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RESEARCH ARTICLE

Finiteness properties of the category of mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

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Abstract
We establish the Bernstein-centre type of results for the category of mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. We treat all the remaining open cases, which occur when p is 2 or 3. Our arguments carry over for all primes p. This allows us to remove the restrictions on the residual representation at p in Lue Pan's recent proof of the Fontaine–Mazur conjecture for Hodge–Tate representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with equal Hodge–Tate weights.

Contents

1. Introduction

Recently, Lue Pan gave a new proof of the Fontaine–Mazur conjecture for Hodge–Tate representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with equal Hodge–Tate weights [38]. One of the ingredients in his proof are the Bernstein-centre type of results for the category of mod p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ proved in [39, 41]. This caused him to impose some restrictions on the Galois representations at p, when $p = 2$ and $p = 3$. In this paper we remove these restrictions by proving the required finiteness results in these remaining cases (Theorem 7.3).

Let L be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and residue field k. Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and let $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ be the category of smooth G-representations on $\mathcal{O}$-torsion modules. Let $\mathrm{Mod}^{\mathrm{l.fin}}_G(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ consisting of representations which are equal to the union of their subrepresentations of finite length. This is equivalent to the requirement that every finitely generated subrepresentation be of finite length, and we call such representations *locally finite*. We fix a character $\zeta : Z \to \mathcal{O}^\times$ of the centre of G, and let $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{l.fin}}_G(\mathcal{O})$ consisting of representations with central character ζ. The category $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ is locally finite and by general results of Gabriel [31] decomposes as a product

$$\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O}) \cong \prod_{\mathfrak{B}} \mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}}$$

of indecomposable subcategories, called blocks. Moreover, each block is antiequivalent to the category of pseudocompact modules over a pseudocompact ring $E_{\mathfrak{B}}$. The centre of the ring $E_{\mathfrak{B}}$, which we denote by $Z_{\mathfrak{B}}$, is naturally isomorphic to the centre of the category $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}}$, which by definition is the ring of natural transformations of the identity functor. This means that $Z_{\mathfrak{B}}$ acts functorially on every object in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$. The finiteness result in the title is an analogue of a result of Bernstein in the theory of smooth representations of p-adic groups on $\mathbb{C}$-vector spaces [5, Proposition 3.3]:

Theorem 1.1. *The ring $Z_{\mathfrak{B}}$ is Noetherian, and $E_{\mathfrak{B}}$ is a finitely generated $Z_{\mathfrak{B}}$-module.*

To prove the theorem we use in an essential way the direct connection between the $\mathrm{GL}_2(\mathbb{Q}_p)$ representations and the representations of the absolute Galois group of $\mathbb{Q}_p$, which we denote by $\mathcal{G}_{\mathbb{Q}_p}$, discovered by Colmez in [19], via his celebrated Montreal functor $\check{\mathbf{V}}$, which we review in Section 4.4.

For each fixed block $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}}$ there is a finite extension L' of L with ring of integers $\mathcal{O}'$, such that $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}'$ decomposes into a finite product of indecomposable subcategories, each of which remains indecomposable after a further extension of scalars. Such absolutely indecomposable blocks have been classified in [40], and they correspond to semisimple representations $\bar{\rho} : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$, which are either absolutely irreducible or a direct sum of two characters. This bijection realises the semisimple mod p local Langlands correspondence established in a visionary paper of Breuil [9].

If $\bar{\rho}$ is absolutely irreducible, then $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}_{\bar{\rho}}}$ contains only one irreducible object π, satisfying $\check{\mathbf{V}}(\pi^\vee) \cong \bar{\rho}$, where $\vee$ denotes the Pontryagin dual. Moreover, π is not a subquotient of any parabolically induced representation; such representations are called *supersingular*.

