty^m$ is the o-minimal site X_{def} as defined in Definition 4.8.

Remark 5.6 (Definable sets with the v+g-site are not $\mathcal{T}$-spaces). Let us say that a subset of a definable set of $V \times \Gamma_\infty^n$ is a v+g-*subset* if it is a finite boolean combination of v+g-open subsets. Note that by quantifier elimination, the v+g-subsets are exactly the definable subsets.

It follows that the v+g-open subsets of a definable set of $V \times \Gamma_\infty^n$ do not form a $\mathcal{T}$-topology unless V is finite, since condition (iii) of Definition 3.2 fails: any v+g-open is always a disjoint union of clopen v+g-subsets (i.e., definable subsets).

For the reader's convenience, below we recall the definition of the stable completion $\widehat{X}$ of a definable set X. When X is moreover a subset of $V \times \Gamma_\infty^n$ for an algebraic variety V, we will equip $\widehat{X}$ with a topology and a site, making $\widehat{X}$ into a $\mathcal{T}$-space.

Let B be any subset of $\mathbb{U}$ and x be a tuple of variables with length $|x|$. We denote as usual by $S_x(B)$ the space of types over B in the variables x: that is, the Stone space of the Boolean algebra of formulas with free variables contained in x and parameters from B up to equivalence over ACVF. If B is a model of ACVF, then $S_x(B)$ can be characterised as the set of ultrafilters of definable subsets of $B^{|x|}$ over B.

Types in $S_x(\mathbb{U})$ are called *global types*. If C is a small substructure, a *global type* $p \in S_x(\mathbb{U})$ *is C-definable* (or definable over C) if and only if for every $\mathcal{L}_\mathcal{G}$-formula $\phi(x, y)$ there is an $\mathcal{L}_\mathcal{G}$-formula $d_p(\phi)(y)$ over C such that for all $b \in \mathbb{U}^{|y|}$, we have $\phi(x, b) \in p$ if and only if $d_p(\phi)(b)$ holds in $\mathbb{U}$. Equivalently, since $\mathbb{U}$ is a model, a global type $p \in S_x(\mathbb{U})$ is C-definable if and only if for every $\emptyset$-definable family $\{X_t\}_{t \in T}$ of definable subsets of $\mathbb{U}^{|x|}$ there is a C-definable subset $S \subseteq T$ such that $X_t \in p$ if and only if $t \in S$.

A type $p \in S_x(B)$ *concentrates on* a C-definable set X if it contains a formula defining X. The notation $S_X(B)$ is used to denote the subset of $S_x(B)$ of all such types. Note if B is a model, then $S_X(B)$ can be identified with $\widetilde{X(B)}$, the set of ultrafilters of definable subsets of $X(B)$ over B; this notation was used in previous sections in the o-minimal setting.

The *stable completion of X over C*, denoted by $\widehat{X}(C)$, can be characterised in ACVF as the set of C-definable global types concentrated on X that are orthogonal to Γ. Recall that a global type p is *orthogonal* to Γ if and only if for every definable function $h\colon X \to \Gamma_\infty$, the pushforward $h_*(p)$ is a global type concentrated on a point of Γ_∞; the pushforward $h_*(p)$ is the global type given by $\{Z \subseteq \Gamma : Z \text{ definable and } h^{-1}(Z) \in p\}$. Also recall that, by ([35, Proposition 2.9.1], a C-definable global type p is orthogonal to Γ if and only if p is stably dominated if and only if p is generically stable. These other characterisations of orthogonality to Γ of an C-definable type will be used when needed.

If $f\colon X \to Y$ is a C-definable map, then we have a map $\widehat{f}\colon \widehat{X}(C) \to \widehat{Y}(C)$ given by $\widehat{f}(p) = f_*(p)$. Indeed, $f_*(p)$ is clearly an C-definable global type on Y, and for any definable $h\colon Y \to \Gamma$, we have that $h \circ f\colon X \to \Gamma$ is definable and $(h \circ f)_*(p) = h_*(f_*(p))$ is a global type concentrated on a point of Γ. However, there are more maps that one has to consider between the stable completions of definable sets. To introduce these maps, we need to introduce the pro-definable structure associated to the stable completion of a definable set.

Let $\text{Pro}(\text{Def}_C)$ be the category of *C-pro-definable sets*: that is, the category whose objects are filtrant projective limits of functors

$$\varprojlim_i \text{Hom}_{\text{Def}_C}(\bullet, X_i),$$

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where $(X_i)_{i\in I}$ is a cofiltering system in Def_C indexed by a small, directed, partially ordered set, and the morphisms are the natural transformations of such functors. By a result of Kamensky [38], the functor of 'taking $\mathbb{U}$-points' induces an equivalence of categories between the category $\mathrm{Pro}(\mathrm{Def}_C)$ and the subcategory of the category of sets whose objects and morphisms are projective limits of $\mathbb{U}$-points of definable sets indexed by a small directed partially ordered set. So one can identify a pro-definable set X represented by $(X_i)_{i\in I}$ with $X(\mathbb{U}) = \varprojlim_i X_i(\mathbb{U})$.

The sets of morphisms are related by

$$\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Def}_C)}(X,Y) \simeq \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Def}_C}(X_i, Y_j),$$

and we call the elements of $\mathrm{Hom}_{\mathrm{Pro}(\mathrm{Def}_C)}(X,Y)$ the *C-pro-definable morphisms* between X and Y.

Below, by *strict C-pro-definable* set, we mean a C-pro-definable set X for which there is a cofiltering system $(X_i)_{i\in I}$ in Def_C representing X with the transition maps $\pi_{i,i'}\colon X_i \to X_{i'}$ being surjective for all $i \geq i'$ in I, or equivalently, with the C-pro-definable projection maps $\pi_i\colon X \to X_i$, represented by the transition maps $\pi_{i,i'}\colon X_i \to X_{i'}$ for all $i \geq i'$, being surjective for all $i \in I$.

Fact 5.7 [35, Theorem 3.1.1] or [16, Theorem 6.4]. For every C-definable set X, there is a canonical strict C-pro-definable set E and a canonical identification $\widehat{X}(F) = E(F)$ for every substructure F containing C. Moreover, if $f\colon X \to Y$ is a morphism in Def_C, then the induced map $\widehat{f}\colon \widehat{X}(F) \to \widehat{Y}(F)$ is a morphism of C-pro-definable sets. □

Thus we have a category

$$\widehat{\mathrm{Def}}_C$$

whose objects are of the form $\widehat{X}(C)$ for X a C-definable set and whose morphisms are the C-pro-definable morphisms between such objects. Furthermore, for every substructure F containing C, we have a functor

$$\mathrm{Def}_C \to \widehat{\mathrm{Def}}_F$$
$$X \mapsto \widehat{X}(F).$$

Example 5.8 (The stable completion of the affine line). Let K be a small elementary substructure of $\mathbb{U}$. Let us describe the stable completion $\widehat{\mathbb{A}^1}(K)$ of the affine line. Let $\eta_{\mathcal{R}}$ denote the *generic type of the valuation ring* $\mathcal{R}$: that is, the global type in $S_x(\mathbb{U})$, where $|x| = 1$, determined by the following rule: a definable set $X \subseteq \mathbb{U}$ is in $\eta_{\mathcal{R}}$ if and only if there is $m \geq 1$, and there are $b_1, \ldots, b_m \in \mathcal{R}(K)$ such that $\mathrm{val}(b_i - b_j) = 0$ for all $i \neq j$ and

$$\mathcal{R} \setminus (\bigcup_{i=1}^{m} b_i + \mathcal{M}) \subseteq X.$$

By quantifier elimination and first-order logic compactness, for every formula $\phi(x,y)$ there is an integer m_ϕ such that for all $t \in \mathbb{U}^{|y|}$, if X_t is defined by $\phi(x,t)$, then for all $m > m_\phi$ and all $b_1, \ldots, b_m \in \mathcal{R}(K)$ such that $\mathrm{val}(b_i - b_j) = 0$ for all $i \neq j$, we have

$$\mathcal{R} \setminus (\bigcup_{i=1}^{m} b_i + \mathcal{M}) \nsubseteq X_t.$$

Therefore, $\eta_{\mathcal{R}}$ is a $\emptyset$-definable global type (see also [32, Lemma 2.3.8]). We let the reader convince her/himself that $\eta_{\mathcal{R}}$ is orthogonal to Γ (see [32, Lemma 2.5.5]).

Given any closed ball

$$B(a,\gamma) = \{x \in \mathbb{A}^1 : \mathrm{val}(x-a) \geq \gamma\}$$

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centred at $a \in K$ with radius $\gamma \in \Gamma_\infty(K)$, by moving the centre and rescaling, there is a K-affine transformation f such that $f(\mathcal{R}) = B(a,\gamma)$. We set $\eta_{B(a,\gamma)} = f_*(\eta_{\mathcal{R}})$, and we call it *the generic type of the closed ball* $B(a,\gamma)$. Thus $\eta_{B(a,\gamma)}$ is orthogonal to Γ. Furthermore, it follows from [32, Lemma 2.3.3] that we have

$$\widehat{\mathbb{A}^1}(K) = \{\eta_{B(a,\gamma)} : a \in K, \gamma \in \Gamma_\infty(K)\},$$

where $B(a,\infty)$, the closed ball with radius ∞ centred at a, is identified with a.

As a set, $\widehat{\mathbb{P}^1}(K)$ consists of the disjoint union of $\widehat{\mathbb{A}^1}(K)$ and the definable type concentrating on the point at infinity in $\mathbb{P}^1(K)$. The description of $\widehat{\mathbb{A}^n}(K)$ for $n > 1$ is more complicated (see [35, Example 3.2.3]).

Remark 5.9. Let K be an elementary substructure of $\mathbb{U}$ and X a K-definable set. Note that $\widehat{X}$ is a functor from the category of all elementary extensions of K into the category of sets (sending an elementary extension K' to $\widehat{X}(K')$). As for definable sets, the category of such functors is equivalent to the category of pro-definable sets of the form $\widehat{X}(\mathbb{U})$ of all global definable types concentrating on X that are orthogonal to Γ. From now on, unless otherwise stated, we will work with $\widehat{X}$, but all the definitions (respectively, results) still make sense (respectively, hold) also for $\widehat{X}(K)$. When we have two small elementary substructures K and K' of $\mathbb{U}$, then we will use the notation $\widehat{X}(K)$ and $\widehat{X}(K')$ to avoid ambiguity.

Let V be an algebraic variety, and let $\mathcal{O}_V$ be the sheaf of regular functions on V. Let $\mathcal{O}_V^{\mathrm{val}}$ be the sheaf of Γ_∞-valued functions defined by

$$\mathcal{O}_V^{\mathrm{val}}(U) = \{\mathrm{val} \circ f \mid f \in \mathcal{O}_V(U)\}$$

for $U \subseteq V$ a Zariski open subset. The *topology on the stable completion* $\widehat{V}$ is the topology having as a basis finite unions of finite intersections of sets of the following form:

$$\{p \in \widehat{U} \mid f_*(p) \in I\},$$

where U is a Zariski open set of V, $f \in \mathcal{O}_V^{\mathrm{val}}(U)$ and I is an open interval on Γ_∞. $\widehat{V \times \Gamma_\infty^n}$ is equipped with the quotient topology induced by the map

$$\widehat{\mathrm{id} \times \mathrm{val}} : \widehat{V \times \mathbb{A}^n} \to \widehat{V \times \Gamma_\infty^n}$$

and, for a definable subset $X \subseteq V \times \Gamma_\infty^m$, we put on $\widehat{X}$ the induced topology from $\widehat{V \times \Gamma_\infty^m}$.

By [35, Lemma 3.5.3], we have:

Fact 5.10. If $Y \subseteq \Gamma_\infty^m$ is a definable subset, then $\widehat{Y} = Y$. The topology on $\widehat{\Gamma_\infty} = \Gamma_\infty$ coincides with the order topology on Γ_∞, and the topology on $\widehat{\Gamma_\infty^n} = \Gamma_\infty^n$ is the product topology.

If X is a definable subset of an algebraic variety V, then there is a canonical bijection from $\widehat{X} \times Y$ to $\widehat{X \times Y}$ such that the topology on $\widehat{X \times Y} = \widehat{X} \times Y$ is the product topology. □

By [35, Proposition 4.2.21 and Corollary 4.2.22], the topology on the stable completion is related to the v+g-site by:

Fact 5.11. Let V be an algebraic variety over K, and let W be a definable subset of $V \times \Gamma_\infty^m$. Then W is v+g-open (respectively, v+g-closed) if and only if $\widehat{W}$ is open (respectively, closed) in $\widehat{V} \times \Gamma_\infty^m$. Moreover, a basis for the topology on the stable completion $\widehat{X}$ of a definable subset of $V \times \Gamma_\infty^m$ is given by

$$\{\widehat{U} : U \in \mathrm{Op}(X_{\mathrm{v+g}})\}. \qquad \square$$

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As we pointed out already, in general, the stable completion topology is not necessarily locally compact:

Remark 5.12 (The stable completion topology and local compactness). Consider the affine line over $\mathbb{C}_p$ (the completion of the algebraic closure of $\mathbb{Q}_p$). The space $\widehat{\mathbb{A}^1}(\mathbb{C}_p)$ is *not* locally compact. Indeed, suppose for a contradiction there is a compact neighbourhood U of $\eta_{\mathcal{R}}$, the generic type of the valuation ring (the Gauss point). Then there is $I \subseteq \mathbb{Q}$ an open interval such that $\widehat{\mathrm{val}}^{-1}(I) \subseteq U$. Let $J \subseteq I$ be a closed sub-interval. Then by continuity of $\widehat{\mathrm{val}}$, $\widehat{\mathrm{val}}^{-1}(J)$ is a closed subset of U and hence a compact subset. Therefore, its image J would be a compact subset of $\mathbb{Q}$.

Due to the failure of local compactness of the stable completion's topology, in order to develop a cohomology theory in the category of stable completions of definable sets, we will replace the stable completion's topology by a site:

Definition 5.13 ($\widehat{\text{v+g}}$-site on stable completions). If $X \subseteq V \times \Gamma_\infty^m$ is a definable subset, the $\widehat{\text{v+g}}$-*site on* $\widehat{X}$, denoted $\widehat{X}_{\widehat{\text{v+g}}}$, is the category $\mathrm{Op}(\widehat{X}_{\widehat{\text{v+g}}})$ whose objects are of the form $\widehat{W}$ with $W \in \mathrm{Op}(X_{\text{v+g}})$, the morphisms are the inclusions, and the admissible covers $\mathrm{Cov}(\widehat{U})$ of $\widehat{U} \in \mathrm{Op}(\widehat{X}_{\widehat{\text{v+g}}})$ are covers by objects of $\mathrm{Op}(\widehat{X}_{\widehat{\text{v+g}}})$ with finite subcovers.

Of course, the v+g-site $X_{\text{v+g}}$ on a definable set X and the $\widehat{\text{v+g}}$-site $\widehat{X}_{\widehat{\text{v+g}}}$ on its stable completion are isomorphic categories, so we will often move from one site to the other whenever convenient.

5.3. *The* $\widehat{\text{v+g}}$*-site and* $\mathcal{T}$*-spaces*

Here we show that the stable completion of a definable subset of $V \times \Gamma_\infty^n$ equipped with the $\widehat{\text{v+g}}$-site is a $\mathcal{T}$-space. For that, we use the following special case of Hrushovski and Loeser's main theorem:

Fact 5.14 [35, Theorem 11.1.1]. Let V be a quasi-projective variety over K, and let X be a definable subset of $V \times \Gamma_\infty^n$. Then there exists a (continuous) pro-definable deformation retraction $H : I \times \widehat{X} \to \widehat{X}$ with image $\mathfrak{X}$ an iso-definable subset definably homeomorphic to a definable subset of some Γ_∞^k. Furthermore, given finitely many definable subsets $X_1, \ldots, X_n$ of X, the pro-definable deformation retraction H can be constructed preserving each of the definable subsets: that is, the restriction $H_| : I \times \widehat{X_i} \to \widehat{X_i}$ is still a continuous pro-definable retraction. □

The I in the theorem is a *generalised interval*: that is, a one-dimensional $\mathbf{\Gamma}_\infty$-definable space obtained by considering finitely many oriented closed sub-intervals of Γ_∞ and glueing them together two by two, end to end, respecting the orientations. The set $\mathfrak{X}$ is called a *skeleton* of $\widehat{X}$ and is often identified, under the definable homeomorphism, with a definable subset of Γ_∞^k.

Also recall ([35, Definition 2.2.2]) that if X is a pro-definable set, then Z is a *pro-definable subset* of X if and only if there are cofiltering systems of definable sets $(X_i)_{i\in I}$ and $(Z_i)_{i\in I}$ representing X and Z, respectively, such that for each i, $Z_i \subseteq X_i$ and for all $i \geq i'$ the transition maps $Z_i \to Z_{i'}$ are the restrictions of the transitions maps $X_i \to X_{i'}$; we say that Z is an *iso-definable subset* of X if furthermore there is i_0 such that restriction maps $Z_i \to Z_{i'}$ are bijections for all $i \geq i' \geq i_0$, or equivalently, the projection maps $Z \to Z_i$ are bijections for all $i \geq i_0$.

Let X be a pro-definable set. A pro-definable subset Y of X is said to be *relatively definable* if for some cofiltering system $(X_i)_{i\in I}$ of definable sets representing X, with transitions $\pi_{i,i'} : X_i \to X_{i'}$ for all $i \geq i'$ in I, there is $i_0 \in I$ and a definable subset $Y_{i_0} \subseteq X_{i_0}$ such that $(Y_i)_{i\geq i_0}$, where $Y_i = \pi_{i,i_0}^{-1}(Y_{i_0})$ represents Y or, equivalently, $Y = \pi_{i_0}^{-1}(Y_{i_0})$, where $\pi_{i_0} : X \to X_{i_0}$ denotes the natural (pro-definable) projection map represented by the transition maps $\pi_{i,i_0} : X_i \to X_{i_0}$ for all $i \geq i_0$.

The main examples of relatively definable subsets are obtained when we have $W \subseteq X \subseteq V \times \Gamma_\infty^m$ are definable sets, since by construction of the pro-definable structure on the stable completions, $\widehat{W}$ is a relatively definable subset of $\widehat{X}$. The following gives another kind of example that we require below:

Remark 5.15 (Simple points). Given $X \subseteq V \times \Gamma_\infty^m$ a definable subset, there is a canonical embedding $\iota_X : X \to \widehat{X}$, taking a point a to the global type $\mathrm{tp}(a/\mathbb{U})$ concentrating on a. The points of the image are called the *simple* points of $\widehat{X}$, and the set $\iota_X(X)$ will be denoted by $\widehat{X}_s$. Note that $\iota_X : X \to \widehat{X}_s$ is a pro-definable bijection. By working on affine charts and using the fact that the induced topology from $\widehat{V \times \Gamma_\infty^m} = \widehat{V} \times \Gamma_\infty^m$ on $V \times \Gamma_\infty^m$ via $\iota^{-1}_{V\times\Gamma_\infty^m}$ coincides with the product topology between the valuation topology on V and the (product) order topology on Γ_∞^m, we see that [35, Lemma 3.6.1] can also be extended to definable subsets of $V \times \Gamma_\infty^m$:

- The set $\widehat{X}_s$ of simple points of $\widehat{X}$ is a dense, iso-definable and relatively definable subset of $\widehat{X}$.

Lemma 5.16. *Let X be a pro-definable set, Z be a strict pro-definable subset of X and W be an iso-definable subset of X. Then:*

(a) *If Y is a relatively definable subset of X, then $Y \cap Z$ is strict pro-definable.*
(b) *If Y is an iso-definable and relatively definable subset of X, then $Y \cap Z$ is iso-definable.*
(c) *If Y is a relatively definable subset of X, then $Y \cap W$ is iso-definable.*

Proof. Suppose that $(X_i)_{i\in I}$ is a cofiltering system of definable sets representing X, with transitions $\pi_{i,i'} : X_i \to X_{i'}$ for all $i \geq i'$ in I. Let $i_0 \in I$ be such that $\pi_{i_0}^{-1}(Y_{i_0}) = Y$ for Y_{i_0} a definable subset of X_{i_0}, so that if for all $i \geq i_0$, $Y_i := \pi_{i,i_0}^{-1}(Y_{i_0})$, then Y is represented by $(Y_i)_{i\geq i_0}$. Without loss of generality, we may also suppose that if for all $i \geq i_0$, $Z_i := \pi_i(Z)$, then Z is represented by $(Z_i)_{i\geq i_0}$ and, if for all $\geq i_0$, $W_i = \pi_i(Z)$, then W is represented by $(W_i)_{i\geq i_0}$.

(a) We show that for all $i \geq i_0$, $\pi_i(Y \cap Z) = Y_i \cap Z_i$. The inclusion from left to right is immediate. For the converse, let $a \in Y_i \cap Z_i$. By assumption, there is $x \in Z$ such that $\pi_i(x) = a$. Since Y is relatively definable, $\pi_i^{-1}(Y_i) = Y$, so $x \in Y$, and therefore $a \in \pi_i(Y \cap Z)$.

(b) It suffices to show that $Y \cap Z = Y \cap \pi_{i_0}^{-1}(Z_{i_0})$, since the right-hand side set $\pi_{i_0}^{-1}(Y_{i_0} \cap Z_{i_0})$, being in pro-definable bijection under π_{i_0} with a definable set, is iso-definable by [35, Corollary 2.2.4]. The left-to-right inclusion is clear. For the converse, suppose $a \in Y$ and $\pi_{i_0}(a) \in Z_{i_0}$. By surjectivity of $\pi_{i_0} : Z \to Z_{i_0}$, there is $b \in Z$ such that $\pi_{i_0}(b) = \pi_{i_0}(a) \in Y_{i_0}$. It follows that $b \in \pi_{i_0}^{-1}(Y_{i_0}) = Y$. Therefore, since $\pi_{i_0} : Y \to Y_{i_0}$ is a bijection, $a = b$ and $a \in Y \cap Z$.

(c) We have $\pi_{i_0}^{-1}(Y_{i_0} \cap W_{i_0}) = \pi_{i_0}^{-1}(Y_{i_0}) \cap \pi_{i_0}^{-1}(W_{i_0}) = Y \cap W$ since $\pi_i : W \to W_i$ is a bijection for all $i \geq i_0$. □

The following lemma is easy and left to the reader.

Lemma 5.17. *Let X and Y be pro-definable sets and $f : X \to Y$ be a pro-definable map. If U is a relatively definable subset of Y, then $f^{-1}(U)$ is a relatively definable subset of X.* □

There is a notion of definable connectedness in the stable completion introduced at the beginning of [35, Section 10.4]. Note that it is introduced there only for stable completions of definable subsets of an algebraic variety V over a valued field, but this notion can be extended to stable completions of definable subsets of $V \times \Gamma_\infty^m$ in the following way:

Definition 5.18 (Definable connectedness in the stable completion). Let V be an algebraic variety. A strict pro-definable subset Z of $\widehat{V}$ is *definably connected* if and only if the only strict pro-definable clopen subsets of Z are $\emptyset$ and Z.

If $X \subseteq V \times \Gamma_\infty^m$ is a definable subset, we say that $\widehat{X}$ is *definably connected* if and only if the pullback of $\widehat{X}$ under $\widehat{\mathrm{id} \times \mathrm{val}} : \widehat{V \times \mathbb{A}^m} \to \widehat{V \times \Gamma_\infty^m}$ is definably connected.

We have the following characterisation of definable connectedness (slightly extending the observation at the beginning of [35, Section 10.4] for the case of definable subsets of V).

Lemma 5.19. *Let V be an algebraic variety, and let X be a definable subset of $V \times \Gamma_\infty^m$. Then $\widehat{X}$ is definably connected if and only if the only* v+g*-clopen subsets of X are $\emptyset$ and X. In fact, the definably connected components of $\widehat{X}$ are of the form $\widehat{U}$ for some* v+g*-clopen subset U of X such that $\widehat{U}$ is definably connected.*

Proof. The left-to-right implication follows directly from Fact 5.11. For the converse, suppose that the only v+g-clopen subsets of X are $\emptyset$ and X, and assume for a contradiction that there is a proper strict pro-definable clopen subset U of $\widehat{X}$. By Remark 5.15 and Lemma 5.16(b), the set $U \cap \widehat{X}_s$ is an iso-definable subset of $\widehat{X}$. Furthermore, by the density of $\widehat{X}_s$ in $\widehat{X}$ and the fact that U is clopen, we have $\mathrm{cl}(U \cap \widehat{X}_s) = U \cap \widehat{X} = U$, where cl denotes the closure in $\widehat{X}$. Abusing notation, let $U \cap X$ denote $\iota_X^{-1}(U \cap \widehat{X}_s)$. Since $\widehat{X}_s$ is iso-definable, $U \cap X$ is a definable subset of X.

Claim 5.20. $\widehat{U \cap X} = U$.

Proof. First note that $(\widehat{U \cap X})_s = U \cap \widehat{X}_s$. Indeed,

$$\begin{aligned} p \in (\widehat{U \cap X})_s &\Leftrightarrow \exists a \in U \cap X \text{ such that } p = \mathrm{tp}(a/\mathbb{U}) \\ &\Leftrightarrow \exists a \in \iota_X^{-1}(U) \text{ such that } p = \mathrm{tp}(a/\mathbb{U}) \\ &\Leftrightarrow p \in U \cap \widehat{X}_s. \end{aligned}$$

By the density of simple points, this yields that

$$\widehat{U \cap X} \subseteq \mathrm{cl}(\widehat{U \cap X}) = \mathrm{cl}((\widehat{U \cap X})_s) = \mathrm{cl}(U \cap \widehat{X}_s) = U,$$

which shows that $\widehat{U \cap X} \subseteq U$. For the converse inclusion, consider $W := \widehat{X} \setminus U$. The set W is clopen, and $W \cap \widehat{X}_s$ is also iso-definable since $W \cap \widehat{X_s} = \widehat{X}_s \setminus U \cap \widehat{X}_s$ is in pro-definable bijection with the definable set $X \setminus U \cap X$. Let $W \cap X$ denote $\iota_X^{-1}(W \cap \widehat{X}_s)$. Then the sets $W \cap X$ and $U \cap X$ form a definable partition of X. By the same argument above, $\widehat{W \cap X} \subseteq W$. This shows the converse inclusion: if there were $x \in U \setminus \widehat{U \cap X}$, since $\widehat{X} = \widehat{U \cap X} \cup \widehat{W \cap X}$, we must have that $x \in \widehat{W \cap X}$, which implies $x \in W$, a contradiction. □

The claim, together with Fact 5.11, contradicts the assumption on X. □

A first consequence of Fact 5.14 is the following:

Lemma 5.21. *Let V be an algebraic variety, and let $X \subseteq V \times \Gamma_\infty^n$ be a definable set. Then $\widehat{X}$ has finitely many definably connected components.*

Proof. Suppose that V is quasi-projective. Then by Fact 5.14, let $H\colon I \times \widehat{X} \to \widehat{X}$ be a continuous pro-definable deformation retraction with image an iso-definable subset $\mathfrak{X}$ of $\widehat{X}$, and let $h\colon \mathfrak{X} \to \mathcal{X}$ be a pro-definable homeomorphism of $\mathfrak{X}$ with a definable subset $\mathcal{X}$ of Γ_∞^k. Let also $i_I, e_I \in I$ be the initial and last end points of I. Let $f\colon \widehat{X} \to \mathfrak{X}$ and $g\colon \widehat{X} \to \mathcal{X}$ be the continuous pro-definable maps given by $f(x) = H(e_I, x)$ and $g = h \circ f$.

Since definable subsets of Γ_∞^k have finitely many definably connected components, let $\mathcal{X}_1, \ldots, \mathcal{X}_k$ be the finitely many definably connected components of $\mathcal{X}$. Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_k$ be the corresponding (under h) finitely many definably connected components of $\mathfrak{X}$, and let $\mathcal{U}$ be the set of definably connected components of $\widehat{X}$.

Now if $U \in \mathcal{U}$ is a definably connected component of $\widehat{X}$, there is $i \in \{1, \ldots, k\}$ such that $g(U) \subseteq \mathcal{X}_i$. Indeed, if i, i' are distinct such that $g(U) \cap \mathcal{X}_i \neq \emptyset$ and $g(U) \cap \mathcal{X}_{i'} \neq \emptyset$, then since $g^{-1}(\mathcal{X}_i)$ and $g^{-1}(\mathcal{X}_{i'})$ are relatively definable (Lemma 5.17) by Lemma 5.16 (a) and continuity of g, both $g^{-1}(\mathcal{X}_i) \cap U$ and $g^{-1}(\mathcal{X}_{i'}) \cap U$ would be clopen strict pro-definable subsets of U distinct from $\emptyset$ and U.

Fix i, and let $\mathcal{U}_i = \{U \in \mathcal{U} : f(U) \subseteq \mathfrak{X}_i\}$. For each $U \in \mathcal{U}_i$, choose $x_U \in U$ and consider the definable path $\gamma_U : I \to \widehat{X}$ given by $\gamma_U(t) = H(t, x_U)$. Let $x'_U = \gamma(e_I) \in f(U) \subseteq \mathfrak{X}_i$. Since cells in Γ^k are definably path connected ([53, Chapter 6, Proposition 3.2]) and cells in Γ_∞^k are definably homeomorphic to cells in Γ^k, by cell decomposition, any definable subset of Γ_∞^k can be partitioned into finitely many definably path connected subsets. So $\mathfrak{X}_i$ can be partitioned into finitely many $\mathfrak{X}_{i,1} \ldots, \mathfrak{X}_{i,l}$ definably path-connected subsets.

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Let j_U be such that $x'_U \in \mathfrak{X}_{i,j_U}$. We claim that the map $\mathcal{U}_i \to \{1,\dots,l\}\colon U \mapsto j_U$ is injective, so $\mathcal{U}_i$ is finite. Hence $\mathcal{U}$ is finite as well. Indeed, suppose that $U, V \in \mathcal{U}$ are distinct and $j_U = j_V = j$. Then $x'_U, x'_V \in \mathfrak{X}_{i,j}$, so there is a definable path in $\mathfrak{X}_{i,j}$ from x'_U to x'_V. It follows that there is a definable path $\delta\colon J \to \widehat{X}$ from $x_U \in U$ to $x_V \in V$. But by Lemma 5.19, both U and V are the stable completions of v+g-clopen subsets of X, so they are relatively definable subsets of $\widehat{X}$. It follows by Lemma 5.17 and continuity of δ that both $\delta^{-1}(U)$ and $\delta^{-1}(V)$ would be clopen (relatively) definable subsets of J distinct from $\emptyset$ and J, contradicting the definable connectedness of J.

If V is not quasi-projective, consider an open immersion $V \to W$, where W is a complete variety and V is Zariski dense. By Chow's lemma, there is an epimorphism $f\colon W' \to W$, where W' is a projective variety. Consider the quasi-projective variety $V' = f^{-1}(V)$. Now if $X' = (f, \mathrm{id})^{-1}(X) \subseteq V' \times \Gamma_\infty^n$, $\widehat{X'}$ has finitely many definably connected components. Since (f, id) pulls back v+g-clopen subsets to v+g-clopen subsets, by Lemma 5.19, we see that $\widehat{X}$ has finitely many definably connected components. □

Proposition 5.22. *Let V be an algebraic variety, and let $X \subseteq V \times \Gamma_\infty^n$ be a definable set. Then the $\widehat{\text{v+g}}$-site $\widehat{X}_{\widehat{\text{v+g}}}$ is a $\mathcal{T}$-space.*

Proof. As we already saw, the $\widehat{\text{v+g}}$-open sets of $\widehat{X}$ form a basis for the topology and are closed under finite unions and intersections, which shows conditions (i) and (ii) in Definition 3.2. Since clopen $\widehat{\text{v+g}}$-subsets are exactly the $\widehat{\text{v+g}}$-clopen subsets, condition (iii) follows from Lemmas 5.19 and 5.21 □

Note that the previous argument does not hold for $X_{\text{v+g}}$ because there are more clopen v+g-subsets than v+g-clopen subsets (for example, the maximal ideal is a clopen v+g-subset but not a v+g-clopen).

5.4. Definable compactness and $\widehat{\text{v+g}}$-normality

In this subsection, we recall the notion of definable compactness in stable completion, we show that the notion of weakly v+g-normality (Definition 3.14) corresponds to $\widehat{\text{v+g}}$-normality (Definition 3.11), and we prove that every v+g-locally closed subset of $V \times \Gamma_\infty^n$ is the union of finitely many v+g-open subsets that are weakly v+g-normal.

Since the notion of a definable type on a definable set can be extended naturally to pro-definable sets, definable compactness of pro-definable topological spaces is defined by replacing curves by definable types; see [35, Section 4.1]. Also recall that if V is an algebraic variety over a valued field, then we say that a subset W of V is *bounded* if there exists an affine cover $V = \bigcup_{i=1}^m U_i$, and subsets $W_i \subseteq U_i$ such that $W = \bigcup_{i=1}^m W_i$ and each W_i is contained in a closed m-ball with radius in Γ; we say that a subset of Γ_∞^m is *bounded* if it is contained in $[a, \infty]^m$ for some a. More generally, we say that a subset W of $V \times \Gamma_\infty^m$ is *bounded* if its pullback under $\mathrm{id} \times \mathrm{val}\colon V \times \mathbb{A}^m \to V \times \Gamma_\infty^m$ is bounded.

A pro-definable topological space Z is *definably compact* if and only if every definable type p on Z has a limit in Z: that is, there is a point $a \in Z$ such that for every open definable subset U of Z, the definable type p concentrates on U.

Remark 5.23 (Definable compactness in the stable completion). By [35, Corollary 4.2.22], if V is an algebraic variety over a valued field F and X is a definable subset of $V \times \Gamma_\infty^m$, then:

- $\widehat{X}$ is definably compact if and only if X is *bounded* and v+g-closed.

Furthermore ([35, Proposition 4.2.30]), an algebraic variety over a valued field is complete if and only if its stable completion is definably compact.

The following is a special case of [35, Lemma 4.2.23]:

Remark 5.24. Let V, V' be varieties, $X \subseteq V \times \Gamma_\infty^m$ and $Y \subseteq V' \times \Gamma_\infty^n$ be definable subsets and $f\colon X \to Y$ be a definable map that is a morphism of v+g-sites. If X is bounded and v+g-closed, then f is v+g-*closed*: that is, f maps v+g-closed subsets of X to v+g-closed subsets of Y.

Indeed, if $Z \subseteq X$ is a v+g-closed subset, then $\widehat{Z} \subseteq \widehat{X}$ is a closed relatively definable subset and $\widehat{f}(\widehat{Z}) \subseteq \widehat{Y}$ is a closed subset ([35, Lemma 4.2.23]). On the other hand, applying [35, Lemma 4.2.6] to

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the surjective definable map $f\colon X \to f(X)$, we get that the map $\widehat{f}\colon \widehat{X} \to \widehat{f(X)}$ is also surjective, which implies that $\widehat{f(Z)} = \widehat{f}(\widehat{Z})$, so $f(Z)$ is v+g-closed (Fact 5.11).

Recall the notion of weakly $\mathcal{T}$-normal from Definition 3.14. Then:

Remark 5.25. Since open (respectively, closed) $\widehat{\text{v+g}}$-subsets of $\widehat{X}$ are exactly the $\widehat{\text{v+g}}$-open (respectively, $\widehat{\text{v+g}}$-closed) subsets of $\widehat{X}$, we have that X is weakly v+g-normal if and only if $\widehat{X}$ is $\widehat{\text{v+g}}$-normal (as in Definition 3.11).

For the next result, we need to recall the schematic distance function from [35, Section 3.12]. First, given $g(x_0, \ldots, x_m)$ a homogeneous polynomial with coefficients in the valuation ring $\mathcal{R}$, we define the function $\mathrm{val}(g)\colon \mathbb{P}^m \to [0, \infty]$ by

$$\mathrm{val}(g)([x_0 : \cdots : x_m]) = \mathrm{val}(g(x_0/x_i, \ldots, x_m/x_i))$$

for some (any) i such that $\mathrm{val}(x_i) = \min\{\mathrm{val}(x_j) : j = 0, \ldots, m\}$.

Let V be a projective variety and Z be a closed subvariety. Fix some embedding $\iota\colon V \to \mathbb{P}^m$, and let $f = (f_1, \ldots, f_r)$ be a tuple of homogeneous polynomials in $\mathcal{R}[X_0, \ldots, X_m]$ such that Z is defined by the zero locus of f in V (note that by rescaling, one can always assume that the polynomials have coefficients in $\mathcal{R}$). The *schematic distance function to* $Z5$ is the function $\varphi_{\iota,f} : V \to [0, \infty]$ defined by

$$\varphi_{\iota,f}(x) = \min\{\mathrm{val}(f_i)(x) : 1 \leq i \leq r\}.$$

Note that $\varphi_{\iota,f}$ is a morphism of v+g-sites and $\varphi_{\iota,f}^{-1}(\infty) = Z$. In addition, if K is an elementary substructure of $\mathbb{U}$ and everything is defined over K, then the function $\varphi_{\iota,f}$ is also K-definable.

Theorem 5.26. *Let V be a variety, and let U be a basic* v+g*-open subset of V. Then U is weakly* v+g*-normal.*

Proof. First we consider the case where V is a quasi-projective variety; then using Chow's lemma, we treat the case V is a complete variety, and the general case will follow by Nagata's theorem.

So suppose that V is a quasi-projective variety. Let V' be a projective variety such that V is an open subset of V'. By fixing an embedding of $\iota\colon V' \to \mathbb{P}^m$, we may suppose without loss of generality that $V' \subseteq \mathbb{P}^m$. Denote by Z the closed subvariety of V' such that $V = V' \setminus Z$,, and let $f = (f_1, \ldots, f_r)$ denote a finite tuple of homogeneous polynomials in $\mathcal{R}[X_0, \ldots, X_m]$ defining Z in V'.

By Definition 5.2 and Fact 5.4, let L be a finite index set such that U is defined by

$$U = \bigcap_{l \in L}\{x \in V : \mathrm{val}(g_l(x)) < \mathrm{val}(h_l(x))\},$$

where g_l, h_l are homogeneous polynomials on V' of the same degree for each $l \in L$. By multiplying by a suitable constant, we may suppose for every $l \in L$ that g_l, h_l have coefficients in $\mathcal{R}$ and their composition under the valuation take values on $[0, \infty]$.

Let C', D' be disjoint v+g-closed subsets of U and C, D be v+g-closed subsets of V' such that $C \cap U = C'$ and $D \cap U = D'$. Again, by Fact 5.4, we may suppose that for some finite index sets I and J, we have

$$C = \bigcup_{i \in I}\bigcap_{j \in J}\{x \in V' : p_{ij}(x) = 0,\ \mathrm{val}(q_{ij}(x)) \leq \mathrm{val}(r_{ij}(x))\},$$

where p_{ij}, q_{ij} and r_{ij} are homogeneous polynomials on V' with q_{ij}s and r_{ij}s of the same degree. As before, we may assume that all such polynomials have coefficients in $\mathcal{R}$ and their compositions under the valuation take values on $[0, \infty]$.

Let $k = 1 + 2|L| + 3|I \times J|$, and consider the definable map $d\colon V' \to [0, \infty]^k$ given by

$$d(x) = (\varphi_{\iota,f}(x), (\mathrm{val}(g_l(x), \mathrm{val}(h_l(x)))_{l \in L}, (\mathrm{val}(p_{ij}(x)), \mathrm{val}(r_{ij}(x)), \mathrm{val}(q_{ij}(x)))_{i \in I, j \in J}),$$

where $\varphi_{\iota,f} : V' \to [0,\infty]$ is the schematic distance to Z as defined above. (Recall that $\varphi_{\iota,f}(x) = \infty$ if and only if $x \in Z$.)

The function d is a morphism of v+g-sites, as every coordinate function has this property. It is also v+g-closed by Remarks 5.23 and 5.24.

Consider the set

$$A := \{(\alpha_l, \gamma_l)_{l\in L} \in ([0,\infty) \times [0,\infty])^{|L|} : \bigwedge_{l\in L} \alpha_l < \gamma_l\},$$

and let $Y := [0,\infty) \times A \times [0,\infty]^{3|I\times J|}$. Note that by the choice of d, we have $d(U) \subseteq Y$.

Claim 5.27. *The sets $d(C')$ and $d(D')$ are closed in Y and, in addition, $d(C') \cap d(D') = \emptyset$.*

Proof. Since d is a v+g-closed morphism of v+g-sites, $d(C)$ and $d(D)$ are closed. Moreover, by the choice of d,

$$d(C') = d(C) \cap d(U) = d(C) \cap Y \text{ and } d(D') = d(D) \cap d(U) = d(D) \cap Y. \tag{3}$$

This show $d(C')$ and $d(D')$ are closed in Y. For the second part, suppose there were $x \in C'$ and $y \in D'$ such that $d(x) = d(y)$. This implies $y \in C'$, contradicting that $C' \cap D' = \emptyset$. This completes the claim. □

To conclude the case V is quasi-projective, it suffices to show that Y is definably normal. Indeed, the definable normality of Y and Claim 5.27 imply there are disjoint open definable sets U_1, U_2 of Y such that $d(C') \subseteq U_1$ and $d(D') \subseteq U_2$. We obtain that $C' \subseteq d^{-1}(U_1)$, $D' \subseteq d^{-1}(U_2)$ and $d^{-1}(U_1) \cap d^{-1}(U_2) = \emptyset$. In addition, both $d^{-1}(U_1)$ and $d^{-1}(U_2)$ are v+g-open since d is a morphism of v+g-sites, which shows U is weakly v+g-normal.

To show the definable normality of Y, first note that the definable map $h\colon A \to A'$ given by $h((\alpha_l, \gamma_l)_{l\in L}) = (\alpha_l, \gamma_l - \alpha_l)_{l\in L},$, where $A' := ([0,\infty) \times [0,\infty])^{|L|}$, is a definable homeomorphism. Therefore, Y is definably normal if and only if the set

$$Y' := [0,\infty) \times A' \times [0,\infty]^{3|I\times J|}$$

is definably normal, and Y' is definably normal by [28, Theorem 2.20]).

Now suppose that V is a complete variety. By Chow's lemma, there are a projective variety V' and a surjective morphism $g\colon V' \to V$. Consider the quasi-projective variety $Y := g^{-1}(V)$ and the basic v+g-open subset $W := g^{-1}(U)$ of Y.

Let U_1' and U_2' be v+g-open subsets of U such that $U_1' \cup U_2' = U$. By (3) of Definition 3.14, it suffices to find $D_1 \subseteq U_1'$ and $D_2 \subseteq U_2'$ v+g-closed subsets of U such that $D_1 \cup D_2 = U$. By definition of being v+g-open in U, there are U_1 and U_2 v+g-open subsets of V such that $U_1' = U_1 \cap U$ and $U_2' = U_2 \cap U$. Since g is a morphism of v+g-sites, $g^{-1}(U_1)$ and $g^{-1}(U_2)$ are v+g-open. For $i = 1, 2$, set $W_i = g^{-1}(U_i')$. Note that $W_i = g^{-1}(U_i) \cap W$ for $i = 1, 2$. In particular, $W_1 \cup W_2 = W$ and W_i is v+g-open in W for $i = 1, 2$. By the quasi-projective case and (3) of Definition 3.14, there are v+g-closed subsets $C_1' \subseteq W_1$ and $C_2' \subseteq W_2$ of W such that $C_1' \cup C_2' = W$. By definition, this implies there are v+g-closed subsets $C_i \subseteq Y$ such that $C_i' = C_i \cap W$.

By Remark 5.24, the map g is v+g-closed. Then $g(C_1), g(C_2)$ are v+g-closed subsets of V. We let the reader check that the sets $D_i := g(C_i) \cap U$ defined for $i = 1, 2$ are v+g-closed subsets of U that satisfy in addition that $D_i \subseteq U_i'$ and $D_1 \cup D_2 = U$.

For the general case, by Nagata's theorem, there is an open immersion $f\colon V \to V'$, where V' is a complete variety. Therefore, V is homeomorphic to the open subset $f(V)$ of V' and $f\colon V \to f(V)$ is an isomorphism of v+g-sites. In particular, since $f(U)$ is a basic v+g-open subset of V' that, by the case for complete varieties, is weakly v+g-normal, the result follows. □

We now need to extend Theorem 5.26 to basic v+g-open subset of $V \times \Gamma_\infty^n$. In order to do that, we require a couple of preliminaries.

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Recall that if $\mathbb{V}$ is an elementary extension of $\mathbb{U}$ and $p \in S_z(B)$ is a type over a subset B of $\mathbb{U}$, then p *is realised in* $\mathbb{V}$ if and only if there is $a \in \mathbb{V}^{|z|}$ such that $p = \mathrm{tp}(a/B)$ where

$$\mathrm{tp}(a/B) = \{\psi(z) : \psi(z) \text{ an } \mathcal{L}_{\mathcal{G}}\text{-formula over } B \text{ and } \psi(a) \text{ holds in } \mathbb{V}\}$$

is the *type of* a *over* B. One says in this case that a *is a realisation of* p. Also recall that any type $p \in S_z(B)$ is realised in some elementary extension of $\mathbb{U}$.

Given definable global types $p \in S_x(\mathbb{U})$ and $q \in S_y(\mathbb{U})$, the tensor product $p \otimes q$ is the definable global type $\mathrm{tp}(a, b/\mathbb{U})$, where a realises p and b realises $q|\mathbb{U} \cup \{a\}$. Recall that for any $B \supseteq \mathbb{U}$, one denotes by $q|B$ *the canonical extension* of q to $S_y(B)$ given by $\phi(y, b) \in q|B$ if and only if $d_q(\phi)(b)$ holds in some elementary extension $\mathbb{V}$ of $\mathbb{U}$ containing B.

Note that $p \otimes q$ is indeed a definable global type with $d_{p\otimes q}(\theta)$ given by $d_p(d_q(\theta))$. On the other hand, by [35, Proposition 2.9.1] and [36, Proposition 3.2], if the definable global types $p \in S_x(\mathbb{U})$ and $q \in S_y(\mathbb{U})$ are orthogonal to Γ, then $p \otimes q$ is orthogonal to Γ.