If $\bar{\rho} = \chi_1 \oplus \chi_2$, where $\chi_1, \chi_2 : \mathcal{G}_{\mathbb{Q}_p} \to k^\times$ are characters, then the irreducible objects in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}_{\bar{\rho}}}$ are the irreducible subquotients of the representation

$$(\mathrm{Ind}^G_B\, \chi_1 \otimes \chi_2\omega^{-1})_{\mathrm{sm}} \oplus (\mathrm{Ind}^G_B\, \chi_2 \otimes \chi_1\omega^{-1})_{\mathrm{sm}},$$

where we consider χ_1, χ_2 as characters of $\mathbb{Q}_p^\times$ via the Artin map $\mathrm{Art}_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \to \mathcal{G}^{\mathrm{ab}}_{\mathbb{Q}_p}$, and $\omega(x) = x|x|$ (mod p) for all $x \in \mathbb{Q}_p^\times$ corresponds to the cyclotomic character modulo p. (See Section 4.1 for an explicit list.)

All the blocks, except when p is either 2 or 3 and $\bar{\rho} = \chi \oplus \chi\omega$, have been well understood in [39, 41]. These *exceptional blocks* are the main focus of this paper, but our arguments work for all p and all blocks.

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The action of $Z_{\mathfrak{B}_{\bar{\rho}}}$ induces a functorial ring homomorphism

$$c_\tau : Z_{\mathfrak{B}_{\bar{\rho}}} \to \mathrm{End}_G(\tau)$$

for every object τ in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}_{\bar{\rho}}}$. Since $\check{\mathbf{V}}$ is a functor, it induces a ring homomorphism

$$\mathrm{End}_G(\tau) \to \mathrm{End}^{\mathrm{cont}}_{\mathscr{G}_{\mathbb{Q}_p}}(\check{\mathbf{V}}(\tau^\vee))^{\mathrm{op}}, \quad \varphi \mapsto \check{\mathbf{V}}(\varphi).$$

We denote the action of $\mathscr{G}_{\mathbb{Q}_p}$ on $\check{\mathbf{V}}(\tau^\vee)$ by $\rho_{\check{\mathbf{V}}(\tau^\vee)}$.

Let $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$ be the universal deformation ring parameterising pseudorepresentations lifting $\mathrm{tr}\,\bar{\rho}$ with determinant $\zeta\varepsilon$, where ε is the p-adic cyclotomic character. We may evaluate the universal pseudorepresentation $T : \mathscr{G}_{\mathbb{Q}_p} \to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$ at $g \in \mathscr{G}_{\mathbb{Q}_p}$ to obtain an element $T(g) \in R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$.

Theorem 1.2. *There is an $\mathcal{O}$-algebra homomorphism*

$$\upsilon : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}} \to Z_{\mathfrak{B}_{\bar{\rho}}}, \tag{1}$$

satisfying the following compatibility property with the Colmez's functor: For all $\tau \in \mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}_{\bar{\rho}}}$ and all $g \in \mathscr{G}_{\mathbb{Q}_p}$, we have

$$\check{\mathbf{V}}(c_\tau(\upsilon(T(g)))) = \rho_{\check{\mathbf{V}}(\tau^\vee)}(g) + \rho_{\check{\mathbf{V}}(\tau^\vee)}(g^{-1})\zeta\varepsilon(g)$$

in $\mathrm{End}^{\mathrm{cont}}_{\mathscr{G}_{\mathbb{Q}_p}}(\check{\mathbf{V}}(\tau^\vee))$.

The construction of the map (1) is the main point of this paper. Outside the exceptional cases, it has been established in [39, 41] using a different argument from ours. Our main result is the following:

Theorem 1.3. *The map (1) makes $Z_{\mathfrak{B}_{\bar{\rho}}}$ and $E_{\mathfrak{B}_{\bar{\rho}}}$ into finitely generated $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$-modules.*

Since $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$ is known to be Noetherian by the work of Chenevier [18], Theorem 1.3 implies Theorem 1.1. Moreover, Theorems 1.2 and 1.3 are sufficient to remove the restrictions in Lue Pan's paper. Further, we can re-prove most of the results concerning Banach-space representations in [39] (see Section 6.2 and Corollary 6.16).