Lemma 5.28. *Let* $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$ *and* $b = (b_1, \ldots, b_n)$ *be a realisation of* $\eta_{\gamma_1} \otimes \cdots \otimes \eta_{\gamma_n}$, *where* $\eta_{\gamma_i} := \eta_{B(0,\gamma_i)}$ *is as defined in Example 5.8. Then for every polynomial* $\sum a_j y^j \in \mathbb{U}[y]$ *with* $y = (y_1, \ldots, y_n)$ *(in multi-index notation), it holds that*

$$\mathrm{val}(\sum_j a_j b^j) = \min_j\{\mathrm{val}(a_j b^j)\}.$$

Proof. Let $c_i \in \mathbb{U}$ be such that $\mathrm{val}(c_i) = \gamma_i$. In particular, $\mathrm{val}(c_i^{-1} b_i) = 0$ for all $i \in \{1, \ldots, n\}$. The definition of the tensor product of definable global types yields that the set

$$A = \{\mathrm{Res}(b_1, c_1), \ldots, \mathrm{Res}(b_n, c_n)\}$$

is algebraically independent over the residue field k. It follows that every finite subset of distinct finite products of elements in A is k-linearly independent. Let $\sum_j e_j(c_1^{-1}b_1 \cdots c_n^{-1}b_n)^j$ be a $\mathbb{U}$-linear combination of distinct finite products of elements in $\{c_1^{-1}b_1, \cdots, c_n^{-1}b_n\}$, where $e_j \in \mathbb{U}$ and we use the multi-index notation

$$(c_1^{-1}b_1 \cdots c_n^{-1}b_n)^j = (c_1^{-1}b_1)^{j_1} \cdots (c_n^{-1}b_n)^{j_n},$$

with $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$. By [30, Lemma 3.2.2], every such $\mathbb{U}$-linear combination satisfies

$$\mathrm{val}(\sum_j e_j(c_1^{-1}b_1 \cdots c_n^{-1}b_n)^j) = \min_j\{\mathrm{val}(e_j(c_1^{-1}b_1 \cdots c_n^{-1}b_n)^j)\}. \tag{4}$$

To conclude, given a polynomial $\sum a_j y^j$ in $\mathbb{U}[y]$, consider the polynomial $\sum_j a_j' y^j$, where $a_j' = a_j(c_1^{j_1} \cdots c_n^{j_n})$. By equation (4),

$$\begin{aligned}\mathrm{val}(\sum a_j b^j) &= \mathrm{val}(\sum_j a_j'(c_1^{-1}b_1 \cdots c_n^{-1}b_n)^j) = \min_j\{\mathrm{val}(a_j'(c_1^{-1}b_1 \cdots c_n^{-1}b_n)^j)\}\\ &= \min_j\{\mathrm{val}(a_j b^j)\}.\end{aligned}$$

□

A minor modification of [35, Lemma 3.5.2] shows the following:

Lemma 5.29. *Let* V *be an algebraic variety. The map*

$$\begin{aligned} s\colon \widehat{V} \times \Gamma_\infty^n &\to \widehat{V \times \mathbb{A}^n}\\ (p, \gamma_1, \ldots, \gamma_n) &\mapsto p \otimes \eta_{\gamma_1} \otimes \cdots \otimes \eta_{\gamma_n},\end{aligned}$$

where $\eta_{\gamma_i} := \eta_{B(0,\gamma_i)}$ is as in Example 5.8, is an injective morphism of $\widehat{\text{v+g}}$-sites, which is a section of

$$\widehat{\mathrm{id}_V \times \mathrm{val}} \colon \widehat{V \times \mathbb{A}^n} \to \widehat{V \times \Gamma_\infty^n} \cong \widehat{V} \times \Gamma_\infty^n.$$

Proof. That s is a section of $\widehat{\mathrm{id}_V \times \mathrm{val}}$ and is injective is straightforward. Let us show it is a morphism of $\widehat{\text{v+g}}$-sites. Since taking preimages preserves Boolean combinations, it is enough to consider the case $V = \mathbb{A}^m$ and show that if U is a v+g-open subset of $\mathbb{A}^{m+n}$ of the form

$$U = \{(x, y) \in \mathbb{A}^{m+n} \mid \mathrm{val}(f_1(x, y)) < \mathrm{val}(f_2(x, y))\}$$

with $f_1, f_2 \in \mathbb{U}[x, y]$, then we have $s^{-1}(\widehat{U}) = \widehat{W}$ for a v+g-open subset W of $\mathbb{A}^m \times \Gamma_\infty^n$.

Suppose that $f_l(x, y) = \sum_j h_{l,j}(x)y^j$ with $y = (y_1, \ldots, y_n)$ (in multi-index notation). Let $p \otimes \eta_{\gamma_1} \otimes \cdots \otimes \eta_{\gamma_n} \in s(\widehat{V} \times \Gamma_\infty^n) \cap \widehat{U}$, and let (a, b) be a realisation of it, $b = (b_1, \ldots, b_n)$ and $\mathrm{val}(b_i) = \gamma_i$. By Lemma 5.28 (applied to the field $\mathbb{U}(a)$), we have

$$\begin{aligned}\mathrm{val}(f_l(a, b)) &= \min_j \mathrm{val}(h_{l,j}(a)b^j) \\ &= P_l((\mathrm{val}(h_{l,j}(a)))_j, \gamma_1, \ldots, \gamma_n),\end{aligned}$$

where the function $P_l \colon \Gamma_\infty^{d_l+m} \to \Gamma_\infty$ is obtained by composition of the natural continuous extensions of min and +. Here d_l is the number of terms in the variable y in the polynomial f_l.

It follows that, if $(p, \gamma) \in \widehat{V} \times \Gamma_\infty^n$ with $\gamma = (\gamma_1, \ldots, \gamma_n)$, then we have

$$\begin{aligned}(p, \gamma) \in s^{-1}(\widehat{U}) &\Leftrightarrow p \otimes \eta_{\gamma_1} \otimes \cdots \otimes \eta_{\gamma_n} \in \widehat{U} \\ &\Leftrightarrow \mathrm{val}(f_1(a, b)) < \mathrm{val}(f_2(a, b)) \ \text{ for all } \ (a, b) \models p \otimes \eta_{\gamma_1} \otimes \cdots \otimes \eta_{\gamma_n} \\ &\Leftrightarrow P_1((\mathrm{val}(h_{1,j}(a)))_j, \gamma) < P_2(\mathrm{val}((h_{2,j}(a)))_j, \gamma) \ \text{ for all } \ a \models p.\end{aligned}$$

Therefore, setting

$$W = \{(a, \gamma) \in \mathbb{A}^m \times \Gamma_\infty^n : P_1((\mathrm{val}(h_{1,j}(a)))_j, \gamma) < P_2((\mathrm{val}(h_{2,j}(a)))_j, \gamma)\},$$

W is a definable subset and $s^{-1}(\widehat{U}) = \widehat{W}$. On the other hand, since the pullback of W under $\mathrm{id} \times \mathrm{val}$ is

$$\{(a, b) \in V \times \mathbb{A}^n \mid P_1(\mathrm{val}((h_{1,j}(a)))_j, \mathrm{val}(b)) < P_2((\mathrm{val}(h_{2,j}(a)))_j, \mathrm{val}(b))\}$$

and min and + are continuous, it follows that $(\mathrm{id} \times \mathrm{val})^{-1}(W)$ is v+g-open subset, so, by definition, W is v+g-open as required. □

Lemma 5.30. *Let $f \colon (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a morphism in $\mathfrak{T}$ with a section $s \colon (Y, \mathcal{T}') \to (X, \mathcal{T})$, which is also a morphism in $\mathfrak{T}$. If U is a $\mathcal{T}$-open subset of X, which is $\mathcal{T}$-normal, then $s^{-1}(U)$ is a $\mathcal{T}'$-open subset of Y, which is $\mathcal{T}'$-normal. In particular, if every $\mathcal{T}$-open subset of X is a finite union of $\mathcal{T}$-open, which are $\mathcal{T}$-normal subsets, then every $\mathcal{T}'$-open subset of Y is a finite union of $\mathcal{T}'$-open, which are $\mathcal{T}'$-normal subsets.*

Proof. Let C_1, C_2 be two disjoint closed $\mathcal{T}'$-subsets of $s^{-1}(U)$. For $i = 1, 2$, let $D_i := f^{-1}(C_i) \cap U$. Since f is a morphism in $\mathfrak{T}$, both D_1 and D_2 are closed $\mathcal{T}$-sets. Hence, there are disjoint open $\mathcal{T}$-subsets $U_1, U_2 \subseteq U$ such that $D_i \subseteq U_i$ for $i = 1, 2$. Since s is a morphism in $\mathfrak{T}$, the sets $W_1 := s^{-1}(U_1)$ and $W_2 := s^{-1}(U_1)$ are disjoint open $\mathcal{T}'$-subsets of $s^{-1}(U)$ and $C_i \subseteq W_i$ for $i = 1, 2$ since $f \circ s = \mathrm{id}$.

For the last part, just note that if $f^{-1}(W) = W_1' \cup \ldots \cup W_l'$, then $W = s^{-1}(W_1') \cup \ldots \cup s^{-1}(W_l')$. □

Corollary 5.31. *Let V be a variety. Then every v+g-locally closed subset X of $V \times \Gamma_\infty^n$ is the union of finitely many basic v+g-open subsets of X that are weakly v+g-normal. In fact, every basic v+g-open subset of $V \times \Gamma_\infty^n$ is weakly v+g-normal.*

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Proof. It suffices to show the result for v+g-open subset as every v+g-closed subset of a weakly v+g-normal set is again weakly v+g-normal. By the isomorphism of sites $(V \times \mathbb{A}^n)_{\text{v+g}}$ and $(\widehat{V \times \mathbb{A}^n})_{\text{v+g}}$ and Theorem 5.26 and Remark 5.25, the result holds for $\widehat{\text{v+g}}$-open subsets of $\widehat{V \times \mathbb{A}^n}$. By Lemmas 5.29 and 5.30, the result holds for $\widehat{\text{v+g}}$-open subsets of $\widehat{V \times \Gamma^n_\infty}$. So by the isomorphism of sites $(V \times \Gamma^n_\infty)_{\text{v+g}}$ and $(\widehat{V \times \Gamma^n_\infty})_{\text{v+g}}$ and Remark 5.25, the result holds in $V \times \Gamma^n_\infty$. This proof actually shows that a basic v+g-open subset of $V \times \Gamma^n_\infty$ is weakly v+g-normal. □

6. Cohomology, finiteness, invariance and vanishing

In this section, we deduce the Eilenberg-Steenrod axioms for the cohomology on the sites introduced in previous sections on definable subsets in $\mathbf{\Gamma}_\infty$, on definable subsets in ACVF and on their stable completions. We also show finiteness, invariance and vanishing results for these cohomologies.

6.1. The Eilenberg-Steenrod axioms

The main work to show the Eilenberg-Steenrod axioms consists in showing the homotopy axiom. To achieve this goal, we will use the Vietoris-Begle theorem (Theorem 3.30) applied to the tilde

$$\widetilde{\pi}\colon \widetilde{X \times [a,b]} \longrightarrow \widetilde{X}$$

of the projection map. In particular, we need to verify the assumptions of the Vietoris-Begle theorem in this context. In $\mathbf{\Gamma}_\infty$, these assumptions have been verified in the literature. For definable sets X and $\widehat{X}$ in ACVF, some work is required.

We start with the following lemma whose proof is exactly the same as that of its o-minimal version given in [28, Proposition 2.33]:

Lemma 6.1. *Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$. Suppose that for every $\alpha \in \widetilde{X}$, any chain of specialisations of α has finite length. If every $\mathcal{T}$-open subset of X is the union of finitely many $\mathcal{T}$-open subsets that are $\mathcal{T}$-normal, then for every $\alpha \in \widetilde{X}$, there is an open, normal and constructible subset U of $\widetilde{X}$ such that $\alpha \in U$ and α is closed in U.* □

In the o-minimal context, the assumption of finiteness of chains of specialisations is verified in [22, Lemma 2.11]. We will now prove an analogue result in ACVF. Let us first make some general observations in this context.

Remark 6.2. Let V be an affine variety, say $V = \operatorname{Spec}(A)$ for $A = \mathbb{U}[T_1, \ldots, T_n]/J$, where J is an ideal of $\mathbb{U}[T_1, \ldots, T_n]$. Note that by quantifier elimination in ACVF, the set $\widetilde{V}$ is in bijection with the set of pairs (I, v), where $I \in V$ and v is a valuation on the fraction field $\operatorname{Frac}(A/I)$ that extends val. Formally, the bijection sends $p \mapsto (\operatorname{supp}(p), v_p)$, where $\operatorname{supp}(p) := \{f \in A : f(x) = 0 \in p\}$, and v_p is determined by setting $v_p(f/g) \geq 0$ if and only if the formula $\operatorname{val}(f(x)) \geq \operatorname{val}(g(x))$ belongs to p (where $x = (x_1, \ldots, x_n)$). We will denote the fraction field $\operatorname{Frac}(A/I)$ by F_p and $\mathcal{R}_p$ denote the valuation ring of F_p with respect to v_p.

As in Definition 3.4, recall that $\widetilde{V}$ is equipped with the topology generated by $\widetilde{U}$, where U is v+g-open in V. Given $p, q \in \widetilde{V}$, we write $p \prec q$ to state that p is a specialisation of q: that is, p belongs to the closure of $\{q\}$. Note that if $p \prec q$, then $\operatorname{supp}(q) \subseteq \operatorname{supp}(p)$ as the set $\{x \in V : \operatorname{val}(f(x)) = \infty\}$ is v+g-closed. In particular, we may assume that $F_p \subseteq F_q$.

Lemma 6.3. *Let V be a variety. Consider $\widetilde{V}$ with the topology generated by $\widetilde{U}$, where U is* v+g*-open in V. Then any chain of specialisations in $\widetilde{V}$ is bounded by* $\dim(V)$.

Proof. Since specialisation is local, it suffices to work with the case when V is affine. For affine V, let $A = \mathcal{O}_V(V)$ be the regular functions on V. Suppose that $p \prec q$ for $p, q \in \widetilde{V}$. As observed in the previous paragraph, we may suppose $F_q \subseteq F_p$. In addition, note that $\mathcal{R}_q \subseteq \mathcal{R}_p$. Indeed, if there was $f \in A$ such

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that $\mathrm{val}(f(x)) \geq 0 \in q$ but $\mathrm{val}(f(x)) < 0 \in p$, since the later formula defines a v+g-open, we must have that $\mathrm{val}(f(x)) < 0 \in q$, a contradiction. Hence, every chain $p_1 \prec \cdots \prec p_n \prec p$ of specialisations of $p \in \widetilde{V}$ induces a chain of valued subfields $(F_p, \mathcal{R}_p) \subseteq (F_{p_n}, \mathcal{R}_{p_n}) \subseteq \cdots \subseteq (F_{p_1}, \mathcal{R}_{p_1})$. By [30, Corollary 3.4.6], the length of every such a chain is bounded by the dimension of V. $\square$

Corollary 6.4. *Let V be a variety, and let $X \subseteq V \times \Gamma_\infty^m$ be a definable subset. Then any chain of specialisations in $\widetilde{X}$ has finite length bounded by* $\dim(V) + m$. *The same holds for $\widetilde{\widetilde{X}}$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \widetilde{V \times \mathbb{A}^m} & \longrightarrow & \widetilde{V \times \Gamma_\infty^m} \\ \downarrow & & \downarrow \\ \widetilde{\widetilde{V \times \mathbb{A}^m}} & \longrightarrow & \widetilde{\widetilde{V \times \Gamma_\infty^m}} \end{array}$$

where the top arrow is $\widetilde{\mathrm{id} \times \mathrm{val}}$, the bottom arrow is $\widetilde{\widetilde{\mathrm{id} \times \mathrm{val}}}$ and the vertical arrows are homeomorphisms induced by isomorphisms of sites (Remark 3.8).

If there were a chain of specialisations in $\widetilde{V \times \Gamma_\infty^m}$ of length $> \dim(V) + m$, then there would be a chain of specialisations in $\widetilde{\widetilde{V \times \Gamma_\infty^m}}$ of length $> \dim(V) + m$. Since the bottom arrow has a section $\widetilde{s} : \widetilde{\widetilde{V \times \Gamma_\infty^m}} \to \widetilde{\widetilde{V \times \mathbb{A}^m}}$, which is the tilde of an injective morphism of $\widetilde{\text{v+g}}$-sites (Lemma 5.29), we would also have a chain of specialisations of length $> \dim(V) + m$ in $\widetilde{V \times \mathbb{A}^m}$, which contradicts Lemma 6.3. $\square$

Remark 6.5. One could improve the previous result by bounding the length of the specialisation chain by an appropriate notion of dimension for definable subsets of $\mathbb{A}^n \times \Gamma_\infty^m$. We would like to point out that a good candidate to play such a role was introduced by F. Martin in his PhD thesis [42], where he more generally defined a dimension function for subsets of $\mathbb{A}^n \times \Gamma_\infty^m \times k^r$.

In the o-minimal context, we know from Fact 4.16 that if $W \subseteq \Gamma_\infty^m$ is definably normal and $[a, b] \subseteq \Gamma_\infty$ is a closed interval, then $W \times [a, b]$ is also definably normal. In the ACVF context, we have:

Lemma 6.6. *Let V be a variety. If W is a basic* v+g*-open subset of $V \times \Gamma_\infty^m$ and $[a, b] \subseteq \Gamma_\infty$ is a closed interval, then $W \times [a, b]$ is weakly* v+g*-normal.*

Proof. We have that the pullback of $W \times [a, b]$ under $\mathrm{id} \times \mathrm{val} \colon V \times \mathbb{A}^{m+1} \to V \times \Gamma_\infty^{m+1}$ is the intersection of the pullback of $W \times \Gamma_\infty$ and the pullback of $V \times \Gamma_\infty^m \times [a, b]$. The first pullback is a basic v+g-open subset of $V \times \mathbb{A}^{m+1}$, so it is weakly v+g-normal by Theorem 5.26. Since the second pullback is a v+g-closed subset $V \times \mathbb{A}^{m+1}$, it follows that $(\mathrm{id} \times \mathrm{val})^{-1}(W \times [a, b])$ is weakly v+g-normal (being a v+g-closed subset of a weakly v+g-normal set). From Lemmas 5.29 and 5.30 and Remark 5.25, it follows that $W \times [a, b]$ is weakly v+g-normal. $\square$

In the o-minimal context, since $[a, b]$ is definably compact, the definable map $[a, b] \to \mathrm{pt}$ is definably proper ([23, Definition 3.10 and Remark 3.11]), and since $\boldsymbol{\Sigma}$ has definable Skolem functions, the map $[a, b] \to \mathrm{pt}$ is proper in Def ([23, Definition 3.3 and Theorem 3.15]). Hence if X is a definable subset of Γ_∞^n, then $\pi \colon X \times [a, b] \longrightarrow X$ maps closed definable subsets to closed definable subsets. In the ACVF context, we have:

Lemma 6.7. *Let V be a variety. Let X be a definable subset of $V \times \Gamma_\infty^n$ and $[a, b] \subseteq \Gamma_\infty$ be a closed interval. Then the projection $\pi \colon X \times [a, b] \to X$ is* v+g*-closed: that is, maps* v+g*-closed subsets into* v+g*-closed subsets.*

Proof. Let $C \subseteq X \times [a, b]$ be a v+g-closed subset. Then $\widehat{C}$ is a closed relatively pro-definable subset of $\widehat{X \times [a, b]}$. It is enough to show that $\widehat{\pi(C)}$ is closed. If it is not closed, then by [35, Proposition 4.2.13],

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there is a definable type $\mathfrak{p}$ on $\widehat{X}$ concentrating on $\widehat{\pi(C)}$ with limit p in the closure of $\widehat{\pi(C)}$ in $\widehat{X}$ such that $p \notin \widehat{\pi(C)}$. Applying [35, Lemma 4.2.6 (2)] twice to the surjective definable map $\pi\colon C \to \pi(C)$, we have that there is a definable type $\mathfrak{q}$ on $\widehat{X\times[a,b]}$ concentrating on $\widehat{C}$ such that $\widehat{\pi}_*(\mathfrak{q}) = \mathfrak{p}$. Since $\widehat{X\times[a,b]} \cong \widehat{X}\times[a,b]$, the topology is the product topology and definable types on $[a,b]$ have limits, $\mathfrak{q}$ must have a limit q in $\widehat{X\times[a,b]}$ (namely, the tensor product of the limits of the corresponding projections). By [35, Lemma 4.2.4], $q \in \widehat{C}$, so $p = \widehat{\pi}(q) \in \widehat{\pi(C)}$. □

Theorem 6.8 (Homotopy axiom in stable completions). *Let V be a variety. Let $X \subseteq V \times \Gamma_\infty^m$ be a* v+g*-locally closed subset, and let $\mathcal{F} \in \mathrm{Mod}(A_{\widehat{X}_{\widehat{\mathrm{v+g}}}})$. Let Ψ be a family of $\widehat{\mathrm{v+g}}$-supports on $\widehat{X}$. Let $[a,b] \subseteq [0,\infty]$ be a closed interval, $\pi\colon X\times[a,b] \to X$ be the projection, and for $d \in [a,b]$, let*

$$i_d\colon X \longrightarrow X\times[a,b]$$

be the continuous definable map given by $i_d(x) = (x,d)$ for all $x \in X$. Then

$$\widehat{i_a}^* = \widehat{i_b}^* : H^n_{\Psi\times[a,b]}(\widehat{X}\times[a,b];\widehat{\pi}^{-1}\mathcal{F}) \longrightarrow H^n_{\Psi}(\widehat{X};\mathcal{F})$$

for all $n \geq 0$.

Proof. The homotopy axiom will follow once we show that the projection map $\pi\colon X\times[a,b] \longrightarrow X$ induces an isomorphism

$$\widehat{\pi}^*\colon H^n_{\Psi}(\widehat{X};\mathcal{F}) \longrightarrow H^n_{\Psi\times[a,b]}(\widehat{X}\times[a,b];\widehat{\pi}^{-1}\mathcal{F})$$

since by functoriality, we obtain

$$\widehat{i_a}^* = \widehat{i_b}^* = (\widehat{\pi}^*)^{-1}\colon H^n_{\Psi\times[a,b]}(\widehat{X}\times[a,b];\widehat{\pi}^{-1}\mathcal{F}) \longrightarrow H^n_{\Psi}(\widehat{X};\mathcal{F})$$

for all $n \geq 0$. Equivalently, we need to show that

$$\widetilde{\widehat{\pi}}^*\colon H^n_{\widetilde{\Psi}}(\widetilde{\widehat{X}};\widetilde{\mathcal{F}}) \longrightarrow H^n_{\widetilde{\Psi\times[a,b]}}(\widetilde{\widehat{X}\times[a,b]};\widetilde{\widehat{\pi}}^{-1}\widetilde{\mathcal{F}})$$

is an isomorphism. For this, we need to verify the hypothesis of the Vietoris-Begle theorem (Theorem 3.30).

By Lemma 6.7 (and the fact that $\widehat{\pi(Z)} = \widehat{\pi}(\widehat{Z})$ ([35, Lemma 4.2.6])), $\widehat{\pi}$ maps closed subsets to closed subsets. It follows that $\widetilde{\widehat{\pi}}$ maps closed constructible subset to closed (constructible) subsets.