To give a flavour of the results on Banach-space representations, we will explain a special case. Let $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ be the category of admissible unitary L-Banach-space representations of G with central character ζ. This category is abelian [44]. By [21], Colmez's functor induces a bijection between the equivalence classes of absolutely irreducible nonordinary $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ and absolutely irreducible Galois representations $\rho : \mathscr{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(L)$. We show that there are no extensions between Π and other irreducible representations in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$; hence $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\Pi}$ is a direct summand of $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}$, where the superscript 'fl' indicates finite length and the subscript Π indicates that all the irreducible subquotients are isomorphic to Π. We show in Corollary 6.16 that $\check{\mathbf{V}}$ induces an antiequivalence of categories between $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\Pi}$ and the category of modules of finite length over $R^{\zeta\varepsilon}_{\rho}$, the universal deformation ring of ρ parameterising deformations of ρ with determinant equal to $\zeta\varepsilon$ to local Artinian L-algebras. Such results were known before for $p \geq 5$ [39], and under assumptions on the reduction modulo p of a G-invariant lattice in Π if $p = 2$ or $p = 3$ [41].

Let $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J$ be the largest quotient of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]$ such that the Cayley–Hamilton theorem holds for the universal pseudorepresentation with determinant $\zeta\varepsilon$ lifting $\mathrm{tr}\,\bar{\rho}$. Such algebras have been studied by Bellaïche and Chenevier [3, 18]; they play a key role in this paper. The subscript 'tf' will indicate the maximal $\mathcal{O}$-torsion-free quotient. Our second main result asserts the following:

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Theorem 1.4. *The essential image of* $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}_{\bar\rho}}$ *under* $\check{\mathbf{V}}$ *is antiequivalent to the category of pseudocompact* $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}}$*-modules. The map (1) induces an isomorphism*

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[1/p] \xrightarrow{\cong} Z_{\mathfrak{B}_{\bar\rho}}[1/p]. \tag{2}$$

Moreover, if $p \neq 2$*, then* $Z_{\mathfrak{B}_{\bar\rho}} = (R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}})_{\mathrm{tf}}$*, and if* $p = 2$*, then the cokernel of map (1) is killed by* 2.

Corollary 1.5. $Z_{\mathfrak{B}_{\bar\rho}}$ *is a complete local Noetherian* $\mathcal{O}$*-algebra with residue field k. It is* $\mathcal{O}$*-torsion-free, and* $Z_{\mathfrak{B}_{\bar\rho}}[1/p]$ *is normal.*

If we are not in the exceptional cases, then Theorem 1.3 is proved in [39, 41] essentially by computing first $E_{\mathfrak{B}}$ and then its centre $Z_{\mathfrak{B}}$. Moreover, it is proved there that map (1) is an isomorphism and $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$ is $\mathcal{O}$-torsion-free. The argument in this paper, sketched in Section 1.1, is different: We do not compute $E_{\mathfrak{B}}$.

Our original strategy for proving formula (2) in this paper was to use an argument of Gabber in [34, Appendix]. We showed that $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[1/p]$ is normal, $Z_{\mathfrak{B}}[1/p]$ is reduced, and map (1) induces a bijection on maximal spectra m-Spec $Z_{\mathfrak{B}}[1/p] \to$ m-Spec $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[1/p]$ and an isomorphism of the residue fields. However, this is replaced by a different argument in the final version, which also proves the first part of Theorem 1.4. One important ingredient in the proof is results of Colmez, Dospinescu and the first author [21, 22] which imply that the universal framed deformation of $\bar\rho$ with determinant $\zeta\varepsilon$ lies in the image of $\check{\mathbf{V}}$. We show in Appendix A that $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}}$ acts faithfully on this representation using the theory of Cayley–Hamilton algebras [43].

The normality of the ring $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[1/p]$ is proved in Appendix A, where we show that if the generic fibre of the framed deformation ring $R^{\square}_{\bar\rho}[1/p]$ is normal, then $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho}[1/p]$ and the corresponding rigid analytic space $(\mathrm{Spf}\, R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho})^{\mathrm{rig}}$ are normal. The same applies to the deformation rings without the fixed-determinant condition. In fact, we prove this statement not only for $\mathcal{G}_{\mathbb{Q}_p}$ but for any profinite group satisfying Mazur's finiteness condition at p. We then show that $R^{\square}_{\bar\rho}[1/p]$ is normal for all 2-dimensional semisimple representations of $\mathcal{G}_{\mathbb{Q}_p}$; the hard cases are precisely those corresponding to the exceptional blocks. If $p = 2$, then the assertion has been shown in [22], and if $p = 3$, then we give a proof in Appendix A based on Böckle's explicit description of the framed deformation ring in [7]. We note that the argument of [22] has been generalised by Iyengar [33], showing that when $\bar\rho$ is the trivial d-dimensional representation of the absolute Galois group of a p-adic field, containing a 4th root of unity if $p = 2$, then $R^{\square}_{\bar\rho}[1/p]$ is normal, so our results also apply in this setting. We expect[1] the rings $R^{\square}_{\bar\rho}[1/p]$ to be normal for any d-dimensional representation $\bar\rho$ of $\mathcal{G}_F$, where F is a finite extension of $\mathbb{Q}_p$.

1.1. A sketch of the proof

We will now explain the construction of the map (1). To fix ideas we will discuss a special case: $p = 2$, $\bar\rho = \mathbf{1} \oplus \omega$ and $\zeta = \mathbf{1}$. Since the cyclotomic character is trivial modulo 2, $\bar\rho = \mathbf{1} \oplus \mathbf{1}$. The corresponding block has two irreducible representations: the trivial $\mathbf{1}$ and the smooth Steinberg representation Sp. Instead of working with representations on $\mathcal{O}$-torsion modules, it is more convenient to use Pontryagin duality and work with representations of G on compact $\mathcal{O}$-modules. We denote by $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ the category antiequivalent to $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})_{\mathfrak{B}}$ under Pontryagin duality.

Let $P_{\mathbf{1}^\vee}$ and $P_{\mathrm{Sp}^\vee}$ be projective envelopes of $\mathbf{1}^\vee$ and $\mathrm{Sp}^\vee$ in $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, respectively. Then $P_{\mathfrak{B}} := P_{\mathbf{1}^\vee} \oplus P_{\mathrm{Sp}^\vee}$ is a projective generator of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, and by results of Gabriel [31] the category $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ is equivalent to the category of pseudocompact modules of $E_{\mathfrak{B}} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}})$. The equivalence is induced by the functors

$$N \mapsto \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}, N), \quad \mathrm{m} \mapsto \mathrm{m}\,\widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}}.$$

The centre $Z_{\mathfrak{B}}$ of the category $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ is naturally isomorphic to the centre of the ring $E_{\mathfrak{B}}$.

[1] This has now been proved in [8, Corollary 4.22].

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Colmez's functor kills all objects on which $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially, and these form a thick subcategory. Thus $\check{\mathbf{V}}$ factors through the quotient category, which we denote by $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$; let $\mathcal{T} : \mathfrak{C}(\mathcal{O})_{\mathfrak{B}} \to \mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ be the quotient functor. Moreover, $\check{\mathbf{V}}$ induces an equivalence of categories between $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ and its essential image under $\check{\mathbf{V}}$. To prove this, one needs to show that $\check{\mathbf{V}}$ sends nonsplit extensions to nonsplit extensions; such arguments are originally due to Colmez, and in the case $p = 2$ this has been shown by the second author [47].