Let $\alpha \in \widetilde{\widehat{X}}$. Given Corollaries 6.4 and 5.31 together with Remark 5.25, it follows from Lemma 6.1 that there is an open, normal constructible subset U of $\widetilde{\widehat{X}}$ such that $\alpha \in U$ and α is closed in U. Furthermore, we may assume that U is of the form $\widetilde{\widehat{W}}$, where W is a basic v+g-open subset of X. From Lemma 6.6, it follows that $W\times[a,b]$ is weakly v+g-normal, and hence, by Remark 5.25, $\widehat{W\times[a,b]} \simeq \widehat{W}\times[a,b]$ is $\widehat{\mathrm{v+g}}$-normal. Since $\widetilde{\widehat{\pi}}^{-1}(\widetilde{\widehat{W}}) = \widetilde{\widehat{\pi}^{-1}(\widehat{W})} = \widetilde{\widehat{W}\times[a,b]}$, this set is normal and constructible by Proposition 3.12.

Since $\widehat{\pi^{-1}(V)} = \widehat{\pi}^{-1}(\widehat{V})$, we have a commutative diagram

$$\begin{array}{ccc} \widehat{X}\times[a,b] & \xrightarrow{\widehat{\pi}} & \widehat{X} \\ \downarrow & & \downarrow \\ X\times[a,b] & \xrightarrow{\pi} & X \end{array}$$

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of morphisms of sites that induces a commutative diagram

$$\begin{array}{ccc} \widetilde{\widehat{X}\times[a,b]} & \xrightarrow{\widetilde{\widehat{\pi}}} & \widetilde{\widehat{X}} \\ \downarrow & & \downarrow \\ \widetilde{X\times[a,b]} & \xrightarrow{\widetilde{\pi}} & \widetilde{X} \end{array}$$

where the vertical arrows are homeomorphisms induced by isomorphisms of sites (Remark 3.8). It follows that $\widetilde{\widehat{\pi}}^{-1}(\alpha)$ is homeomorphic to $\widetilde{\pi}^{-1}(\beta)$ for some $\beta\in\widetilde{X}$. On the other hand, as in the proof of [22, Claim 4.5], and since the value group $\mathbf{\Gamma}_\infty$ is stably embedded, $\widetilde{\pi}^{-1}(\beta)$ is homeomorphic to $\widetilde{[a,b]}(\mathbf{\Sigma}')$, where $\mathbf{\Sigma}'$ is the definable closure of $\Sigma\cup\{b\}$ with b an element realising the type β, and therefore, $\widetilde{\pi}^{-1}(\beta)$ is connected and $H^q(\widetilde{\pi}^{-1}(\beta);\widetilde{\pi}^{-1}\widetilde{\mathcal{F}}_{|\widetilde{\pi}^{-1}(\beta)})=0$ for all $q>0$. So we conclude that $\widetilde{\widehat{\pi}}^{-1}(\alpha)$ is connected and acyclic as required. □

Corollary 6.9 (Homotopy axiom in ACVF). *Let V be a variety. Let $X\subseteq V\times\Gamma_\infty^m$ be a* v+g*-locally closed subset, and let $\mathcal{F}\in\mathrm{Mod}(A_{X_{\mathrm{v+g}}})$. Let Ψ a family of* v+g*-supports on X. Let $[a,b]\subseteq[0,\infty]$ be a closed interval, let $\pi\colon X\times[a,b]\to X$ be the projection, and, for $d\in[a,b]$, let*

$$i_d\colon X\longrightarrow X\times[a,b]$$

be the continuous definable map given by $i_d(x)=(x,d)$ for all $x\in X$. Then

$$i_a^*=i_b^*\colon H^n_{\Psi\times[a,b]}(X\times[a,b];\pi^{-1}\mathcal{F})\longrightarrow H^n_\Psi(X;\mathcal{F})$$

for all $n\geq 0$.

Proof. As explained in the previous proof, the homotopy axiom will follow once we show that the projection map $\pi\colon X\times[a,b]\longrightarrow X$ induces an isomorphism

$$\pi^*\colon H^n_\Psi(X;\mathcal{F})\longrightarrow H^n_{\Psi\times[a,b]}(X\times[a,b];\pi^{-1}\mathcal{F}).$$

But this follows from the isomorphism

$$\widehat{\pi}^*\colon H^n_\Psi(\widehat{X};\mathcal{F})\longrightarrow H^n_{\Psi\times[a,b]}(\widehat{X}\times[a,b];\widehat{\pi}^{-1}\mathcal{F})$$

proven in Theorem 6.8 and the commutative diagram

$$\begin{array}{ccc} \widehat{X}\times[a,b] & \xrightarrow{\widehat{\pi}} & \widehat{X} \\ \downarrow & & \downarrow \\ X\times[a,b] & \xrightarrow{\pi} & X \end{array}$$

of morphisms of sites, with the vertical arrows being isomorphisms of sites. □

Using Theorem 4.28 and the $\mathbf{\Gamma}_\infty$ analogues of Lemmas 6.3, 6.6 and 6.7 mentioned above, arguing as in the proof of Theorem 6.8, we obtain:

Theorem 6.10 (Homotopy axiom in $\mathbf{\Gamma}_\infty$). *Let $\mathbf{\Gamma}=(\Gamma,<,+,\dots)$ be an arbitrary o-minimal expansion of an ordered group. Suppose that $X\subseteq\Gamma_\infty^m$ is a definably locally closed set and $\mathcal{F}\in\mathrm{Mod}(A_{X_{\mathrm{def}}})$. Let Ψ*

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be a family of definable supports on X. Let $[a, b] \subseteq [0, \infty]$ *be a closed interval,* $\pi\colon X \times [a, b] \to X$ *the projection and, for* $d \in [a, b]$,

$$i_d\colon X \longrightarrow X \times [a, b]$$

be the continuous definable map given by $i_d(x) = (x, d)$ *for all* $x \in X$. *Then*

$$i_a^* = i_b^*\colon H^n_{\Psi\times[a,b]}(X \times [a, b]; \pi^{-1}\mathcal{F}) \longrightarrow H^n_\Psi(X; \mathcal{F})$$

for all $n \geq 0$.

Theorem 6.11. *The sheaf cohomology associated to*

(1) *the category of definable sets in* $\mathbf{\Gamma}_\infty$ *with their associated o-minimal site,*
(2) *the full subcategory of definable sets in* ACVF *with their associated* v+g*-site and*
(3) *the category of stable completions with their associated* $\widehat{\text{v+g}}$*-site*

satisfies the Eilenberg-Steenrod axioms.

Proof. The homotopy axiom corresponds to Theorem 6.10 for (1), to Corollary 6.9 for (2) and to Theorem 6.8 for (3). For all (1)–(3), the remaining axioms follow by standard arguments. Indeed, once we apply the tilde functor in each case, the proofs of the exactness ([14, Chapter II, Section 12, (22)]) and excision axioms ([14, Chapter II, Section 12, 12.8 or 12.9]) are purely algebraic. The dimension axiom is also immediate. □

Remark 6.12. From the Eilenberg-Steenrod axioms for cohomology, one obtains as usual the exactness for triples ([14, Chapter II, Section 12, (24)]) and the Mayer-Vietoris long exact sequence ([14, Chapter II, Section 13, (32)]).

6.2. Finiteness and invariance results in $\mathbf{\Gamma}_\infty$

Here we will show finiteness and invariance results for o-minimal cohomology with definably compact supports of definably locally closed subsets of Γ^m_∞. Using Hrushovski and Loeser's main theorem ([35, Theorem 11.1.1]), we will deduce in the next subsection finiteness and invariance results for cohomology in ACVF and in stable completions.

For the finiteness and invariance results in $\mathbf{\Gamma}_\infty$, we first show that the arguments from [7] for $\mathbf{\Gamma} = (\Gamma, <, +, \ldots)$ and for definably compact sets can be extended to $\mathbf{\Gamma}_\infty$ and to definably locally closed sets with only minor modifications. We include the details in the main technical lemma (Lemma 6.15) and refer the reader to [7] for the other results that simply follow from the Eilenberg-Steenrod axioms. Concerning invariance, the extension to arbitrary sheaves adapts techniques from [26], and we include the details as the same argument will be used in the ACVF case.

Below, we let L be an A-module, where A is a commutative ring with unit.

In [7, Lemma 3.2], it is shown that for every *bounded* cell in Γ^m, there is a deformation retraction of C onto a cell of strictly lower dimension via a Γ-definable homotopy. As observed in [7], the proof extends to $\mathbf{\Gamma}_\infty$.

Lemma 6.13. *If* $C \subseteq [0, \infty]^m$ *is a cell of dimension* $n > 0$, *then there is a deformation retraction of* C *onto a cell of strictly lower dimension. So by induction, every cell in* $[0, \infty]^m$ *is* $\mathbf{\Gamma}_\infty$*-definably contractible to a point.*

Proof. If C is the graph of a function, we can reason by induction on m. So assume $C = (f, g)_B$. If $g < \infty$ and C is of maximal dimension, then $C \subseteq \Gamma^m$, and we can apply [7, Lemma 3.2]. If $g < \infty$ and C is not of maximal dimension, then one can apply induction. If $g = \infty$, let $h = f + 1$. Then $f < h < g$,

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and the $\mathbf{\Gamma}_\infty$-definable map given $H\colon [0,\infty] \times C \to C$ by

$$H(t,(x,y)) = \begin{cases} (x, h(x)-t), & \text{if } y < h(x)-t \\ (x,y), & \text{if } h(x)-t \leq y \leq h(x)+t \\ (x, h(x)+t), & \text{if } y > h(x)+t \end{cases}$$

is a deformation retract of C onto the cell $\Gamma(h)$. □

As observed after the proof of [7, Corollary 3.3] that result (for bounded cell in Γ^n) extends to $\mathbf{\Gamma}_\infty$ due to Lemma 6.13 and the homotopy axiom (Theorem 6.10), recall that cells are definably locally closed (Remark 4.12):

Lemma 6.14. *Let $C \subseteq [0,\infty]^m$ be a cell. Then C is acyclic: that is, $H^p(C; L_C) = 0$ for $p > 0$ and $H^0(C; L_C) = L$.* □

We also have the analogue of [7, Lemma 7.1] with small modifications in the construction:

Lemma 6.15. *Let $C \subseteq [0,\infty]^m$ be a cell of dimension r. There is a $\mathbf{\Gamma}_\infty$-definable family $\{C_{(t,s)} : 0 < t, s < \infty\}$ of closed and bounded subsets $C_{(t,s)} \subseteq C$ such that:*

(1) $C = \bigcup_{(t,s)} C_{(t,s)}$.
(2) *If $r > 0$ and $0 < t' < t < \frac{s}{2} < s < s'$, then $C_{(t,s)} \subseteq C_{(t',s')}$, and this inclusion induces an isomorphism*

$$H^p(C \backslash C_{(t,s)}; L_C) \simeq H^p(C \backslash C_{(t',s')}; L_C).$$

(3) *If $r > 0$, then the o-minimal cohomology of $C \backslash C_{(t,s)}$ is given by*

$$H^p(C \backslash C_{(t,s)}; L_C) = \begin{cases} L^{1+\chi_1(r)} & \text{if} \quad p \in \{0, r-1\} \\ & \\ 0 & \text{if} \quad p \notin \{0, r-1\} \end{cases}$$

where $\chi_1 \colon \mathbb{Z} \to \{0,1\}$ is the characteristic function of the subset $\{1\}$.
(4) *If $K \subseteq C$ is a definably compact subset, the there are $0 < t, s$ such that $K \subseteq C_{(t,s)}$.*

Proof. We define the definable family $\{C_{(t,s)} : 0 < t, s < \infty\}$ by induction on $l \in \{1, \ldots, m-1\}$ and the definition of the $\mathbf{\Gamma}_\infty$-cell C in the following way:

- If $l = 1$ and C is a singleton $\{d\}$ in Γ_∞, we define $C_{(t,s)} = C$. Clearly $C_{(t,s)}$ is a closed and bounded subset.
- If $l = 1$ and $C = (d, e) \subseteq \Gamma_\infty$, then

$$C_{(t,s)} = \begin{cases} [d + \gamma(t,s), e - \gamma(t,s)] & \text{if} \quad e < \infty \\ & \\ [d + \gamma(t,s), (d+s) - \gamma(t,s)] & \text{otherwise} \end{cases}$$

where

$$\gamma(t,s) = \begin{cases} \min\{\frac{e-d}{2}, t\} & \text{if} \quad e < \infty \\ & \\ \min\{\frac{s}{2}, t\} & \text{otherwise} \end{cases}$$

(in this way $C_{(t,s)}$ is non empty). Clearly $C_{(t,s)}$ is a closed and bounded subset.

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- If $l > 1$ and $C = \Gamma(f)$, where $B \subseteq [0,\infty]^l$ is $\mathbf{\Gamma}_\infty$-cell. By induction, $B_{(t,s)}$ is defined and is a closed and bounded subset. We put $C_{(t,s)} = \Gamma(f_{|B_{(t,s)}})$. Clearly $C_{(t,s)}$ is a closed and bounded subset.
- If $l > 1$ and $C = (f,g)_B$, where $B \subseteq [0,\infty]^l$ is $\mathbf{\Gamma}_\infty$-cell and $f < g$. By induction, $B_{(t,s)}$ is defined and is a closed and bounded subset. Recall that either $g < \infty$ or $g = \infty$. We put

$$C_{(t,s)} = \begin{cases} [f+\gamma(t,s), g-\gamma(t,s)]_{B_{(t,s)}} & \text{if} \quad g < \infty \\ \\ [f+\gamma(t,s), (f+s)-\gamma(t,s)]_{B_{(t,s)}} & \text{otherwise} \end{cases}$$

where

$$\gamma(t,s) = \begin{cases} \min\{\frac{g-f}{2}, t\} & \text{if} \quad g < \infty \\ \\ \min\{\frac{s}{2}, t\} & \text{otherwise.} \end{cases}$$

By induction, $B_{(t,s)}$ is a closed and bounded subset of B. Also, for each $x \in B_{(t,s)}$, the fibre $(\pi_{|C})^{-1}(x) \cap C_{(t,s)}$ is closed and bounded. Let $(x,y) \in C$ be an element in the closure of $C_{(t,s)}$. Then $x \in B_{(t,s)}$ and $(x,y) \in (\pi_{|C})^{-1}(x) \cap C_{(t,s)} \subseteq C_{(t,s)}$. So $C_{(t,s)}$ is a closed and bounded subset of C.

We observe that from this construction, we obtain:

Claim 6.16. *For (t,s) as above, there is a covering $\mathcal{U}_C = \{U_i : i \in I\}$ of $C \backslash C_{(t,s)}$ by relatively open bounded subsets of C such that:*

(1) *The index set I is the family of the closed faces of an r-dimensional cube. (So $|I| = 2r$).*
(2) *If $E \subseteq I$, then $U_E := \bigcap_{i\in E} U_i$ is either empty or a $\mathbf{\Gamma}_\infty$-cell. (So in particular $H^p(U_E; L_C) = 0$ for $p > 0$ and, if $U_E \neq \emptyset$, $H^0(U_E; L_C) = L$.)*
(3) *For $E \subseteq I$, $U_E \neq \emptyset$ iff the faces of the cubes belonging to E have a non-empty intersection.*

So the nerve of $\mathcal{U}_C$ is isomorphic to the nerve of a covering of an r-cube by its closed faces.

Proof. To show that there is a covering satisfying the properties above, we define $\mathcal{U}_C$ by induction on $l \in \{1,\dots,m-1\}$. We distinguish four cases according to the definition of the $C_{(t,s)}$:

- If $l = 1$ and C is a singleton in Γ_∞, then $\mathcal{U}_C = \{C\}$.
- If $l = 1$ and $C = (d,e) \subseteq \Gamma_\infty$, then

$$\mathcal{U}_C = \begin{cases} \{(d, d+\gamma(t,s)), (e-\gamma(t,s), e)\} & \text{if} \quad e < \infty \\ \\ \{(d, d+\gamma(t,s)), ((d+s)-\gamma(t,s), \infty)\} & \text{otherwise} \end{cases}$$

- If $l > 1$ and $C = \Gamma(f)$, where $B \subseteq [0,\infty]^l$ is $\mathbf{\Gamma}_\infty$-cell. By definition, $C_{(t,s)} = \Gamma(f_{|B_{(t,s)}})$. By induction, we have a covering $\mathcal{V}_B$ of $B \backslash B_{(t,s)}$ with the stated properties, and we define $\mathcal{U}_C$ to be a covering of $C \backslash C_{(t,s)}$ induced by the natural homeomorphism between the graph of f and its domain.
- If $l > 1$ and $C = (f,g)_B$, where $B \subseteq [0,\infty]^l$ is $\mathbf{\Gamma}_\infty$-cell and $f < g$. By definition,

$$C_{(t,s)} = \begin{cases} [f+\gamma(t,s), g-\gamma(t,s)]_{B_{(t,s)}} & \text{if} \quad g < \infty \\ \\ [f+\gamma(t,s), (f+s)-\gamma(t,s)]_{B_{(t,s)}} & \text{otherwise.} \end{cases}$$

By induction, we have that $B \backslash B_{(t,s)}$ has a covering $\mathcal{V}_B = \{V_j : j \in J\}$ with the stated properties, where J is the set of closed faces of the cube $[0,1]^{r-1}$. Define a covering $\mathcal{U}_C = \{U_i : i \in I\}$ of $C \backslash C_{(t,s)}$ as follows. As index set I, we take the closed faces of the cube $[0,1]^r$. Thus $|I| = |J| + 2$, with the two extra faces corresponding to the 'top' and 'bottom' faces of $[0,1]^r$. We associate to the

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top face the open set $(g - \gamma(t,s), g)_B$ if $g < \infty$ or $((f+s) - \gamma(t,s), \infty)_B$ if $g = \infty$ and the bottom face the open set $(f, f + \gamma(t,s))_B$. The other open sets of the covering are the preimages of the sets V_j under the restriction of the projection $\Gamma_\infty^{l+1} \to \Gamma_\infty^l$. This defines a covering of $C \backslash C_{(t,s)}$ with the stated properties. □

Property (1) of the lemma is clear. By (the proof of) Claim 6.16, there are open covers $\mathcal{U}'_C$ of $C \backslash C_{(t',s')}$ and $\mathcal{U}_C$ of $C \backslash C_{(t,s)}$ satisfying the assumptions of [7, Lemma 5.5]. Hence property (2) of the lemma holds. Finally, if $r > 1$, then property (3) follows from Claim 6.16 and [7, Corollary 5.2]. On the other hand, if $r = 1$, then $C \setminus C_{(t,s)}$ is by construction a disjoint union $D \sqcup E$ of two $\mathbf{\Gamma}_\infty$-cells in $[0, \infty]^m$ of dimension $r = 1$. Therefore, in this case, the result follows from Lemma 6.14, since $H^*(C \backslash C_{(t,s)}; L_C) \simeq H^*(D; L_D) \oplus H^*(E; L_E)$.

It remains to show property (4). Note that if $0 < t' < t < s < s'$, then by construction, $C_{(t',s')}$ contains the interior relative to C of $C_{(t,s)}$. Thus C is a directed union $\bigcup_{(t,s)} U_{(t,s)}$ of a $\mathbf{\Gamma}_\infty$-definable family of relatively open $\mathbf{\Gamma}_\infty$-definable subsets. In particular, if $K \subseteq C$ is a $\mathbf{\Gamma}_\infty$-definably compact subset, then $\{K \cap U_{(t,s)} : 0 < t < s\}$ is a $\mathbf{\Gamma}_\infty$-definable family of open $\mathbf{\Gamma}_\infty$-definable subsets of K with the property that every finite subset of K is contained in one of the $K \cap U_{(t,s)}$. Therefore, since K is closed and bounded (Remark 4.11), by [44, Corollary 2.2 (ii)], there are $0 < t < s$ such that $K \subseteq U_{(t,s)} \subseteq C_{(t,s)}$. □

From Lemma 6.15 and computations using excision axiom ([14, Chapter II, 12.9]) and long exactness sequence ([14, Chapter II, Section 12 (22)]), we obtain just as in [7, Lemma 7.2]:

Lemma 6.17. *Let $X \subseteq [0, \infty]^m$ be a definable set and $C \subseteq X$ be a cell of maximal dimension. Then for every $0 < t' < t < \frac{s}{2} < s < s'$, we have isomorphisms induced by inclusions:*

$$H^*(X \backslash C_{(t,s)}; L_X) \simeq H^*(X \backslash C_{(t',s')}; L_X).$$ □

The next result is the analogue of [7, Corollary 7.3], but we have to explain how to use [7, Lemma 6.7] in our context:

Lemma 6.18. *Let $X \subseteq [0, \infty]^m$ be a closed definable set and $C \subseteq X$ be a cell of maximal dimension. Then for every $0 < t < s$, we have isomorphisms induced by inclusions:*

$$H^*(X \backslash C_{(t,s)}; L_X) \simeq H^*(X \backslash C; L_X).$$

Proof. Let $Z = X \backslash C$, and for $0 < t < s$, let $Y_{(t,s)} = X \backslash C_{(t,s)}$. Let V be an open definable neighbourhood of Z in X. Since $K = X \setminus V \subseteq C$ is a definably compact subset, by Lemma 6.15 (4), there are $0 < t < s$ such that $K \subseteq C_{(t,s)} \subseteq C$. Therefore, $Z \subseteq Y_{(t,s)} \subseteq V$. On the other hand, $\widetilde{Y_{(t,s)}}$ for all $0 < t < s$ are taut in $\widetilde{X}$, being open subsets ([14, page 73]); and since X is definably normal (being definably compact) and Z is a closed subset, the family of all closed subsets of $\widetilde{X}$ is a normal and constructible family of supports on $\widetilde{X}$, so by Corollary 3.28, $\widetilde{Z}$ is taut in $\widetilde{X}$. Therefore, by the purely topological [7, Lemma 6.4], together with Lemma 6.17 and [7, Remark 6.6], we have

$$H^*(X \backslash C; L_X) \simeq \varinjlim_{0<t<s} H^*(X \backslash C_{(t,s)}; L_X) \simeq H^*(X \backslash C_{(t,s)}; L_X).$$ □

Remark 6.19. Let $X \subseteq \Gamma_\infty^m$ be a definably locally closed subset. By Remark 4.13, there is a definably compact subset $P \subseteq \Gamma_\infty^m$ such that X is an open definable subset of P. Since by Remark 4.14, P is definably normal and every definably compact subset of X is a closed definable subset of P, it follows from Definition 3.11 (2) that the family, which we will denote by c, of all definably compact definable subsets of X is a family of definably normal supports on X.

Below, we will assume some properties of the corresponding cohomology with definably compact supports; see, for instance, [27]. In particular, we observe that if $Z \subseteq X$ is definably compact, then $H_c^*(Z; L_Z) \simeq H^*(Z; L_Z)$.

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Theorem 6.20. *Let A be a noetherian ring, and let L be a finitely generated A-module. If $X \subseteq \Gamma_\infty^m$ is a definably locally closed subset, then $H_c^p(X; L_X)$ is finitely generated for each p. Moreover, $H_c^p(X; L_X) = 0$ for $p > \dim X$.*

Proof. Let $i\colon X \hookrightarrow P \subseteq [0,\infty]^{2m}$ be a definable completion of X (Remark 4.13). Since P is a closed subset of $[0,\infty]^{2m}$, from Lemma 6.18 and computations using Mayer-Vietoris sequence ([14, Chapter II, Section 13 (32)]; see Remark 6.12), we obtain just as in [7, Theorem 7.4] that $H^p(P; L_P)$ is finitely generated for each p and $H^p(P; L_P) = 0$ for $p > \dim P$. Since X is an open definable subset of P, $P \setminus X$ is a closed subset of $[0,\infty]^{2m}$, and similarly, we have that $H^p(P \setminus X; L_{P\setminus X})$ is finitely generated for each p and $H^p(P \setminus X; L_{P\setminus X}) = 0$ for $p > \dim P \setminus X$.

By Equation (1) on page 5, if $Z \subseteq P$ is a definably locally closed subset, then we have $L_Z = i_!(L_{P\,|Z})$. Then Corollary 3.27 together with the short exact coefficients sequence

$$0 \to L_X \to L_P \to L_{P\setminus X} \to 0$$

yields the long exact cohomology sequence

$$\ldots \to H^{l-1}(P \setminus X; L_{P\setminus X}) \xrightarrow{\alpha} H_c^l(X; L_X) \xrightarrow{\beta} H^l(P; L_P) \to H^l(P \setminus X; L_{P\setminus X}) \to \ldots.$$

Since P is the closure of X, we have $\dim X = \dim P$ and $\dim P \setminus X < \dim P$ ([53, Chapter 4, (1.8)]). It follows that $H_c^p(X; L_X) = 0$ for all $p > \dim X$. It also follows that $\ker\beta = \operatorname{Im}\alpha$ is finitely generated and $\operatorname{Im}\beta$ is finitely generated. Thus we see that $H_c^p(X; L_X)$ is finitely generated for each p. □

Below, we will require the following, which is obtained by going to $\widetilde{\mathfrak{T}}$ and applying the corresponding criterion in the topological case ([14, Chapter II, 16.12]) (point (iii) is a little stronger here), observing that $\widetilde{U}$ with $U \in \operatorname{Op}(X_{\mathcal{T}})$ forms a filtrant basis for the topology of $\widetilde{X}_{\mathcal{T}}$.

Lemma 6.21. *Let $(X, \mathcal{T})$ be an object of $\mathfrak{T}$, and let $\mathfrak{R}$ be a class of objects of $\operatorname{Mod}(A_{X_{\mathcal{T}}})$. Suppose that $\mathfrak{R}$ satisfies:*

(i) *For each exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ with $\mathcal{F}' \in \mathfrak{R}$, we have $\mathcal{F} \in \mathfrak{R}$ if and only if $\mathcal{F}'' \in \mathfrak{R}$.*
(ii) *$\mathfrak{R}$ is stable under filtrant $\varinjlim$.*
(iii) *$A_V \in \mathfrak{R}$ for any $V \in \operatorname{Op}(X_{\mathcal{T}})$.*

Then $\mathfrak{R} = \operatorname{Mod}(A_{X_{\mathcal{T}}})$. □

Recall that if $\mathbf{\Gamma}'$ is an elementary extension of $\mathbf{\Gamma}$ or an o-minimal expansion of $\mathbf{\Gamma}$ and $Y \subseteq \Gamma_\infty^m$ is a definable subset, then we have a natural morphism of sites

$$Y(\mathbf{\Gamma}'_\infty)_{\mathrm{def}} \to Y_{\mathrm{def}}.$$

If $\mathcal{F} \in \operatorname{Mod}(A_{Y_{\mathrm{def}}})$, then we have a corresponding $\mathcal{F}(\mathbf{\Gamma}'_\infty) \in \operatorname{Mod}(A_{Y(\mathbf{\Gamma}'_\infty)_{\mathrm{def}}})$ given by inverse image.

Theorem 6.22. *Let $\mathbf{\Gamma}'$ be an elementary extension of $\mathbf{\Gamma}$ or an o-minimal expansion of $\mathbf{\Gamma}$. If $X \subseteq \Gamma_\infty^m$ is a definably locally closed subset, then for every $\mathcal{F} \in \operatorname{Mod}(A_{X_{\mathrm{def}}})$, we have an isomorphism*

$$H_c^*(X; \mathcal{F}) \simeq H_c^*(X(\mathbf{\Gamma}'_\infty); \mathcal{F}(\mathbf{\Gamma}'_\infty)).$$

Proof. Let $i\colon X \hookrightarrow P \subseteq [0,\infty]^{2m}$ be a definable completion of X (Remark 4.13). Since P is a closed subset of $[0,\infty]^{2m}$, for any closed definable subset $Z \subseteq P$, from Lemma 6.18 and computations using Mayer-Vietoris sequence ([14, Chapter II, Section 13 (32)]; see Remark 6.12), we obtain just as in [7, Theorems 8.1 and 8.3] an isomorphism

$$H^*(Z; A_Z) \simeq H^*(Z(\mathbf{\Gamma}'_\infty); A_{Z(\mathbf{\Gamma}'_\infty)}).$$

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Since P is definably compact, it is definably normal, so the family of closed definable subsets of P is a family of definably normal supports on P. Also X is an open definable subset of P, and hence by Corollary 3.27, if $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, then

$$H^*(P; i_!\mathcal{F}) = H^*_c(X; \mathcal{F}).$$

By invariance of definably compactness (see Remark 4.11) and Corollary 3.27 in $\mathbf{\Gamma}'_\infty$, we also have

$$H^*(P(\mathbf{\Gamma}'_\infty); i_!^{\mathbf{\Gamma}'_\infty}\mathcal{F}(\mathbf{\Gamma}'_\infty)) = H^*_c(X(\mathbf{\Gamma}'_\infty); \mathcal{F}(\mathbf{\Gamma}'_\infty)).$$

Moreover, by Equation (2) on page 5 applied to the commutative diagram (with tildes omitted)

$$\begin{array}{ccc} X(\mathbf{\Gamma}'_\infty) & \xhookrightarrow{i^{\mathbf{\Gamma}'_\infty}} & P(\mathbf{\Gamma}'_\infty) \\ \downarrow{\scriptstyle r_|} & & \downarrow{\scriptstyle r} \\ X & \xhookrightarrow{\quad i \quad} & P \end{array}$$

we get $(i_!\mathcal{F})(\mathbf{\Gamma}'_\infty) = i_!^{\mathbf{\Gamma}'_\infty}\mathcal{F}(\mathbf{\Gamma}'_\infty)$, so

$$H^*(P(\mathbf{\Gamma}'_\infty); (i_!\mathcal{F})(\mathbf{\Gamma}'_\infty)) = H^*_c(X(\mathbf{\Gamma}'_\infty); \mathcal{F}(\mathbf{\Gamma}'_\infty)).$$

The result will follow once we show that, for every $\mathcal{G} \in \mathrm{Mod}(A_{P_{\mathrm{def}}})$, we have an isomorphism

$$H^*(P; \mathcal{G}) \simeq H^*(P(\mathbf{\Gamma}'_\infty); \mathcal{G}(\mathbf{\Gamma}'_\infty)).$$

But this is obtained by applying Lemma 6.21 in the following way. For (i), the exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ implies the following chain of morphisms in cohomology ($j \in \mathbb{Z}$):

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^j(P; \mathcal{F}') & \longrightarrow & H^j(P; \mathcal{F}) & \longrightarrow & H^j(P; \mathcal{F}'') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^j(P(\mathbf{\Gamma}'_\infty); \mathcal{F}'(\mathbf{\Gamma}'_\infty)) & \longrightarrow & H^j(P(\mathbf{\Gamma}'_\infty); \mathcal{F}(\mathbf{\Gamma}'_\infty)) & \longrightarrow & H^j(P(\mathbf{\Gamma}'_\infty); \mathcal{F}''(\mathbf{\Gamma}'_\infty)) & \longrightarrow & \cdots \end{array}$$

Therefore, if we have isomorphisms for $\mathcal{F}'$, then using the five lemmas, we have isomorphisms for $\mathcal{F}$ if and only if we have isomorphisms for $\mathcal{F}''$.

For (ii), first note that (a) sections commute with filtrant $\varinjlim$ ([27, Example 1.1.4 and Proposition 1.2.12]); (b) filtrant $\varinjlim$ of $\mathcal{T}$-flabby sheaves is $\mathcal{T}$-flabby ([27, Proposition 2.3.4 (i)]); (c) the full additive subcategory of $\mathcal{T}$-flabby objects is $\Gamma(U; \bullet)$-injective for every $U \in \mathrm{Op}(P_{\mathcal{T}})$ ([27, Proposition 2.3.5]). This implies that $H^*(P; \varinjlim \mathcal{F}_i) = \varinjlim H^*(P; \mathcal{F}_i)$. Since the same facts hold in $\mathbf{\Gamma}'_\infty$, we have $H^*(P(\mathbf{\Gamma}'_\infty); \varinjlim(\mathcal{F}_i(\mathbf{\Gamma}'_\infty))) = \varinjlim H^*(P(\mathbf{\Gamma}'_\infty); \mathcal{F}_i(\mathbf{\Gamma}'_\infty))$, and (ii) follows since $\varinjlim(\mathcal{F}_i(\mathbf{\Gamma}'_\infty)) = (\varinjlim \mathcal{F}_i)(\mathbf{\Gamma}'_\infty)$ (inverse image commutes with $\varinjlim$).

For (iii), if $V \in \mathrm{Op}(P_{\mathcal{T}})$, then the exact sequence $0 \to A_V \to A_P \to A_{P\setminus V} \to 0$ implies the following chain of morphisms in cohomology ($j \in \mathbb{Z}$)

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H^j(P; A_V) & \longrightarrow & H^j(P; A_P) & \longrightarrow & H^j(P; A_{P\setminus V}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H^j(P(\mathbf{\Gamma}'_\infty); A_{V(\mathbf{\Gamma}'_\infty)}) & \longrightarrow & H^j(P(\mathbf{\Gamma}'_\infty); A_{P(\mathbf{\Gamma}'_\infty)}) & \longrightarrow & H^j(P(\mathbf{\Gamma}'_\infty); A_{(P\setminus V)(\mathbf{\Gamma}'_\infty)}) & \longrightarrow & \cdots \end{array}$$

where we are using that $P(\mathbf{\Gamma}'_\infty) \setminus V(\mathbf{\Gamma}'_\infty) = (P \setminus V)(\mathbf{\Gamma}'_\infty)$. Since $P \setminus V$ is a closed definable subset of P, we have as observed above $H^*(P \setminus V; A_{P\setminus V}) \simeq H^*((P \setminus V)(\mathbf{\Gamma}'_\infty); A_{(P\setminus V)(\mathbf{\Gamma}'_\infty)})$. But by Corollary 3.27 $H^*(P; A_{P\setminus V}) \simeq H^*(P \setminus V; A_{P\setminus V})$ and by the same result in $\mathbf{\Gamma}'_\infty$, we have $H^*(P(\mathbf{\Gamma}'_\infty); A_{(P\setminus V)(\mathbf{\Gamma}'_\infty)}) \simeq H^*((P \setminus V)(\mathbf{\Gamma}'_\infty); A_{(P\setminus V)(\mathbf{\Gamma}'_\infty)})$. So (iii) follows now using the five lemmas. □

Remark 6.23. In the paper [26], it was shown that in an arbitrary o-minimal structure $\mathbf{M}$ with definable Skolem functions, on full subcategories $\widetilde{\mathbf{A}}$ of $\widetilde{\mathrm{Def}}$, the category of o-minimal spectra of definable spaces equipped with the o-minimal site satisfying the following properties, one can develop the formalism of the six Grothendieck operations:

(A0) Cartesian products (in $\widetilde{\mathrm{Def}}$) of objects of $\widetilde{\mathbf{A}}$ are objects of $\widetilde{\mathbf{A}}$, and locally closed constructible subsets of objects of $\widetilde{\mathbf{A}}$ are objects of $\widetilde{\mathbf{A}}$;
(A1) In every object of $\widetilde{\mathbf{A}}$, every open constructible subset is a finite union of open and normal constructible subsets;
(A2) Every object of $\widetilde{\mathbf{A}}$ has a completion in $\widetilde{\mathbf{A}}$.

For some of the results about the proper direct image (base change formula, derived base change formula, the Künneth formula and dual base change formula), we required that the morphisms $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ in $\widetilde{\mathbf{A}}$ involved satisfy the following:

(A3) If $u \in Y$, then for every elementary extension $\mathbf{S}$ of $\mathbf{M}$ and every $F \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, we have an isomorphism

$$H^*_c(f^{-1}(u); F_{|f^{-1}(u)}) \simeq H^*_c((f^{\mathbf{S}})^{-1}(u); F(\mathbf{S})_{|(f^{\mathbf{S}})^{-1}(u)})$$

of cohomology with definably compact supports, where $\widetilde{F}(\mathbf{S}) = r^{-1}\widetilde{F}$ and $r : \widetilde{X(\mathbf{S})} \to \widetilde{X}$ is the canonical restriction.

The work done here gives a new example where we have this formalism: namely, for the category of definably locally closed definable sets in $\mathbf{\Gamma}_\infty$. Of course, in this case, (A0) is obvious. For (A1), (A2) and (A3), see, respectively, Corollary 5.31, Remark 4.13 and Theorem 6.22.

6.3. Finiteness and invariance results in ACVF and stable completions

Here we will use the finiteness and invariance results for o-minimal cohomology with definably compact supports of definably locally closed subsets of Γ^m_∞ and Hrushovski and Loeser's main theorem ([35, Theorem 11.1.1]) to deduce finiteness and invariance results for cohomology in ACVF and in stable completions.

First we recall a few things. Let V and W be algebraic varieties, a model of ACVF. Let $Y \subseteq V \times \Gamma^m_\infty$ and $Z \subseteq W \times \Gamma^n_\infty$ be definable subsets. Then:

(1) A pro-definable map $g : Y \to \widehat{Z}$ is v+g-*continuous* if and only if the pullback of an open definable subset of $\widehat{Z}$ is a v+g-open subset of Y (see page 48 in [35]). Since open definable subsets of $\widehat{Z}$ are exactly the $\widetilde{\text{v+g}}$-open subsets of $\widehat{Z}$, in our terminology, $g : Y \to \widehat{Z}$ is v+g-continuous if and only if g is a morphism $g : Y_{\text{v+g}} \to \widehat{Z}_{\widehat{\text{v+g}}}$ of sites.
(2) A pro-definable map $g : Y \to \widehat{W}$ is *g-continuous* (respectively, *v-continuous*) if and only if for any regular function $h \in \mathcal{O}_W(W)$, the pullback under $\widehat{(\mathrm{val} \circ h)} \circ g$ of a g-open (respectively, v-open) subset of Γ_∞ is g-open (respectively, v-open) subset of Y, where a subset of Γ^m_∞ is g-open (respectively, v-open) if its pullback under $\mathrm{val} : \mathbb{A}^m \to \Gamma^m_\infty$ is g-open (respectively, v-open). It follows that if $g : Y \to \widehat{W}$ is both g-continuous and v-continuous, then g is v+g-continuous.
(3) Any pro-definable map $g : Y \to \widehat{Z}$ has a *canonical extension* $G : \widehat{Y} \to \widehat{Z}$ that is a pro-definable map ([35, Lemma 3.8.1]). G is given by: if $p \in \widehat{Y}$ and $p = \mathrm{tp}(c/\mathbb{U})$, then $G(p) \in \widehat{Z}$ is such that $G(p)|\mathbb{U}(c) = \mathrm{tp}(d/\mathbb{U})$ with d a realisation of $\mathrm{tp}(g(c)/\mathbb{U}(c))$. It follows that if $g : Y \to \widehat{Z}$ is

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v+g-continuous and U is a v+g-open subset of Z, then $\widehat{g^{-1}(\widehat{U}))} = G^{-1}(\widehat{U})$, and the canonical extension G is a morphism of $\widehat{\text{v+g}}$-sites.

Let V be a quasi-projective variety. Let $X \subseteq V \times \Gamma_\infty^m$ be a definable subset. By [35, Theorem 11.1.1] (see Fact 5.14), let $H\colon I \times \widehat{X} \to \widehat{X}$ be the continuous pro-definable deformation retraction with image an iso-definable subset $\mathfrak{X}$ of $\widehat{X}$ definably homeomorphic to a definable subset of some Γ_∞^k. Let $h\colon \mathfrak{X} \to \mathcal{X}$ be a pro-definable homeomorphism of $\mathfrak{X}$ with a definable subset of Γ_∞^k.

We now make a couple of observations required below that are not explicitly stated in [35].

Lemma 6.24. *The continuous pro-definable deformation retraction $H\colon I \times \widehat{X} \to \widehat{X}$ is a morphism of $\widehat{\text{v+g}}$-sites.*

Proof. The statement of [35, Theorem 11.1.1] is more general: the deformation retraction respects finitely many definable maps $\zeta_i\colon X \to \Gamma_\infty$ with canonical extension $\zeta_i\colon \widehat{X} \to \Gamma_\infty$ and respects the action of a finite algebraic group acting on V, leaving X globally invariant. For this reason, one can make several reductions such as assuming that $m = 0$; $X = V$ and V is a projective and equidimensional variety. Then the proof proceeds by induction on $\dim(V)$. The case $\dim(V) = 0$ being trivial, for the case $\dim(V) > 0$, several preliminary reductions are performed, allowing one to essentially reduce the proof to the case of curve fibration. In the end, the homotopy H is the concatenation $H_\Gamma^\alpha \circ ((H_{\widetilde{base}} \circ H_{curves}) \circ H_{inf})$ of the relative curve homotopy H_{curves}, the liftable base homotopy $H_{\widetilde{base}}$, the purely combinatorial tropical homotopy H_Γ^α and the inflation homotopy H_{inf}.

Furthermore, we have:

(i) The inflation homotopy $H_{inf}\colon [0,\infty] \times \widehat{V} \to \widehat{V}$ is given by

$$H_{inf}(t,x) = \begin{cases} F(t,x) & x \in \widehat{V \setminus D} \\ x & x \in \widehat{D} \end{cases}$$

where D is a closed subvariety of V, $F\colon [0,\infty] \times \widehat{V \setminus D} \to \widehat{V \setminus D}$ is the canonical extension of a v+g-continuous pro-definable map $f\colon [0,\infty] \times V \setminus D \to \widehat{V \setminus D}$.
Let $U \subseteq V$ be a v+g-open subset. Then $U \setminus D$ is a v+g-open subset of $V \setminus D$. Also

$$\begin{aligned} H_{inf}^{-1}(\widehat{U} \setminus \widehat{D}) &= H_{inf}^{-1}(\widehat{U \setminus D}) \\ &= F^{-1}(\widehat{U \setminus D}) \\ &= \widehat{f^{-1}(\widehat{U \setminus D})} \end{aligned}$$

and so $H_{inf}^{-1}(\widehat{U} \setminus \widehat{D})$ is a $\widehat{\text{v+g}}$-open subset of $[0,\infty] \times \widehat{V \setminus D}$. It follows that $H_{inf}^{-1}(\widehat{U}) = H_{inf}^{-1}(\widehat{U} \setminus \widehat{D}) \cup ([0,\infty] \times (\widehat{U \cap D}))$ is a $\widehat{\text{v+g}}$-subset of $[0,\infty] \times \widehat{V}$. By [35, Lemma 10.3.2], H_{inf} is continuous, so $H_{inf}^{-1}(\widehat{U})$ is an open subset of $[0,\infty] \times \widehat{V}$. Therefore, by Fact 5.11, $H_{inf}^{-1}(\widehat{U})$ is $\widehat{\text{v+g}}$-open subset. Hence H_{inf} is a morphism of $\widehat{\text{v+g}}$-sites.

(ii) The relative curve homotopy $H_{curve}\colon [0,\infty] \times \widehat{V_0 \cup D_0} \to \widehat{V_0 \cup D_0}$ is the canonical extension of a g-continuous and v-continuous homotopy $h_{curves}\colon [0,\infty] \times V_0 \cup D_0 \to \widehat{V_0 \cup D_0}$. Hence by the remarks made above, H_{curves} is a morphism of $\widehat{\text{v+g}}$-sites.

(iii) The base homotopy $H_{\widetilde{base}}$ is obtained from the canonical extension of a homotopy $h_{base}\colon I \times U \to \widehat{U}$ that is obtained by the induction procedure. Hence, by induction, $H_{\widetilde{base}}$ is also a morphism. By the remarks made above, H_{curves} is a morphism of $\widehat{\text{v+g}}$-sites.

(iv) The tropical homotopy H_Γ^α is constructed in the Γ_∞ setting where v+g-continuous is the same as definable and continuous, so this homotopy is a also a morphism of $\widehat{\text{v+g}}$-sites.

We can therefore conclude that H is a morphism of $\widehat{\text{v+g}}$-sites as required. □

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Lemma 6.25. *The pro-definable homeomorphism $h\colon \mathfrak{X} \to \mathcal{X}$ of $\mathfrak{X}$ with a definable subset of Γ_∞^k is a morphism of $\widehat{\text{v+g}}$-sites.*

Proof. Let U be a v+g-open subset of X. Then $\widehat{U}$ is a relatively definable subset of $\widehat{X}$. So, since $\mathfrak{X}$ is an iso-definable subset, $\widehat{U} \cap \mathfrak{X}$ is a relatively definable subset of $\mathfrak{X}$ (Lemma 5.16 (c)). By Lemma 5.17, $(h^{-1})^{-1}(\widehat{U})$ is a relatively definable subset of $\mathcal{X}$, and hence it is an open definable subset and so a $\widehat{\text{v+g}}$-open subset (Facts 5.11 and 5.10). Since $h^{-1} : \mathcal{X} \to \mathfrak{X}$ is a morphism of $\widehat{\text{v+g}}$-sites, so is h. □

Lemma 6.26. *If $X \subseteq V \times \Gamma_\infty^m$ is a* v+g-*locally closed subset, then there is a bounded* v+g-*closed subset $P \subseteq V' \times \Gamma_\infty^m$ (for V' a projective variety in which V is open) such that $X \subseteq P$ and X is* v+g-*open in P. In addition, if K is an elementary substructure of $\mathbb{U}$, V is over K and X is K-definable, then P is also K-definable.*

Proof. Indeed, let V' be a projective variety such that V is an open subset of V'. Then X is still a v+g-locally closed subset of $V' \times \Gamma_\infty^m$. Consider the natural definable map $l\colon V' \times \Gamma_\infty^m \to V' \times [0, \infty]^{2m}$, which is $\mathrm{id}_{V'}$ on the first coordinate and on the second coordinate is given as in Remark 4.13. Then $l\colon V' \times \Gamma_\infty^m \to l(V' \times \Gamma_\infty^m) \subseteq V' \times [0, \infty]^{2m}$ is a definable homeomorphism, a morphism of v+g-sites and $l(X)$ is v+g-locally closed in $l(V' \times \Gamma_\infty^m)$. So there is a v+g-closed subset Z of $l(V' \times \Gamma_\infty^m)$ such that $l(X)$ is v+g-open in Z since $l(V' \times \Gamma_\infty^m)$ is v+g-closed in $V' \times [0, \infty]^{2m}$, and Z and $Z \setminus l(X)$ are bounded and v+g-closed subsets of $V' \times \Gamma_\infty^{2m}$. Let $P = l^{-1}(Z)$ and $Q = l^{-1}(Z \setminus l(X))$. Then applying [35, Lemma 4.2.6] to the surjective definable maps $l^{-1} : Z \to l^{-1}(Z)$ and $l^{-1} : Z \setminus l(X) \to l^{-1}(Z \setminus l(X))$, we see that $\widehat{P} = \widehat{l^{-1}}(\widehat{Z})$ and $\widehat{Q} = \widehat{l^{-1}}(\widehat{Z \setminus l(X)})$; hence they are definably compact by Remark 5.23 and [35, Proposition 4.2.9]. Hence, P and Q are bounded v+g-closed subsets (Remark 5.23), X is v+g-open in P and $Q = P \setminus X$.