Since $\mathcal{T}(\mathbf{1}^\vee) = 0$, then $\mathcal{T}(\mathrm{Sp}^\vee)$ is the only irreducible object in $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ up to isomorphism. Moreover, it is shown in [39] that $\mathcal{T}(P_{\mathrm{Sp}^\vee})$ is a projective envelope of $\mathcal{T}(\mathrm{Sp}^\vee)$, and $\mathcal{T}$ induces an isomorphism

$$E'_{\mathfrak{B}} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathrm{Sp}^\vee}) \cong \mathrm{End}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} P_{\mathrm{Sp}^\vee}).$$

Since $\mathcal{T}(\mathrm{Sp}^\vee)$ is the only irreducible object in $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$, then $\mathcal{T}(P_{\mathrm{Sp}^\vee})$ is a projective generator of $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$, and thus $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ is equivalent to the category of pseudocompact $E'_{\mathfrak{B}}$-modules. This implies that $\mathcal{T}(P_{\mathrm{Sp}^\vee})$ – and hence, by equivalence of categories, $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ – is flat over $E'_{\mathfrak{B}}$. Since $\check{\mathbf{V}}(\mathrm{Sp}^\vee)$ is a 1-dimensional representation of $\mathcal{G}_{\mathbb{Q}_p}$, in fact the trivial representation with our normalisations, an application of Nakayama's lemma shows that $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ is a free $E'_{\mathfrak{B}}$-module of rank 1. The action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ commutes with the action of $E'_{\mathfrak{B}}$ and so induces a homomorphism

$$\alpha : \mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!] \to \mathrm{End}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})) \cong (E'_{\mathfrak{B}})^{\mathrm{op}}.$$

We show that this map is surjective. In general, the argument is carried out in Section 2 in an abstract setting, and then in Proposition 4.18 we verify that the conditions of the abstract setting are satisfied. However, in the special case under consideration, the argument is easier: Since $\check{\mathbf{V}}$ induces an equivalence of categories between $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ and its essential image, the $E'_{\mathfrak{B}}$-cosocle and $\mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]$-cosocle of $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ coincide. This implies that there is $v \in \check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$, which is a generator of $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ as both an $E'_{\mathfrak{B}}$- and an $\mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]$-module. This implies that α is surjective.

We also show in Section 3 that the natural map

$$\beta : \mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!] \to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$$

is surjective, where $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$ is the largest quotient of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]$ such that the Cayley–Hamilton theorem holds for the universal pseudorepresentation with determinant $\zeta\varepsilon$ lifting $\mathrm{tr}\,\bar{\rho}$.

The idea is to show that $\mathrm{Ker}\,\alpha$ contains $\mathrm{Ker}\,\beta$, since this implies that the action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ induces surjections:

$$\mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!] \twoheadrightarrow R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J \twoheadrightarrow E'_{\mathfrak{B}}. \tag{3}$$

This is proved using the results of Berger and Breuil [4] on universal unitary completions of locally algebraic principal series and density arguments already used in [21] and also in [46]. Morally, the argument should be that $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$ injects into the product of all 2-dimensional crystabelline representations of $\mathcal{G}_{\mathbb{Q}_p}$ with mod p reduction isomorphic to $\bar{\rho}$ and determinant $\zeta\varepsilon$; then an element in $\mathrm{Ker}\,\beta$ would kill this product, and hence $\check{\mathbf{V}}(P_{\mathrm{Sp}^\vee})$. In reality the argument is technically a bit more complicated: We also have to consider deformations of such representations to local Artinian L-algebras (see Section 5).

Wang-Erickson proved in [48] that $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$ is a finitely generated $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$-module. The surjection (3) implies that the image of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$ is contained in the centre of $E'_{\mathfrak{B}}$, which we denote by $Z'_{\mathfrak{B}}$. Moreover, both $E'_{\mathfrak{B}}$ and $Z'_{\mathfrak{B}}$ are finite $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}$-modules. We show that formula (3) induces an isomorphism

$$(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}} \xrightarrow{\cong} E'_{\mathfrak{B}}$$

by showing that the universal framed deformation of $\bar{\rho}$ with determinant $\zeta\varepsilon$ lies in the image of $\check{\mathbf{V}}$ using [21, 22], and $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}}$ acts faithfully on it (see Proposition A.11). The assertions about the

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centre in Theorem 1.4, with $Z'_{\mathfrak{B}}$ instead of $Z_{\mathfrak{B}}$, are proved by studying the centre of $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}}$. This argument is carried out in Appendix A for d-dimensional representations of any profinite group, satisfying Mazur's finiteness condition at p.

We then transfer this result from $E'_{\mathfrak{B}}$ and $Z'_{\mathfrak{B}}$ to $E_{\mathfrak{B}}$ and $Z_{\mathfrak{B}}$. Let $M_{\mathfrak{B}}$ be the kernel of $P_{\mathfrak{B}} \to (P_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)}$. We show that

$$\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}) \cong \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}) = E_{\mathfrak{B}}$$

by examining various exact sequences and showing that certain Ext-groups vanish. We also show that the cosocle of $M_{\mathfrak{B}}$ is a direct sum of finitely many copies of $\mathrm{Sp}^{\vee}$. Thus $M_{\mathfrak{B}}$ is a quotient of $(P'_{\mathfrak{B}})^{\oplus n}$ for some $n \geq 1$. This allows us to conclude that $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})$ and its centre are finitely generated $Z'_{\mathfrak{B}}$-modules, which finishes the proof of Theorem 1.3. The arguments showing the finiteness of $E_{\mathfrak{B}}$ and $Z_{\mathfrak{B}}$ over $Z'_{\mathfrak{B}}$ are carried out in Section 4.3. Then with some more effort we are able to show that $Z'_{\mathfrak{B}} = Z_{\mathfrak{B}}$ (Corollary 6.15).

1.2. What is left to do?

Although we believe that our results will suffice for most number-theoretic applications – for example, [38] – to complete the programme started in [39], one would have to compute the ring $E_{\mathfrak{B}}$ in the exceptional cases. This will be harder than [39, Section 10.5], which is already quite involved. We expect that the map (3) induces isomorphisms

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J \xrightarrow{\cong} E'_{\mathfrak{B}}, \qquad R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}} \xrightarrow{\cong} Z_{\mathfrak{B}}.$$

This is known to hold for all blocks except for the exceptional ones. Theorem 1.4 implies that to prove this result, it would be enough to show that $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$ is $\mathcal{O}$-torsion-free, and for the second isomorphism in the case $p = 3$ it would be enough to show[2] that $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}}$ is $\mathcal{O}$-torsion-free.

It seems likely that using the results of this paper, one can remove the restriction on the prime p in Lue Pan's work [37] on the Fontaine–Mazur conjecture in the residually reducible case, which generalises the work of Skinner and Wiles. We hope to return to these questions in future work.

2. Endomorphism rings

Let E be a pseudocompact $\mathcal{O}$-algebra and let $\mathrm{PC}(E)$ be the category of left pseudocompact E-modules (see [12], [31, Section IV.3]. Let $\mathrm{Irr}(E)$ be the set of equivalence classes of irreducible objects in $\mathrm{PC}(E)$.

Let M be in $\mathrm{PC}(E)$. We assume that we are given a continuous E-linear action of a profinite group $\mathcal{G}$ on M, which makes M into a pseudocompact module over the completed group algebra $\mathcal{O}[\![\mathcal{G}]\!]$. The action induces a homomorphism of $\mathcal{O}$-algebras $\mathcal{O}[\![\mathcal{G}]\!] \to \mathrm{End}^{\mathrm{cont}}_E(M)$. In this section we will study when this map is surjective, as well as its kernel.

If N is a pseudocompact E-module, which is finitely generated as an E-module, then we may present it as

$$\prod_{i\in I} E \to E^{\oplus n} \to N \to 0.$$

By applying $\mathrm{Hom}^{\mathrm{cont}}_E(*, M)$ we obtain an exact sequence

$$0 \to \mathrm{Hom}^{\mathrm{cont}}_E(N, M) \to M^{\oplus n} \to \oplus_{i\in I} M.$$

We thus may identify $\mathrm{Hom}^{\mathrm{cont}}_E(N, M)$ with a closed submodule of $M^{\oplus n}$, which makes $\mathrm{Hom}^{\mathrm{cont}}_E(N, M)$ into a pseudocompact left $\mathcal{O}[\![\mathcal{G}]\!]$-module.