The last statement of the lemma is clear. □

Lemma 6.27. *If $X \subseteq V \times \Gamma_\infty^m$ is a* v+g-*locally closed subset, then $\mathcal{X} \subseteq \Gamma_\infty^k$ is a definably locally closed subset.*

Proof. Let V' be a projective variety such that V is an open subset of V'. Then X is still a v+g-locally closed subset of $V' \times \Gamma_\infty^m$. By Lemma 6.26, let P and Q be bounded v+g-closed subsets such that X is v+g-open in P and $Q = P \setminus X$.

Now we may assume that the pro-definable deformation retraction $H : I \times \widehat{X} \to \widehat{X}$ is the restriction of a pro-definable deformation retraction $H : I \times \widehat{V'} \to \widehat{V'}$ preserving V, P, X and Q. Let $\mathfrak{P}$, $\mathfrak{Q}$ (and $\mathfrak{X}$) be the images under H of $\widehat{P}$, $\widehat{Q}$ (and $\widehat{X}$), respectively. Then $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{X}$ are iso-definable subsets of $\widehat{V'}$, so $\mathfrak{Q}$ and $\mathfrak{X}$ are iso-definable subsets of $\mathfrak{P}$. Let $h : \mathfrak{P} \to \mathcal{P}$ be a pro-definable homeomorphism between $\mathfrak{P}$ and a definable subset $\mathcal{P}$ of Γ_∞^k. Then $\mathcal{Q} = h(\mathfrak{Q})$ and $\mathcal{X} = h(\mathfrak{X})$ are definable subsets of $\mathcal{P}$. By [35, Proposition 4.2.9], $\mathcal{P}$ and $\mathcal{Q}$ are definably compact definable subsets of Γ_∞^k; in particular, they are closed. Thus, $\mathcal{X} = \mathcal{P} \setminus \mathcal{Q}$ is an open definable subset of a closed definable subset: that is, it is definably locally closed. □

As above, let $H\colon I \times \widehat{X} \to \widehat{X}$ be a continuous pro-definable deformation retraction with image an iso-definable subset $\mathfrak{X}$ of $\widehat{X}$ for which there is a pro-definable homeomorphism $h\colon \mathfrak{X} \to \mathcal{X}$ with a definable subset $\mathcal{X}$ of Γ_∞^k. Then we have a retraction $r = H \circ i_1\colon \widehat{X} \to \mathfrak{X}$, where $i_1\colon \widehat{X} \to I \times \widehat{X}$ is the natural inclusion $\widehat{X} \to \{e_I\} \times \widehat{X}$, where e_I is the end point of I.

As in [35, Remark 11.1.3 (5)], we have:

Remark 6.28. The pro-definable retraction $\widehat{X} \to \mathfrak{X}$ and the pro-definable homeomorphism $h\colon \mathfrak{X} \to \mathcal{X}$ can be assumed to be definably proper: that is, the pullback of a definably compact set is definably compact.

Indeed, since $h\colon \mathfrak{X} \to \mathcal{X}$ is a pro-definable homeomorphism, it is definably proper. On the other hand, take V' a projective variety such that V is an open subset of V', and assume that the pro-definable deformation retraction $H : I \times \widehat{X} \to \widehat{X}$ is the restriction of a pro-definable deformation retraction

$H : I\times\widehat{V'} \to \widehat{V'}$. If $\mathfrak{B}'$ is the image of $H : I\times\widehat{V'} \to \widehat{V'}$, then the retraction $\widehat{V'} \to \mathfrak{B}'$ is definably proper; hence its restriction $\widehat{X} \to \mathfrak{X}$ is also.

Remark 6.29. Let $X \subseteq V\times\Gamma_\infty^m$ be a definable subset. Below, we denote by c the family of v+g-supports on X whose elements are the closed v+g-subsets of X each contained in some bounded v+g-closed subset of X. We will also denote by c the family of $\widehat{\text{v+g}}$-supports on $\widehat{X}$ whose elements are the definably compact definable subsets of $\widehat{X}$.

Note the following:

- The isomorphism of sites $\widehat{X}_{\widehat{\text{v+g}}} \to X_{\text{v+g}}$ is a morphism in $\mathfrak{T}$, and the inverse image of the family c of v+g-supports of X is the family c of $\widehat{\text{v+g}}$-supports on $\widehat{X}$. This follows from Remark 5.23 and the fact that a definable subset of $\widehat{X}$ is the same as a $\widehat{\text{v+g}}$-subset of $\widehat{X}$. Hence the isomorphism of sites $\widehat{X}_{\widehat{\text{v+g}}} \to X_{\text{v+g}}$ induces an isomorphism in cohomology

$$H_c^*(X;\mathcal{F}) \simeq H_c^*(\widehat{X};\widehat{\mathcal{F}}).$$

- If X is a v+g-locally closed subset, then the family c of $\widehat{\text{v+g}}$-supports on $\widehat{X}$ is a family of $\widehat{\text{v+g}}$-normal supports on $\widehat{X}$. Indeed, let V' be a projective variety having V as an open subset. By Corollary 5.31, $V'\times\Gamma_\infty^m$ is weakly v+g-normal, so P, being a v+g-closed subset of $V'\times\Gamma_\infty^m$ (by Lemma 6.26), is also weakly v+g-normal. Since X is a v+g-open subset of P and every bounded v+g-closed subset of X is also a v+g-closed subset of P, the result follows from Definition 3.14 (2).

It follows from the last two remarks that:

Remark 6.30. If $X \subseteq V\times\Gamma_\infty^m$ is a v+g-locally closed definable subset, then we have induced homomorphisms

$$r^*\colon H_c^*(\mathfrak{X};L_{\mathfrak{X}}) \to H_c^*(\widehat{X};L_{\widehat{X}})$$

and

$$h^*\colon H_c^*(\mathcal{X};L_{\mathcal{X}}) \to H_c^*(\mathfrak{X};L_{\mathfrak{X}}),$$

which are isomorphisms. In particular,

$$H_c^*(\widehat{X};L_{\widehat{X}}) \simeq H_c^*(\mathcal{X};L_{\mathcal{X}}).$$

Indeed, let $j\colon \mathfrak{X}\to\widehat{X}$ be the inclusion. Since $\mathfrak{X} = H(e_I,\widehat{X})$, we have $r\circ j = \mathrm{id}_{\mathfrak{X}}$; and as usual, H is a homotopy between $j\circ r$ and $\mathrm{id}_{\widehat{X}}$. Moreover, since H is a morphism of $\widehat{\text{v+g}}$-sites (Lemma 6.24), we also have that $r\circ j$ and $j\circ r$ are morphisms of $\widehat{\text{v+g}}$-sites. Since $\mathfrak{X} = r(\widehat{X})$, we get that r^* is an isomorphism by homotopy axiom (Theorem 6.8); h^* is an isomorphism since h is a pro-definable homomorphism.

Note that by the same argument, taking the family of v+g-supports given by the v+g-closed subsets, we get, for any definable subset $X \subseteq V\times\Gamma_\infty^m$, an isomorphism

$$H^*(\widehat{X};L_{\widehat{X}}) \simeq H^*(\mathcal{X};L_{\mathcal{X}}).$$

Theorem 6.31. *Let A be a noetherian ring, and let L be a finitely generated A-module. Let V be a variety. If $X \subseteq V\times\Gamma_\infty^m$ is a* v+g*-locally closed definable subset, then $H_c^p(X;L_X) \simeq H_c^p(\widehat{X};L_{\widehat{X}})$ is finitely generated for each p.*

Proof. The variety V is a finite union of open quasi-projective subvarieties $V_1,\ldots,V_n$, and the result follows by induction on n. In the case $n=1$, the variety V is quasi-projective, and since $\mathcal{X}$ is definably locally closed by Lemma 6.27, the result follows from Theorem 6.20 and the isomorphism of Remark 6.30.

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Now suppose that $V = V_1 \cup \cdots \cup V_{n+1}$, and, by the induction hypothesis, the result holds for $U = V_1 \cup \cdots \cup V_n$. Let $Y = X \cap (U \times \Gamma_\infty^m)$ and $Z = X \cap (V_{n+1} \times \Gamma_\infty^m)$. Then Y (respectively, Z) is a v+g-locally closed definable subset of $U \times \Gamma_\infty^m$ (respectively, $V_{n+1} \times \Gamma_\infty^m$) and $H_c^p(Y; L_Y)$ (respectively, is $H_c^p(Z; L_Z)$) finitely generated for each p. Also note that $Y \cap Z$ is a v+g-locally closed definable subset of $V_{n+1} \times \Gamma_\infty^m$ and $H_c^p(Y \cap Z; L_{Y\cap Z})$ finitely generated for each p.

Since Y, Z and $Y \cap Z$ are v+g-locally closed subsets of X, the short exact coefficients sequence

$$0 \to L_{Y\cap Z} \to L_Y \oplus L_Z \to L_X \to 0$$

together with Corollary 3.27 gives the Mayer-Vietoris sequence

$$\ldots \to H_c^l(Y; L_Y) \oplus H_c^l(Z; L_Z) \xrightarrow{\alpha} H_c^l(X; L_X) \xrightarrow{\beta} H_c^{l+1}(Y \cap Z; L_{Y\cap Z}) \to \ldots.$$

It also follows that $\ker\beta = \operatorname{Im}\alpha$ is finitely generated and $\operatorname{Im}\beta$ is finitely generated. Thus we see that $H_c^p(X; L_X)$ is finitely generated for each p. □

Let K and K' be small elementary substructures of $\mathbb{U}$, with K' an elementary extension of K. Given a K-definable set X, let, as before, $H: I \times \widehat{X} \to \widehat{X}$ be a continuous pro-definable deformation retraction with image an iso-definable subset $\mathfrak{X}$ of $\widehat{X}$ for which there is a pro-definable homeomorphism $h: \mathfrak{X} \to \mathcal{X}$ with a K-definable subset $\mathcal{X}$ of Γ_∞^k. Then we have commutative diagrams of morphisms of $\widehat{\text{v+g}}$-sites

$$\begin{array}{ccc} I(K') \times \widehat{X}(K') & \xrightarrow{H^{K'}} & \widehat{X}(K') \\ \downarrow & & \downarrow \\ I(K) \times \widehat{X}(K) & \xrightarrow{H^{K}} & \widehat{X}(K) \end{array}$$

and

$$\begin{array}{ccc} \mathfrak{X}(K') & \xrightarrow{h^{K'}} & \mathcal{X}(K') \\ \downarrow & & \downarrow \\ \mathfrak{X}(K) & \xrightarrow{h^{K}} & \mathcal{X}(K) \end{array}$$

where the vertical arrows are the morphisms of $\widehat{\text{v+g}}$-sites taking $\widehat{U}(K)$ to $\widehat{U}(K')$.

Note that $H^{K'}: I(K') \times \widehat{X}(K') \to \widehat{X}(K')$ is a continuous pro-definable deformation retraction with image an iso-definable subset $\mathfrak{X}(K')$ of $\widehat{X}(K')$ and $h^{K'}: \mathfrak{X}(K') \to \mathcal{X}(K')$ is a pro-definable homeomorphism to a K-definable subset $\mathcal{X}(K')$ of $\Gamma_\infty^k(K')$. Furthermore, if X is a v+g-locally closed definable subset, then $X(K')$ is also. Similar remarks hold for K in place of K'.

Theorem 6.32. *Let V be a variety over K. Let K' be an elementary extension of K. If $X \subseteq V \times \Gamma_\infty^m$ is a* v+g*-locally closed K-definable subset, then for every $\mathcal{F} \in \operatorname{Mod}(A_{X(K)_{\text{v+g}}})$, we have an isomorphism*

$$H_c^*(X(K); \mathcal{F}) \simeq H_c^*(X(K'); \mathcal{F}(K')).$$

Proof. First let us prove that for every v+g-locally closed K-definable subset $X \subseteq V \times \Gamma_\infty^m$, we have an isomorphism

$$H_c^*(X(K); A_{X(K)}) \simeq H_c^*(X(K'); A_{X(K')}).$$

The variety V is a finite union of open quasi-projective subvarieties $V_1, \ldots, V_n$, and the result follows by induction on n. In the case $n = 1$, the variety V is quasi-projective, and the result follows from the observations above, Remark 6.30 and Theorem 6.22 taken in K and in K'.

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The inductive step is obtained from the following commutative diagram with the Mayer-Vietoris sequences as in the proof of Theorem 6.31 taken in K and in K', the inductive hypothesis and the five lemmas:

$$
\begin{array}{ccccccc}
\longrightarrow & H_c^l(Y(K);A_{Y(K)}) \oplus H_c^l(Z(K);A_{Z(K)}) & \longrightarrow & H_c^l(X(K);A_{X(K)}) & \longrightarrow & H_c^{l+1}((Y\cap Z)(K);A_{(Y\cap Z)(K)}) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H_c^l(Y(K');A_{Y(K')}) \oplus H_c^l(Z(K');A_{Z(K')}) & \longrightarrow & H_c^l(X(K');A_{X(K')}) & \longrightarrow & H_c^{l+1}((Y\cap Z)(K');A_{(Y\cap Z)(K')}) & \longrightarrow
\end{array}
$$

By Lemma 6.26, let P be a definably compact v+g-closed subset such that $X \subseteq P$ is a v+g-open subset. Note that P is defined over K. It follows from the above that for every v+g-closed subset Z of P, defined over K, we have

$$H^*(Z(K);A_{Z(K)}) \simeq H^*(Z(K');A_{Z(K')}).$$

With this setup, the rest of the proof is exactly the same as that of Theorem 6.22 using Corollary 3.27, Remark 5.23 instead of Remark 4.11 and Lemma 6.21. □

Remark 6.33. Note that for the cohomologies $H^*(X;L_X)$ and $H^*(X;\mathcal{F})$ without supports, we could also get finiteness and invariance results, respectively, if we knew the corresponding results in $\boldsymbol{\Gamma}_\infty$. Indeed, by the isomorphism $H^*(X;L_X) \simeq H^*(\mathcal{X};L_{\mathcal{X}})$ of Remark 6.30, in the quasi-projective case, we would get the finiteness and invariance results for any definable subset $X \subseteq V \times \Gamma_\infty^m$. For the general case, X would have to be assumed v+g-normal (in order to use Corollary 3.27), and the result would follow by exactly the same arguments as above.

6.4. *Vanishing of cohomology*

Below are a couple of quick remarks about vanishing of cohomology in our setting.

By the isomorphisms of Remark 6.30 and Theorem 6.20, we immediately obtain:

Theorem 6.34. *Let V be a quasi-projective variety. If $X \subseteq V \times \Gamma_\infty^m$ is a* v+g*-locally closed subset, then $H_c^p(X;L_X) = 0$ for all $p > \dim \mathcal{X}$. In particular, $H_c^p(X;L_X) = 0$ for all $p > \dim X$.* □

When there is a good notion of dimension (recall Remark 6.5), we can obtain the following results.

Theorem 6.35. *Let X be a definable subset of a variety V. If X is* v+g*-normal, then $H^p(X;\mathcal{F}) = 0$ for every $p > \dim(\overline{X}^{\mathrm{Zar}})$ and every $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathrm{v+g}}})$, where $\overline{X}^{\mathrm{Zar}}$ denotes the Zariski closure of X.*

Proof. Going to $\widetilde{\widehat{X}}$, we obtain a normal spectral space, and the result follows from [15, Corollary 6] since $\dim_{\mathrm{Krull}}(\widetilde{\widehat{X}})$: that is, the maximal length of a chain of proper specialisations of points in $\widetilde{\widehat{X}}$ is bounded by $\dim(\overline{X}^{\mathrm{Zar}})$ by Corollary 6.4. □

Based on Propositions 3.24, 3.26 and 3.31 together with Theorem 6.35, the proof of the following result is classical. Compare with [25, Theorem 3.12] in the o-minimal case, with [17, Corollary 9.4] in the locally semi-algebraic case or with [14, Chapter II, 16.2 and 16.4] in the topological case.

Theorem 6.36. *Let V be a variety. If X is a* v+g*-locally closed subset of V, then $H_c^p(X;\mathcal{F}) = 0$ for every $p > \dim(\overline{X}^{\mathrm{Zar}})$ and every $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathrm{v+g}}})$.*

Proof. We work in $\widetilde{\widehat{X}}$ and let $n = \dim(\overline{X}^{\mathrm{Zar}})$. Since the functor $\Gamma_c(X;\bullet)$ is left exact and the full additive subcategory of $\mathrm{Mod}(A_{\widetilde{\widehat{X}}})$ of c-soft sheaves is $\Gamma_c(X;\bullet)$-injective (Proposition 3.26), by the general result of homological algebra [39, Exercise I.19], it is enough to show that if $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \cdots \longrightarrow \mathcal{I}^n \longrightarrow 0$ is an exact sequence of sheaves in $\mathrm{Mod}(A_{\widetilde{\widehat{X}}})$ such that $\mathcal{I}^k$ is c-soft for $0 \le k \le n-1$, then $\mathcal{I}^n$ is c-soft.

By Proposition 3.24 (4), it suffices to prove that $\mathcal{I}^n_{|Z}$ is soft for every constructible subset Z of $\widetilde{\widehat{X}}$, which is in the normal and constructible family of supports c. Since c is normal, there is a constructible neighbourhood Y of Z in $\widetilde{\widehat{X}}$ that is in c. If we show that $\mathcal{I}^n_{|Y}$ is soft, then it will follow that $\mathcal{I}^n_{|Z}$ is soft (Proposition 3.24 (2)).

Let U be an open and constructible subset of Y. By hypothesis and Proposition 3.31, each $(\mathcal{I}^k_{|Y})_U$ is acyclic for $0 \leq k \leq n-1$. Let $\mathcal{Z}^k = \ker((\mathcal{I}^k_{|Y})_U \longrightarrow (\mathcal{I}^{k+1}_{|Y})_U)$. Then the long exact cohomology sequences of the short exact sequences $0 \longrightarrow \mathcal{Z}^k \longrightarrow (\mathcal{I}^k_{|Y})_U \longrightarrow \mathcal{Z}^{k+1} \longrightarrow 0$ show that

$$H^q(Y;(\mathcal{I}^n_{|Y})_U) = H^q(Y;\mathcal{Z}^n) = H^{q+1}(Y;\mathcal{Z}^{n-1}) = \cdots = H^{q+n}(Y;\mathcal{Z}^0) = H^{q+n}(Y;(\mathcal{F}_{|Y})_U).$$

Since Y is a normal, constructible open subset of $\widetilde{\widehat{X}}$, we have $H^p(Y;\mathcal{G}) = 0$ for $p > n$ and every sheaf $\mathcal{G}$ on Y by Theorem 6.35 applied to an appropriate v+g-normal open definable subset of X whose tilde of the hat is Y. Thus $H^1(Y;(\mathcal{I}^n_{|Y})_U) = 0$. Since U was an arbitrary open and constructible subset of Y, it follows from Proposition 3.31 that $\mathcal{I}^n_{|Y}$ is soft as required. □

7. Relation with Berkovich spaces

In this section, we relate our results to classical results of Berkovich spaces. Let us first recall the relation – already established in [35] – between the stable completion of algebraic varieties and their analytification.

Let (F, val) be a nonarchimedean valued field of rank 1 (i.e., $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$). We allow F to be trivially valued and do not assume it to be complete. Consider $\mathbf{F} := (F, \mathbb{R})$ as a substructure of a model of ACVF. Let V be an algebraic variety over F and X be an $\mathbf{F}$-definable subset of V. As in [35, Chapter 14], we define $B_{\mathbf{F}}(X)$, the *model-theoretic Berkovich space of* X, to be the space of types over $\mathbf{F}$, concentrated on X, which are almost orthogonal to $\mathbb{R}$. The last condition means that for any $\mathbf{F}$-definable map $h\colon X \longrightarrow \mathbb{R}_\infty$ and for any $p \in B_{\mathbf{F}}(X)$, the push-forward $h_*(p)$ is concentrated on a point of $\mathbb{R}_\infty$ that is denoted by $h(p)$. The topology on $B_{\mathbf{F}}(X)$ is the topology having as a basis finite unions of finite intersections of sets of the form

$$\{p \in X \cap U \mid f(p) \in I\},$$

where U is a Zariski open set of V, $f \in \mathcal{O}^{\mathrm{val}}_V(U)$ and I is an open interval on $\mathbb{R}_\infty$. In addition, the construction is functorial: that is, given a variety V' over F and $\mathbf{F}$-definable subset X' of V' and an $\mathbf{F}$-definable map $f\colon X \longrightarrow X'$, there is an induced map $B_{\mathbf{F}}(f)\colon B_{\mathbf{F}}(X) \longrightarrow B_{\mathbf{F}}(X')$.

Suppose now that F is complete. As a set, Berkovich's analytification V^{an} of V can be described as pairs (x, u_x) with x a point (in the schematic sense) of V and $u_x\colon F(x) \longrightarrow \mathbb{R}_\infty$ a valuation extending val on the residue field $F(x)$ of the stalk at x. The topology on V^{an} is the coarsest topology such that for every $f \in \mathcal{O}_V(U)$, the maps $x \mapsto u_x \circ f_x$, where $f_x \in \mathcal{O}_{V,x} \simeq F(x)$ are all continuous. By results in [8], the topological space V^{an} is Hausdorff (recall V is assumed to be separated), locally compact and locally path connected. The induced topology on the subset $V(F)$ coincides with the valuation topology, and if F is algebraically closed, then this subset is dense.

As explained in [35, Section 14.1], we have:

Fact 7.1. If F is complete, then the model-theoretic Berkovich space $B_{\mathbf{F}}(V)$ is canonically homeomorphic to V^{an}. Moreover, the image of $B_{\mathbf{F}}(X)$ in V^{an} is a semi-algebraic subset in the sense of [20], and every semi-algebraic subset of V^{an} is of this form. □

Let us now recall the relation between $B_{\mathbf{F}}(V)$ and $\widehat{V}$. Let F^{max} be the unique, up to isomorphism over $\mathbf{F}$, maximally complete algebraically closed field containing F, with value group $\mathbb{R}$ and residue field equal to the algebraic closure of the residue field of F. Recall that maximally complete means every family of balls with the finite intersection property has a non-empty intersection. By [35, Lemma 14.1.1 and Proposition 14.1.2], we have:

Fact 7.2. There is a continuous surjective closed map

$$\pi\colon \widehat{X}(F^{max}) \to B_{\mathbf{F}}(X).$$

If $F = F^{max}$, then π is a homeomorphism. □

For the rest of this section, we assume that $\mathbb{U}$ contains $\mathbf{F}$ and consider only small elementary substructures of $\mathbb{U}$ containing $\mathbf{F}$.