[2]This follows from [8, Corollary 5.11].

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Lemma 2.1. *Let N be a finitely generated projective E-module and let* m *be a right pseudocompact $\mathcal{O}[\![\mathcal{G}]\!]$-module. Then the natural map*

$$\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]}\,\mathrm{Hom}_E^{\mathrm{cont}}(N, M) \to \mathrm{Hom}_E^{\mathrm{cont}}(N, \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]}\, M) \tag{4}$$

is an isomorphism.

Proof. Since N is finitely generated and projective, we may present it as

$$F \xrightarrow{e} F \to N \to 0,$$

where $F \cong E^{\oplus n}$ and e is an idempotent. In particular, N is a pseudocompact E-module. The map (4) is induced by a continuous bilinear map

$$(v, \phi) \mapsto [w \mapsto v \,\widehat{\otimes}\, \phi(w)].$$

It is an isomorphism if $N = F$. The general case follows by applying the idempotent e to the isomorphism obtained for $N = F$. □

Lemma 2.2. *Let $\{\rho_i\}_{i\in I}$ be a family of pairwise distinct absolutely irreducible right pseudocompact $\mathcal{O}[\![\mathcal{G}]\!]$-modules. Then the map*

$$\mathcal{O}[\![\mathcal{G}]\!] \to \prod_{i\in I} \mathrm{End}_k(\rho_i)^{\mathrm{op}} \tag{5}$$

is surjective.

Proof. Since $\mathcal{G}$ is profinite, each ρ_i is a finite-dimensional k-vector space. Thus $\varphi_i : \mathcal{O}[\![\mathcal{G}]\!] \to \mathrm{End}_k(\rho_i)^{\mathrm{op}}$, given by the action, is continuous for the discrete topology on the target. Since ρ_i is absolutely irreducible, φ_i is surjective. Moreover, $\mathrm{Ker}\,\varphi_i$ is an open maximal two-sided ideal of $\mathcal{O}[\![\mathcal{G}]\!]$. If $i \neq j$, then $\rho_i \not\cong \rho_j$, and thus $\mathrm{Ker}\,\varphi_i + \mathrm{Ker}\,\varphi_j = \mathcal{O}[\![\mathcal{G}]\!]$. This implies that for every finite subset F of I, the map $\mathcal{O}[\![\mathcal{G}]\!] \to \prod_{i\in F} \mathrm{End}_k(\rho_i)^{\mathrm{op}}$ is surjective. Thus the image of map (5) is dense for the product topology on the target. On the other hand, map (5) is a continuous map between pseudocompact $\mathcal{O}$-modules, and thus its image is closed, which implies surjectivity. □

If M is in $\mathrm{PC}(E)$, then we let $\mathfrak{r}(M)$ be the intersection of open maximal submodules of M. Then $\mathfrak{r}(E)$ is a closed two-sided ideal of E and $\mathfrak{r}(M)$ is the closure of $\mathfrak{r}(E)M$ inside M.

Proposition 2.3. *Let us assume that the following hold:*

1. *M is a finitely generated projective E-module.*
2. $M/\mathfrak{r}(M) = \mathrm{cosoc}_{\mathcal{G}}\, M$.
3. *For all $S \in \mathrm{Irr}(E)$ such that*

$$\rho_S := \mathrm{Hom}_E^{\mathrm{cont}}(M, S) \neq 0,$$

   $\dim_k \rho_S$ *is finite and* ρ_S *is an absolutely irreducible representation of* $\mathcal{G}$.
4. *If $S, S' \in \mathrm{Irr}(E)$ and $S \not\cong S'$, then* $\mathrm{Hom}_{\mathcal{G}}(\rho_S, \rho_{S'}) = 0$.

Then the map $\mathcal{O}[\![\mathcal{G}]\!] \to \mathrm{End}_E^{\mathrm{cont}}(M)$ is surjective.