Remark 7.3. Let X be an $\mathbf{F}$-definable subset, and let K and K' be elementary substructures of $\mathbb{U}$ containing $\mathbf{F}$. Observe that one can express (over $\mathbf{F}$) the fact that $X(K)$ is covered by two disjoint v+g-open subsets. Therefore, if two such disjoint v+g-open subsets exist in K, they exist in any elementary extension of K containing $\mathbf{F}$. By Fact 5.11, this shows that $\widehat{X}(K)$ is definably connected if and only if $\widehat{V}(K')$ is definably connected in K'.

Proposition 7.4. *Let X be an $\mathbf{F}$-definable subset of an algebraic variety V over F. Then the following are equivalent:*

(1) *$\widehat{X}$ is definably connected.*
(2) *$\widehat{X}(F^{max})$ is connected.*

In addition, if $\widehat{X}(F^{max})$ is connected, so is $B_{\mathbf{F}}(X)$. In particular, when F is complete, if $\widehat{V}(F^{max})$ is connected, then V^{an} is connected.

Proof. By Remark 7.3, (1) above is equivalent to $\widehat{X}(F^{max})$ being definably connected. Suppose first that V is quasi-projective. By the main theorem of Hrushovski and Loeser [35, Theorem 11.1.1], there is a pro-definable deformation retraction of $\widehat{X}(F^{max})$ to a definable subset $\mathcal{X}$ of $\mathbb{R}^n_\infty$. Hence, $\widehat{X}(F^{max})$ is definably connected (respectively, connected) if and only if $\mathcal{X}$ is definably connected (respectively, connected). But $\mathcal{X}$ is definably connected if and only if it is connected (the proof in o-minimal expansions of $(\mathbb{R}, <)$ given in [53, Chapter III, (2.18) and (2.19) Exercise 7] uses only cell decomposition and works in $\mathbb{R}_\infty$).

If V is not quasi-projective, consider an open immersion $V \to W$, where W is a complete variety and V is Zariski dense. By Chow's lemma, there is an epimorphism $f\colon W' \to W$, where W' is a projective variety. Consider the quasi-projective variety $V' = f^{-1}(V)$, and let $X' = f^{-1}(X) \subseteq V'$. By Lemma 5.21, $\widehat{X'}(F^{max})$ has finitely many definably connected components, say $C'_1, \ldots, C'_k$, which by the above are also the finitely many connected components of the topological space $\widehat{X'}(F^{max})$. The image C_i of each connected component C'_i of $\widehat{X'}(F^{max})$ under the restriction of $f_|$ is a connected subset of $\widehat{X}(F^{max})$. Note that these connected subsets of $\widehat{X}(F^{max})$ are also definably connected subsets.

Now we follow the idea in the proof of [53, Chapter III, (2.18)], as above in $\mathbb{R}_\infty$ with cells for the C_is, to show that some union of the C_is is both a definably connected component and a connected component of $\widehat{X}(F^{max})$. From there, it follows that $\widehat{X}(F^{max})$ is definably connected if and only if it is connected. Let us construct inductively a sequence $i_1, \ldots, i_j, \ldots$ such that for each j, $\bigcup_{l\leq j} C_{i_l}$ is both definably connected and connected. Set $i_1 = 1$, and if $i_1, \ldots, i_q$ are given, let $i_{q+1} = \min\{i : i \notin \{i_1, \ldots, i_q\}$ and $C_i \cap (\bigcup_{l\leq q} C_{i_l}) \neq \emptyset\}$. Let p be the maximal length of such a sequence. We claim that $C := \bigcup_{j=1}^{p} C_{i_j}$ is both a definably connected component and a connected component of $\widehat{X}(F^{max})$.

To see the claim, it is enough to show that if $Y \subseteq \widehat{X}(F^{max})$ is a definably connected (respectively, connected) subset such that $Y \cap C \neq \emptyset$, then $Y \subseteq C$. Indeed, if this holds, taking Y a definably connected (respectively, connected) open neighbourhood of a point of C (respectively, a point of the complement of C), we see that C is open (respectively, closed). Now let $I = \{i : C_i \cap Y \neq \emptyset\}$ and $C_Y = \bigcup_{i\in I} C_i$. Since the C_is cover $\widehat{X}(F^{max})$, we have $Y \subseteq C_Y$. So C_Y is definably connected (respectively, connected), being of the form $Y \cup \bigcup_{i\in I} C_i$. Since $\emptyset \neq C \cap Y \subseteq C \cap C_Y$, it follows that $C \cup C_Y$ is definably connected (respectively, connected). By maximality of C, we have $C \cup C_Y = C$, so $Y \subseteq C$ as required.

The last statement of the Proposition follows from Fact 7.2. □

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Remark 7.5. In general, the connectedness of $B_{\mathbf{F}}(X)$ does not imply that $\widehat{X}(F^{max})$ is connected. Take F to be $\mathbb{R}$ with the trivial valuation, and let X be defined by x^2+x+1. The space $B_{\mathbf{F}}(X)$ is connected, consisting of a single point that corresponds to the Galois orbit of the two 3rd roots of unity. In contrast, $\widehat{X}(F^{max})$ consists of two isolated points and is therefore not connected.

As an application, we recover a result of Ducros [20] about the number of connected components of semi-algebraic subsets of V^{an}. Let ρ denote the homeomorphism from $B_{\mathbf{F}}(V)$ to V^{an}.

Theorem 7.6 (Ducros [20]). *Suppose F is complete. Let V be a variety over F, and let Y be a semi-algebraic subset of V^{an}. Then Y has finitely many connected components.*

Proof. Since Y is a semi-algebraic subset of V^{an}, there is an $\mathbf{F}$-definable subset X of V such that $Y=\rho(B_{\mathbf{F}}(X))$. By Lemma 5.21, Proposition 7.4 and Fact 7.2, $\widehat{X}(F^{max})$ has finitely many connected components. Now the image under π (as defined in Fact 7.2) of a connected component of $\widehat{X}(F^{max})$ lies in a connected component of $B_{\mathbf{F}}(X)$. Since π is surjective, this shows the result. □

Also note that if F is algebraically closed, then each connected component of Y is also a semi-algebraically connected component. Indeed, by Lemmas 5.19 and 5.21, $\widehat{X}(F^{max})$ has finitely many definably connected components, each of which is of the form $\widehat{U_i}(F^{max})$ for some v+g-clopen subsets U_i $(i=1,\dots,k)$ of X. By Proposition 7.4, each $\widehat{U_i}(F^{max})$ is also a connected component of $\widehat{X}(F^{max})$. Since $\pi:\widehat{X}(F^{max})\to B_F(X)$ (as defined in Fact 7.2) is continuous, surjective and closed and $\pi(\widehat{U_i}(F^{max}))=B_F(U_i)$, each $B_F(U_i)$ is a closed connected subset of $B_F(X)$. Let $I_1,\dots,I_q$ be a finite partition of $\{1,\dots,k\}$ such that for each $j\le q$, $\bigcup_{l\in I_j}B_F(U_l)$ is connected and I_j is maximal with this property. Then as in the last part of the proof of Proposition 7.4, each $\bigcup_{l\in I_j}B_F(U_l)=B_F(\bigcup_{l\in I_j}U_l)$ is a connected component of $\widehat{X}(F^{max})$ with $V_j=\bigcup_{l\in I_j}U_l$ a v+g-clopen subset of X. Since F is a model of ACVF, by Remark 7.3, each V_j is $\mathbf{F}$-definable.

Fact 7.7 ([35, Proposition 14.1.2]). Let X be an $\mathbf{F}$-definable subset of an algebraic variety V over F. Then the following are equivalent:

- $\widehat{X}$ is definably compact.
- $\widehat{X}(F^{max})$ is compact.
- $B_{\mathbf{F}}(X)$ is compact.

In particular, when F is complete, $\widehat{V}$ is definably compact if and only if V^{an} is compact. □

The proof of the previous fact in [35] does not seem to include the proof that $B_{\mathbf{F}}(X)$ is Hausdorff. For the reader's convenience, we include an argument.

Lemma 7.8. *Let X be an $\mathbf{F}$-definable subset of an algebraic variety V over F. Then $B_{\mathbf{F}}(X)$ is a Hausdorff topological space.*

Proof. Note that it suffices to show that when $X=V$ is a complete variety over F, $B_{\mathbf{F}}(V)$ is Hausdorff. Indeed, by Nagata's theorem, there is an open immersion $V\to V'$, where V' is a complete variety. Since $B_{\mathbf{F}}(X)$ is a subspace of $B_{\mathbf{F}}(V')$, it is also Hausdorff.

We first show that points in $B_{\mathbf{F}}(V)$ are closed. Note that for each affine open subset $W\subseteq V$, $B_{\mathbf{F}}(W)$ is an open subspace of $B_{\mathbf{F}}(V)$. Hence it suffices to show that points in $B_{\mathbf{F}}(W)$ are closed for affine W. Actually, this is true since $B_{\mathbf{F}}(W)$ is even Hausdorff. Let p,q be two distinct points in $B_{\mathbf{F}}(W)$: by quantifier elimination in ACVF, there is a regular function f on W such that $\text{val}\circ f_*(p)=r_1\neq \text{val}\circ f_*(q)=r_2$ for some $r_1,r_2\in\mathbb{R}_\infty$. Then take two disjoint open subsets $r_1\in U_1$ and $r_2\in U_2$. Their inverse image under $\text{val}\circ f$ will be two disjoint opens in $B_{\mathbf{F}}(W)$ separating p,q.

Suppose now that V is a complete variety. By Remark 5.23, $\widehat{V}$ is definably compact (and Hausdorff by [35, Proposition 3.7.8]) so, by Fact 7.7, $B_F(V)$ is quasi-compact. Consider the restriction map $\pi:\widehat{V}(F^{max})\to B_{\mathbf{F}}(V)$; note that the map is closed. On the other hand, given $p_1,p_2\in B_{\mathbf{F}}(V)$, the fibres $\pi^{-1}(p_1),\pi^{-1}(p_2)$ are two disjoint compact subsets of $\widehat{V}(F^{max})$and hence can be separated by open subsets U_1 and U_2; then $p_1\in B_{\mathbf{F}}(V)\backslash\pi(\widehat{V}(F^{max})\backslash U_1)$ and $p_2\in B_{\mathbf{F}}(V)\backslash\pi(\widehat{V}(F^{max})\backslash U_2)$ are two disjoint open subsets of $B_{\mathbf{F}}(V)$ by the fact that π is closed. □

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Recall that in general, $\widehat{X}$ may fail to be a locally compact space (Remark 5.12). However, since $F^{max} = (F^{max})^{max}$, by Fact 7.2, $\widehat{X}(F^{max})$ is homeomorphic to $B_{F^{max}}(X)$; and by the next result, $\widehat{X}(F^{max})$ is a locally compact topological space when X is an $\mathbf{F}$-definable v+g-locally closed subset of an algebraic variety V over F.

Lemma 7.9. *If X is an $\mathbf{F}$-definable* v+g*-locally closed subset of an algebraic variety V over F, then $B_{\mathbf{F}}(X)$ is a locally closed subset of $B_{\mathbf{F}}(V)$. In particular, $B_{\mathbf{F}}(X)$ is a locally compact topological space.*

Proof. Let U be an $\mathbf{F}$-definable v+g-open subset of V, and let Y be an $\mathbf{F}$-definable v+g-closed subset of V such that $X = Y \cap U$. Then $\widehat{Y}(F^{max})$ is a closed subset of $\widehat{V}(F^{max})$. By Fact 7.2, $\pi(\widehat{Y}(F^{max})) = B_{\mathbf{F}}(Y)$, and $B_{\mathbf{F}}(Y)$ is a closed subset of $B_{\mathbf{F}}(V)$ since π is a closed map.

On the other hand, since $V \setminus U$ is an $\mathbf{F}$-definable v+g-closed subset of V, by the above and the fact that $B_{\mathbf{F}}(V) \setminus B_{\mathbf{F}}(U) = B_{\mathbf{F}}(V \setminus U)$, $B_{\mathbf{F}}(U)$ is an open subset of $B_{\mathbf{F}}(V)$. Therefore, $B_{\mathbf{F}}(X) = B_{\mathbf{F}}(U) \cap B_{\mathbf{F}}(Y)$ is a locally closed subset of $B_{\mathbf{F}}(V)$.

By Nagata's theorem, there is an open immersion $V \to V'$, where V' is a complete variety (over F). Since V' is complete, $\widehat{V'}$ is definably compact (Remark 5.23), so, by Fact 7.7, $B_{\mathbf{F}}(V')$ is compact. Thus the result follows since (the homeomorphic image of) $B_{\mathbf{F}}(X)$ is also locally closed in $B_{\mathbf{F}}(V')$. □

Remark 7.10. Let V be an algebraic variety over F. Then $B_{\mathbf{F}}(V)$ is paracompact. Since V is a scheme of finite type, we may assume V is affine. Furthermore, since $B_{\mathbf{F}}(V)$ is locally compact by Lemma 7.9, it suffices to say it is σ-compact (see [14, Page 21]). Now $B_{\mathbf{F}}(V)$ is covered by the sets $B_{\mathbf{F}}(X) \subseteq U$, where X is a closed ball with radius in $\mathbb{Q}$. Each such set is compact by Fact 7.7.

Fact 7.11. Let V be a quasi-projective variety over F and X an $\mathbf{F}$-definable subset. Then there is a strong deformation retraction $\mathbf{I} \times B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)$ with image a subset $\mathfrak{X}$ homeomorphic to a semi-algebraic subset $\mathcal{X}$ of some $\mathbb{R}^k$. This deformation retraction is induced by a strong deformation retraction $H : I \times \widehat{X} \to \widehat{X}$ (of [35, Theorem 11.1.1]) in the sense that there is a commutative diagram

$$\begin{array}{ccc} I(F^{max}) \times \widehat{X}(F^{max}) & \longrightarrow & \widehat{X}(F^{max}) \\ \downarrow{\scriptstyle i\times\pi} & & \downarrow{\scriptstyle \pi} \\ \mathbf{I} \times B_{\mathbf{F}}(X) & \longrightarrow & B_{\mathbf{F}}(X) \end{array}$$

of continuous maps, where $i : I(F^{max}) = \widehat{I}(F^{max}) \to \mathbf{I} = I(\mathbf{F}) = I(\mathbb{R}_\infty)$ is the canonical identification. See [35, Corollary 14.1.6, Proposition 14.1.3 and Theorem 14.2.1].

If $F \le F''$ is a field extension, then we have a commutative diagram of strong deformation retractions

$$\begin{array}{ccc} \mathbf{I} \times B_{\mathbf{F}''}(X) & \longrightarrow & B_{\mathbf{F}''}(X) \\ \downarrow & & \downarrow \\ \mathbf{I} \times B_{\mathbf{F}}(X) & \longrightarrow & B_{\mathbf{F}}(X) \end{array}$$

where the vertical arrows are the canonical restrictions. Furthermore, there is a finite Galois extension $F \le F'$ such that:

(1) If $F' \le F''$, then the image $\mathfrak{X}''$ of $\mathbf{I} \times B_{\mathbf{F}''}(X) \to B_{\mathbf{F}''}(X)$ is homeomorphic to the image $\mathfrak{X}'$ of $\mathbf{I} \times B_{\mathbf{F}'}(X) \to B_{\mathbf{F}'}(X)$.
(2) The image $\mathfrak{X}$ of $\mathbf{I} \times B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)$ is homeomorphic to $\mathfrak{X}'/\mathrm{Gal}(F'/F)$.

See [35, Theorem 14.2.3 and the discussion at the beginning of Section 14.2]. □

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An immediate consequence of this fact is the following results about cohomology:

Theorem 7.12. *If X is an $\mathbf{F}$-definable subset of a quasi-projective variety V over F, then:*

(1) $H^p(B_{\mathbf{F}}(X); L_{B_{\mathbf{F}}(X)})$ *is finitely generated for every* $p \geq 0$.
(2) *There is a finite Galois extension* $F \leq F'$ *such that for every* $F' \leq F''$*, we have isomorphisms*

$$H^*(B_{\mathbf{F}'}(X); L_{B_{\mathbf{F}'}(X)}) \simeq H^*(B_{\mathbf{F}''}(X); L_{B_{\mathbf{F}''}(X)}).$$

Proof. The result follows from Fact 7.11, the homotopy axiom in topology ([14, Chapter II, 11.8]) and the fact that a semi-algebraic subset of $\mathbb{R}^k$ is homeomorphic to a finite simplicial complex. □

Note that if $B_{\mathbf{F}}(X)$ is paracompact (see Remark 7.10), then these results can be extended to an arbitrary algebraic variety V over F using the topological analogue of Corollary 3.27 ([14, Chapter II, 10.2]) and the corresponding Mayer-Vietoris sequence [14, Chapter II, (27)] as in the proof of Theorem 6.31 and Theorem 6.32, respectively.

Theorem 7.13. *If X is an $\mathbf{F}$-definable* v+g*-locally closed subset of an algebraic variety V over F, then:*

(1) $H^p_c(B_{\mathbf{F}}(X); L_{B_{\mathbf{F}}(X)})$ *is finitely generated for every* $p \geq 0$.
(2) *There is a finite Galois extension* $F \leq F'$ *such that for every* $F' \leq F''$*, we have isomorphisms*

$$H^*_c(B_{\mathbf{F}'}(X); L_{B_{\mathbf{F}'}(X)}) \simeq H^*_c(B_{\mathbf{F}''}(X); L_{B_{\mathbf{F}''}(X)}).$$

Proof. Suppose first that V is a quasi-projective variety over F. Note that since X is an $\mathbf{F}$-definable v+g-locally closed subset, $\mathcal{X}$ is locally closed. The proof of this is similar to that of Lemma 6.27 using $B_{\mathbf{F}}(X)$ instead of $\widehat{X}$ and Fact 7.7 instead of [35, Prposition 4.2.9]. In this case, the result follows from the homotopy axiom in topology ([14, Chapter II, 11.8]) and the fact that a semi-algebraic subset of $\mathbb{R}^k$ is homeomorphic to a finite simplicial complex.

In the general case, since the family of compact supports is paracompactifying, the results are obtained using the topological analogue of Corollary 3.27 ([14, Chapter II, 10.2]) and the corresponding Mayer-Vietoris sequence [14, Chapter II, (27)] as in the proof of Theorem 6.31 and Theorem 6.32, respectively. □

Let K be a model of ACVF extending $\mathbf{F}$. Below, when we write $\widehat{X}(K)$, we mean $\widehat{X}(K)$ equipped with the $\widetilde{\mathrm{v+g}}$-site; and when we write $\widehat{X}(K)_{\mathrm{top}}$, we mean the underlying topological space. Observe that since we have a homeomorphism between $\widetilde{X(K)}$ and $\widetilde{\widehat{X}(K)}$ induced by the isomorphism of sites (Remark 3.8), the natural inclusion $\widehat{X}(K) \to \widetilde{X(K)}$ induces a natural inclusion $i : \widehat{X}(K) \to \widetilde{\widehat{X}(K)}$ and $\widehat{X}(K)$ with the induced topology is $\widehat{X}(K)_{\mathrm{top}}$.

Given $\mathcal{F} \in \mathrm{Mod}(A_{\widehat{X}(K)_{\widetilde{\mathrm{v+g}}}})$, we let $\mathcal{F}_{\mathrm{top}} \in \mathrm{Mod}(A_{\widehat{X}(K)_{\mathrm{top}}})$ be the sheaf induced by the natural morphism of sites $\widehat{X}(K)_{\mathrm{top}} \to \widehat{X}(K)_{\widetilde{\mathrm{v+g}}}$. Note that this is the same as $\widetilde{\mathcal{F}}_{|\widehat{X}(K)_{\mathrm{top}}}$.

As observed above, when X is an $\mathbf{F}$-definable v+g-locally closed subset of an algebraic variety V over F, then $\widehat{X}(F^{max})_{\mathrm{top}}$ is a locally compact topological space. Below, we use c to denote either the family of compact supports in $\widehat{X}(K)_{\mathrm{top}}$, as usual, or the family of definably compact supports in $\widehat{X}(K)$ as before.

Remark 7.14. If $D \in \widetilde{c}$ is constructible, then the following hold: (1) $D \cap \widehat{X}(F^{max})_{\mathrm{top}}$ is a quasi-compact subset of $\widetilde{\widehat{X}(F^{max})}$; (2) $D \cap \widehat{X}(F^{max})_{\mathrm{top}}$ has a fundamental system of normal and constructible locally closed neighbourhoods in $\widetilde{\widehat{X}(F^{max})}$; and (3) $D \cap \widehat{X}(F^{max})_{\mathrm{top}}$ is a compact subset of $\widehat{X}(F^{max})_{\mathrm{top}}$.

Indeed, there is $C \in c$ (in $\widehat{X}(F^{max})$) such that $D = \widetilde{C}$, and hence $D \cap \widehat{X}(F^{max})_{\mathrm{top}} = \widetilde{C} \cap X = C$. Since C is a definably compact subset of $\widehat{X}(F^{max})$, it is a compact subset of $\widehat{X}(F^{max})_{\mathrm{top}}$ (Fact 7.7). Hence C is a quasi-compact subset of $\widetilde{\widehat{X}(F^{max})}$. Finally, since c (in $\widehat{X}(F^{max})$) is a definably normal family of supports on $\widehat{X}(F^{max})$, it follows that C has a fundamental system of normal and constructible locally closed neighbourhoods in $\widehat{X}(F^{max})$.

The above verifies the assumptions of Corollary 3.28 for the space $\widehat{X}(F^{max})$ and the subspace $\widehat{X}(F^{max})_{\mathrm{top}}$. Therefore, for every $\mathcal{F} \in \mathrm{Mod}(A_{\widehat{X}(F^{max})_{\widehat{\mathrm{v+g}}}})$, we have an isomorphism

$$H^*_c(\widehat{X}(F^{max}); \mathcal{F}) \simeq H^*_c(\widehat{X}(F^{max})_{\mathrm{top}}; \mathcal{F}_{\mathrm{top}}).$$

Also note that if X is v+g-normal, then considering the family of v+g-supports given by the v+g-closed subsets, we get a family of v+g-normal supports on X, and, as above, we verify the assumptions of Corollary 3.28 for the space $\widehat{X}(F^{max})$ and the subspace $\widehat{X}(F^{max})_{\mathrm{top}}$. Hence for every $\mathcal{F} \in \mathrm{Mod}(A_{\widehat{X}(F^{max})_{\widehat{\mathrm{v+g}}}})$, we have an isomorphism

$$H^*(\widehat{X}(F^{max}); \mathcal{F}) \simeq H^*(\widehat{X}(F^{max})_{\mathrm{top}}; \mathcal{F}_{\mathrm{top}}).$$

In the o-minimal context, $\mathbb{R}$ plays the analogous role of F^{max} in Remark 7.14, and in a similar way, we have:

Remark 7.15. Let $\mathbf{\Gamma} = (\mathbb{R}, <, +, \ldots)$ be an o-minimal expansion of $(\mathbb{R}, <, +)$, and let $X \subseteq \mathbb{R}^m_\infty$ be a definably locally closed subset. Let X_{top} be X equipped with the topology generated by the open definable subsets. Then X_{top} is a locally closed subset, in particular a locally compact topological space. Also note that if we consider the natural inclusion $i : X \to \widetilde{X}$, where $\widetilde{X} = \widetilde{X}_{\mathrm{def}}$ is the o-minimal spectrum of X, then X with the induced topology is X_{top}.

Given $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, let $\mathcal{F}_{\mathrm{top}} \in \mathrm{Mod}(A_{X_{\mathrm{top}}})$ denote the sheaf induced by the natural morphism of sites $X_{\mathrm{top}} \to X_{\mathrm{def}}$. Note that this is the same as $\widetilde{\mathcal{F}}_{|X_{\mathrm{top}}}$. Also let c denote either the family of compact supports on X_{top}, as usual, or the family of definably compact supports on X as before. Using Remark 4.11 instead of Fact 7.7, we verify the assumptions of Corollary 3.28 for $\widetilde{X}$ and the subspace X_{top} exactly in the same way as in Remark 7.14. Therefore for every $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, we have an isomorphism

$$H^*_c(X; \mathcal{F}) \simeq H^*_c(X_{\mathrm{top}}; \mathcal{F}_{\mathrm{top}}).$$

Similary, as above, if X is definably normal, then for every $\mathcal{F} \in \mathrm{Mod}(A_{X_{\mathrm{def}}})$, we have an isomorphism

$$H^*(X; \mathcal{F}) \simeq H^*(X_{\mathrm{top}}; \mathcal{F}_{\mathrm{top}}).$$

The topological cohomology of the model-theoretic Berkovich spaces and the v+g-cohomology of their corresponding stable completions are related by the following results:

Theorem 7.16. *Let X be an* $\mathbf{F}$*-definable* v+g*-locally closed subset of an algebraic variety V over F, and let $\mathcal{F} \in \mathrm{Mod}(A_{\widehat{X}(F^{max})_{\widehat{\mathrm{v+g}}}})$. Then we have a spectral sequence*

$$H^p_c(B_{\mathbf{F}}(X); R^q\pi_*(\mathcal{F}_{\mathrm{top}})) \Rightarrow H^{p+q}_c(\widehat{X}(F^{max}); \mathcal{F})$$

which, when $F = F^{max}$, induces isomorphisms

$$H^p_c(B_{\mathbf{F}}(X); \pi_*(\mathcal{F}_{\mathrm{top}})) \simeq H^p_c(\widehat{X}(F^{max}); \mathcal{F}).$$

Proof. Considering the continuous map $\pi\colon \widehat{X}(F^{max})_{\mathrm{top}} \to B_{\mathbf{F}}(X)$ of locally compact topological spaces (Fact 7.2 and Lemma 7.9), we have the corresponding Leray spectral sequence ([14, Chapter IV, 6.1])

$$H^p_c(B_{\mathbf{F}}(X); R^q\pi_*(\mathcal{F}_{\mathrm{top}})) \Rightarrow H^{p+q}_c(\widehat{X}(F^{max})_{\mathrm{top}}; \mathcal{F}_{\mathrm{top}})$$

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and, by Remark 7.14, we have an isomorphism

$$H^*_c(\widehat{X}(F^{max})_{\text{top}}; \mathcal{F}_{\text{top}}) \simeq H^*_c(\widehat{X}(F^{max}); \mathcal{F}).$$

On the other hand, when $F = F^{max}$, we get isomorphisms

$$H^p_c(B_{\mathbf{F}}(X); \pi_*(\mathcal{F}_{\text{top}})) \simeq H^p_c(\widehat{X}(F^{max})_{\text{top}}; \mathcal{F}_{\text{top}})$$

since π is a homeomorphism (Fact 7.2). □

Combining Theorems 7.16 and 7.13, we have:

Corollary 7.17. *Let X be an $\mathbf{F}$-definable* v+g*-locally closed subset of an algebraic variety V over F. Then there is a finite Galois extension $F \leq F'$ such that we have an isomorphism*

$$H^*_c(B_{\mathbf{F}'}(X); L_{B_{\mathbf{F}'}(X)}) \simeq H^*_c(\widehat{X}(F^{max}); L_{\widehat{X}(F^{max})}).$$

In particular, when F is complete, we have an isomorphism

$$H^*_c(V^{\text{an}}\widehat{\otimes}_F F'; \mathbb{Z}) \simeq H^*_c(\widehat{V}(F^{max}); \mathbb{Z}).$$

Proof. By Theorem 7.13, we have

$$H^*_c(B_{\mathbf{F}'}(X); L_{B_{\mathbf{F}'}(X)}) \simeq H^*_c(B_{F'^{max}}(X); L_{B_{F'^{max}}(X)}).$$

Since $F'^{max} = F^{max}$ and $F^{max} = (F^{max})^{max}$, by Theorem 7.18, we have

$$H^*_c(B_{F'^{max}}(X); L_{B_{F'^{max}}(X)}) \simeq H^*_c(\widehat{X}(F^{max}); L_{\widehat{X}(F^{max})}).$$ □

Similarly, using the second part of Remark 7.14 and the Leray spectral sequence ([14, Chapter IV, 6.1]), we have:

Theorem 7.18. *Let X be an $\mathbf{F}$-definable* v+g*-normal subset of an algebraic variety V over F, and let $\mathcal{F} \in \text{Mod}(A_{\widehat{X}(F^{max})_{\widehat{\text{v+g}}}})$. Then we have a spectral sequence*

$$H^p(B_{\mathbf{F}}(X); R^q\pi_*(\mathcal{F}_{\text{top}})) \Rightarrow H^{p+q}(\widehat{X}(F^{max}); \mathcal{F})$$

which, when $F = F^{max}$, induces isomorphisms

$$H^p(B_{\mathbf{F}}(X); \pi_*(\mathcal{F}_{\text{top}})) \simeq H^p(\widehat{X}(F^{max}); \mathcal{F}).$$ □

Combining this with Theorem 7.12, we obtain:

Corollary 7.19. *Let X be an $\mathbf{F}$-definable* v+g*-normal subset of a quasi-projective variety V over F. Then there is a finite Galois extension $F \leq F'$ such that we have an isomorphism*

$$H^*(B_{\mathbf{F}'}(X); L_{B_{\mathbf{F}'}(X)}) \simeq H^*(\widehat{X}(F^{max}); L_{\widehat{X}(F^{max})}).$$

In particular, when F is complete, we have an isomorphism

$$H^*(V^{\text{an}}\widehat{\otimes}_F F'; \mathbb{Z}) \simeq H^*(\widehat{V}(F^{max}); \mathbb{Z}).$$ □

We conclude the paper with a couple of remarks.

Let X be an $\mathbf{F}$-definable subset of an algebraic variety V over F. If $F \leq F'' \leq F^{max}$ is a field extension, then the natural restriction $\pi_{F'',F} : B_{\mathbf{F}''}(X) \longrightarrow B_{\mathbf{F}}(X)$ is a proper map. Indeed, let V' be a

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complete variety over F such that V is an open subset. Then $\pi_{F'',F} : B_{\mathbf{F}''}(V') \longrightarrow B_{\mathbf{F}}(V')$ is proper (by Fact 7.7), and we have a commutative diagram

$$\begin{array}{ccc} \widehat{V'}(F^{max}) & & \\ \downarrow{\scriptstyle \pi} & \searrow{\scriptstyle \pi} & \\ B_{\mathbf{F}''}(V') & \xrightarrow{\pi_{F'',F}} & B_{\mathbf{F}}(V') \end{array}$$

of surjective restrictions such that $\pi^{-1}(B_{\mathbf{F}}(X)) = \pi^{-1}(B_{\mathbf{F}''}(X)) = \widehat{X}(F^{max})$ ([35, Proposition 14.1.2]). If $C \subseteq B_{\mathbf{F}}(X)$ is compact, then it is a compact subset of $B_{\mathbf{F}}(V')$, so $\pi^{-1}_{F'',F}(C)$ is a compact subset of $B_{\mathbf{F}''}(V')$ and $\pi^{-1}(\pi_{F'',F}(C))$ is a compact subset of $\widehat{X}(F^{max})$. Therefore, $\pi(\pi^{-1}(\pi^{-1}_{F'',F}(C))) = \pi^{-1}_{F'',F}(C)$ is a compact subset of $B_{\mathbf{F}''}(X)$ since $\pi : \widehat{X}(F^{max}) \longrightarrow B_{\mathbf{F}''}(X)$ is surjective. It follows that:

Remark 7.20. If X is an $\mathbf{F}$-definable v+g-locally closed subset of an algebraic variety V over F and $\mathcal{F} \in \mathrm{Mod}(A_{B_{\mathbf{F}}(X)})$, then we have an isomorphism

$$\varinjlim_{F''/F} H^*_c(B_{\mathbf{F}''}(X); \mathcal{F}_{F''}) \simeq H^*_c(B_{\mathbf{F}^{\mathrm{alg}}}(X); \mathcal{F}_{F^{\mathrm{alg}}}),$$

where the limit is taken over all finite Galois extensions $F \leq F''$ contained in F^{alg}, $\mathcal{F}_{F^{\mathrm{alg}}} = \pi^{-1}_{F^{\mathrm{alg}},F}\mathcal{F}$ and $\mathcal{F}_{F''} = \pi^{-1}_{F'',F}\mathcal{F}$.

Indeed, since $B_{\mathbf{F}^{\mathrm{alg}}}(X)$ and $\varprojlim_{F''/F} B_{\mathbf{F}''}(X)$ are homeomorphic, the result follows from [14, Chapter II, 14.5].

Now let $G = \mathrm{Aut}(F^{\mathrm{alg}}/F^h)$ be the absolute Galois group of the Henselisation F^h of F. Recall that this is the group of valued field automorphisms of F^{alg} over F. As observed in [35, page 205], G acts continuously on $B_{\mathbf{F}^{\mathrm{alg}}}(V)$ and $B_{\mathbf{F}}(V) = B_{\mathbf{F}^{\mathrm{alg}}}(V)/G$. Therefore, since this action leaves $B_{\mathbf{F}^{\mathrm{alg}}}(X)$ invariant, G acts continuously on $B_{\mathbf{F}^{\mathrm{alg}}}(X)$ and $B_{\mathbf{F}}(X) = B_{\mathbf{F}^{\mathrm{alg}}}(X)/G$.

Since $G = \varprojlim_{F''/F^h} \mathrm{Gal}(F''/F^h)$, where the limit is taken over all finite Galois extensions $F^h \leq F''$ contained in F^{alg}, then when $F = F^h$ (i.e., it is Henselian), the isomorphism of Remark 7.20 is an isomorphism of G-modules. However, unlike the étale cohomology, we do not have the Cartan-Leray spectral sequence relating the cohomologies of $B_{\mathbf{F}}(X)$ and $B_{\mathbf{F}^{\mathrm{alg}}}(X)$ through the absolute Galois group $G = \mathrm{Gal}(F^{\mathrm{alg}}/F^h)$. Indeed, in this case, G does not act freely and properly on $B_{\mathbf{F}^{\mathrm{alg}}}(X)$.

Regarding the vanishing of cohomology, we have:

Remark 7.21. Let X be an $\mathbf{F}$-definable subset of a quasi-projective variety V over F. Since the finite simplicial complex $\mathcal{X}$ of Fact 7.11 to can be assumed to have dimension less or equal to $\dim(\overline{X}^{\mathrm{Zar}})$, then by the homotopy axiom in topology ([14, Chapter II, 11.8]), we have:

$$H^p(B_{\mathbf{F}}(X); L_{B_{\mathbf{F}}(X)}) = 0 \text{ for every } p > \dim(\overline{X}^{\mathrm{Zar}})$$

and, similarly, when X is a v+g-locally closed subset, we have

$$H^p_c(B_{\mathbf{F}}(X); L_{B_{\mathbf{F}}(X)}) = 0 \text{ for every } p > \dim(\overline{X}^{\mathrm{Zar}}).$$

It is worthwhile to point out that the first of these vanishing results was also observed in the paper [4], where it plays a crucial role.

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8. Some tameness results in families

In [35] Hrushovski and Loeser show finiteness of homotopy (even homeomorphism) types of uniform families of (model-theoretic) Berkovich spaces (see [35, Theorems 14.3.1 and 14.4.4]). In higher ranks, one has to replace the topological homotopy (respectively, homeomorphism) type by the definable homotopy (respectively, homeomorphism) type. As observed by Hrushovski and Loeser at the beginning [35, Section 14.3], in higher ranks, we no longer have such finiteness results. In fact, one can have a uniform family of triangles in Γ^2, corresponding to skeleta of a uniform family of elliptic curves, without finitely many definable homotopy (equivalently definable homeomorphism) types.

Taking advantage of the invariance results for our cohomology theories with definably compact supports, we can prove the finiteness of cohomological complexity in uniform families. Namely:

Theorem 8.1. *Let K be an algebraically closed, nontrivially valued nonarchimedean field. Let V and V' be algebraic varieties over K. Let $W \subseteq V' \times \Gamma_\infty^k$ be a K-definable subset, and let $Z \subseteq V \times W$ be a* v+g*-locally closed K-definable subset. For $w \in W$, let $Z_w = \{v \in V : (v, w) \in Z\}$. Then as w runs through W, there are finitely many possibilities for the $\widehat{\mathrm{v+g}}$-cohomology*

$$H_c^*(\widehat{Z}_w; L_{\widehat{Z}_w})$$

with definably compact supports.

Proof. First, notice that it is enough to prove the result assuming that V is a quasi-projective variety. In fact, in general, if V is a variety, then V is a finite union of open quasi-projective subvarieties $V_1, \dots, V_n$, and the result follows by induction on n using the Mayer-Vietoris sequence as in the proof of Theorem 6.31.

By the uniform version of the main theorem of Hrushovski and Loeser ([35, Proposition 11.7.1]), there is a uniformly pro-definable family $H_w : I \times \widehat{Z_w} \to \widehat{Z_w}$, a definable subset $\mathcal{Z}_w \subseteq \Gamma_\infty^{\dim V}$, an iso-definable subset $\mathfrak{Z}_w$ of $\widehat{Z}_w$ and $h_w : \mathfrak{Z}_w \to \mathcal{Z}_w$, pro-definable uniformly in w, such that for each w, H_w is a deformation retraction of $\widehat{Z}_w$ onto $\mathfrak{Z}_w$, and h_w is a pro-definable homeomorphism.

By stable embeddedness of Γ_∞ (see, for example, [35, Proposition 2.7.1]) and elimination of imaginaries in Γ_∞, there is a K-definable subset $D \subseteq \Gamma_\infty^{m'}$ (for some m'), a K-definable surjection $\tau\colon W \to D$ and a K-definable family $\mathcal{Z}'_\gamma$ with γ ranging over D, such that for every $w \in W$,

$$\mathcal{Z}_w = \mathcal{Z}'_{\tau(w)}.$$

Let K' be a sufficiently saturated elementary extension of K such that its value group $\Gamma(K')$ expands to a model $\mathbf{\Gamma}''$ of real closed fields. Then we have a commutative diagram of morphisms of $\widehat{\mathrm{v+g}}$-sites

$$\begin{array}{ccc}
I(K') \times \widehat{Z}_w(K') & \xrightarrow{H_w^{K'}} & \widehat{Z}_w(K') \\
\downarrow & & \downarrow \\
I(K) \times \widehat{Z}_w(K) & \xrightarrow{H_w^{K}} & \widehat{Z}_w(K)
\end{array}$$

and

$$\begin{array}{ccc}
\mathfrak{Z}_w(K') & \xrightarrow{h_w^{K'}} & \mathcal{Z}_w(K') \\
\downarrow & & \downarrow \\
\mathfrak{Z}_w(K) & \xrightarrow{h_w^{K}} & \mathcal{Z}_w(K)
\end{array}$$

uniformly in w, with $H_w^{K'}$ a pro-definable deformation retraction and $h_w^{K'}$ a pro-definable homeomorphism and where the vertical arrows are the morphisms of $\widehat{\mathrm{v+g}}$-sites taking $\widehat{U}(K)$ to $\widehat{U}(K')$.

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Note that since Z is v+g-locally closed, each Z_w is also a v+g-locally closed subset. By Lemma 6.27, each $\mathcal{Z}_w$ is a definably locally closed subset. It follows that we take $\widehat{\text{v+g}}$-cohomologies with definably compact supports and obtain the following commutative diagram of isomorphisms

$$\begin{array}{ccccc}
 & & & & H^*_c(\mathcal{Z}'_\gamma(\Gamma''); L_{\mathcal{Z}'_\gamma(\Gamma'')}) \\
 & & & & \downarrow \wr \\
H^*_c(\widehat{Z}_w(K'); L_{\widehat{Z}_w(K')}) & \xrightarrow{\sim} & H^*_c(\mathcal{Z}_w(K'); L_{\mathcal{Z}_w(K')}) & \xrightarrow{\sim} & H^*_c(\mathcal{Z}'_\gamma(\Gamma_\infty(K')); L_{\mathcal{Z}'_\gamma(\Gamma_\infty(K'))}) \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
H^*_c(\widehat{Z}_w(K); L_{\widehat{Z}_w(K)}) & \xrightarrow{\sim} & H^*_c(\mathcal{Z}_w(K); L_{\mathcal{Z}_w(K)}) & \xrightarrow{\sim} & H^*_c(\mathcal{Z}'_\gamma(\Gamma_\infty(K)); L_{\mathcal{Z}'_\gamma(\Gamma_\infty(K))})
\end{array}$$

where $\gamma = \tau(w)$, the vertical isomorphisms are given by the invariance results (Theorems 6.32 and 6.22) and the horizontal isomorphisms are induced by the pro-definable deformation retractions (Remark 6.30).

By the definable trivialisation theorem in the real closed field $\mathbf{\Gamma}''$ ([53, Chapter 9]), there is a partition of $D(\Gamma'')$ by finitely many $\mathbf{\Gamma}''$-definable subsets $D_1, \ldots, D_r$ such that for every $1 \le i \le r$ and for all $\gamma, \gamma' \in D_i$, we have that $\mathcal{Z}_\gamma$ is $\mathbf{\Gamma}''$-definably homeomorphic to $\mathcal{Z}_{\gamma'}$. In particular, we have

$$H^*_c(\mathcal{Z}_\gamma(\Gamma''); L_{\mathcal{Z}_\gamma(\Gamma'')}) \simeq H^*_c(\mathcal{Z}_{\gamma'}(\Gamma''); L_{\mathcal{Z}_{\gamma'}(\Gamma'')})$$

for all $\gamma, \gamma' \in D_i$ and for every $1 \le i \le r$, and the result follows. □

While Theorem 8.1 above can be see as a higher-rank analogue of [35, Theorem 14.3.1] concerning finiteness of homotopy types in uniform families in the rank-one case, the next result is the analogue of [35, Theorem 14.4.4].

Corollary 8.2. *Let K be an algebraically closed nontrivially valued nonarchimedean field. Let V be an algebraic variety over K. Let $X \subseteq V$ be a* v+g*-locally closed definable subset, and let $G\colon X \to \Gamma_\infty$ be a K-definable map. Then there is a finite partition of Γ_∞ into intervals such that the fibres of $\widehat{G} : \widehat{X} \to \Gamma_\infty$ over each interval have canonically isomorphic* $\widehat{\text{v+g}}$*-cohomology groups with definably compact supports.*

Proof. Let $Z \subseteq V \times \Gamma_\infty$ be the graph of $G : X \to \Gamma_\infty$. Then for each $w \in \Gamma_\infty$, we have $Z_w = G^{-1}(w)$ and $\widehat{Z}_w = \widehat{G}^{-1}(w)$. Note that the W and D of the proof of Theorem 8.1 are both equal to Γ_∞ here; in particular, τ is the identity. So that proof gives us that, by the definable trivialisation theorem in the real closed field Γ'' ([53, Chapter 9]), there is a partition of Γ''_∞ by finitely many $\mathbf{\Gamma}''$-definable subsets $D_1, \ldots, D_r$ such that for every $1 \le i \le r$ and for all $\gamma, \gamma' \in D_i$, we have that $\mathcal{Z}_\gamma$ is $\mathbf{\Gamma}''$-definably homeomorphic to $\mathcal{Z}_{\gamma'}$. In particular, we have

$$H^*_c(\mathcal{Z}_\gamma(\Gamma''); L_{\mathcal{Z}_\gamma(\Gamma'')}) \simeq H^*_c(\mathcal{Z}_{\gamma'}(\Gamma''); L_{\mathcal{Z}_{\gamma'}(\Gamma'')})$$

for all $\gamma, \gamma' \in D_i$ and every $1 \le i \le r$. Now the definable trivialisation theorem ensures that we can take the D_is to be $\mathbf{\Gamma}''$-definable with parameters in Γ_∞; hence, by o-minimality, they are intervals in Γ''_∞ with end points in Γ_∞. To conclude, take the intervals $D_1(\Gamma_\infty), \ldots, D_r(\Gamma_\infty)$ partitioning Γ_∞. □

Given a definable set $X \subseteq V \times \Gamma^n_\infty$, let π_0^{def} be the functor taking $\widehat{X}$ into its set of definably connected components. Note that by Lemma 5.21, $\pi_0^{\text{def}}(\widehat{X})$ is a finite set. Moreover, π_0^{def} is invariant under elementary extensions of K (Lemma 5.19) and in $\mathbf{\Gamma}_\infty$ is also invariant under o-minimal expansions. Since π_0^{def} is furthermore invariant under pro-definable deformation retractions, using the same transfer arguments as above, we can prove the following higher-rank analogue of results by A. Abbes and T. Saito ([1, Theorem 5.1]) and by J. Poineau ([48, Théorème 2]).

Corollary 8.3. *Let K be an algebraically closed, nontrivially valued nonarchimedean field. Let V be an algebraic variety over K. Let $X \subseteq V$ be a K-definable subset, and let $G\colon X \longrightarrow \Gamma_\infty$ be a K-definable map. Then there is a finite partition of Γ_∞ into intervals such that over each interval I, for any $\epsilon', \epsilon \in I$, we have a canonical bijection between $\pi_0^{\mathrm{def}}(\widehat{G}^{-1}(\epsilon'))$ and $\pi_0^{\mathrm{def}}(\widehat{G}^{-1}(\epsilon))$.*

We end with some remarks containing open questions that will be considered in the sequel to this paper:

Remark 8.4. One can ask if there is also a higher-rank analogue of local contractability ([35, Theorem 14.4.1]): for example, definable local contractibility or local acyclicity with respect to v+g-cohomology groups. Following the proof in [35], this question reduces to the corresponding question in $\mathbf{\Gamma}_\infty$. See [29] for such results in $\mathbf{\Gamma}$.

Remark 8.5. An issue that was not settled here is that of finiteness and invariance for the $\widehat{\mathrm{v{+}g}}$-cohomology without supports. By Remark 6.30, that reduces to knowing if we have such results for o-minimal cohomology without supports in $\mathbf{\Gamma}_\infty$. Such a result would also give us the finiteness of cohomological complexity as above but without supports.

Moreover, we could also obtain in higher ranks uniform sharp bounds on $\widehat{\mathrm{v{+}g}}$-Betti numbers as in a result obtained by S. Basu and D. Patel ([4, Theorem 2]) in the case where the valued field has rank one, using the usual topological (singular) Betti numbers. Using the strategy above, such a result could be obtained by reduction of the problem to o-minimal expansions of real closed fields and then using the analogue result in that setting proved earlier by Basu ([3, Theorem 2.2]).

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