Proof. For $S \in \mathrm{Irr}(E)$, we let $\overline{M}_S$ be the smallest quotient of M such that

$$\rho_S = \mathrm{Hom}_E^{\mathrm{cont}}(M, S) = \mathrm{Hom}_E^{\mathrm{cont}}(\overline{M}_S, S).$$

It follows from assumption (3) that these subspaces are finite-dimensional. Thus $\overline{M}_S \cong S^{\oplus d}$ such that $d \cdot \dim_k \mathrm{End}_E^{\mathrm{cont}}(S) = \dim_k \rho_S$. Since S is irreducible, $\mathrm{End}_E^{\mathrm{cont}}(S)$ is a skew field. It acts on ρ_S,

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and this action commutes with the action of $\mathcal{G}$. Since ρ_S is absolutely irreducible, we conclude that $\operatorname{End}_E^{\mathrm{cont}}(S) = k$ and $\dim_k \rho_S = d$. Thus $\operatorname{End}_E^{\mathrm{cont}}(\overline{M}_S) \cong M_d(k)$ and thus does not have nontrivial two-sided ideals. Hence, the natural right action of $\operatorname{End}_E^{\mathrm{cont}}(\overline{M}_S)$ on ρ_S induces an injective ring homomorphism

$$\operatorname{End}_E^{\mathrm{cont}}(\overline{M}_S) \to \operatorname{End}_k(\rho_S)^{\mathrm{op}}, \tag{6}$$

which has to be surjective, as both k-vector spaces have dimension equal to d^2.

The isomorphism $M/\mathfrak{r}(M) \cong \prod_{S\in\operatorname{Irr}(E)} \overline{M}_S$ induces an isomorphism

$$\operatorname{End}_E^{\mathrm{cont}}(M/\mathfrak{r}(M)) \cong \prod_{S\in\operatorname{Irr}(E)} \operatorname{End}_E^{\mathrm{cont}}(\overline{M}_S).$$

Since $\rho_S \not\cong \rho_{S'}$ if $S \not\cong S'$, it follows from Lemma 2.2 together with isomorphism (6) that the action of $\mathcal{O}[\![\mathcal{G}]\!]$ on $M/\mathfrak{r}(M)$ induces a surjection

$$\mathcal{O}[\![\mathcal{G}]\!] \twoheadrightarrow \operatorname{End}_E^{\mathrm{cont}}(M/\mathfrak{r}(M)). \tag{7}$$

Since M is projective and $M/\mathfrak{r}(M)$ is prosemisimple, we have

$$\operatorname{End}_E^{\mathrm{cont}}(M) \twoheadrightarrow \operatorname{Hom}_E^{\mathrm{cont}}(M, M/\mathfrak{r}(M)) \cong \operatorname{End}_E^{\mathrm{cont}}(M/\mathfrak{r}(M)). \tag{8}$$

If m is an irreducible right pseudocompact $\mathcal{O}[\![\mathcal{G}]\!]$-module and $\mathfrak{a}$ is its annihilator, then $\mathcal{O}[\![\mathcal{G}]\!]/\mathfrak{a}$ is a finite-dimensional simple k-algebra, and thus $\mathcal{O}[\![\mathcal{G}]\!]/\mathfrak{a}$ is semisimple as a left $\mathcal{O}[\![\mathcal{G}]\!]$-module, and thus $(\mathcal{O}[\![\mathcal{G}]\!]/\mathfrak{a})\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M$ is semisimple as a left $\mathcal{O}[\![\mathcal{G}]\!]$-module. Hence, the surjection $M \twoheadrightarrow (\mathcal{O}[\![\mathcal{G}]\!]/\mathfrak{a})\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M$ factors through as

$$M \twoheadrightarrow \operatorname{cosoc}_{\mathcal{G}} M \twoheadrightarrow (\mathcal{O}[\![\mathcal{G}]\!]/\mathfrak{a})\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M.$$

Moreover, the maps become isomorphisms after application of $\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]}$. Thus

$$\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M \cong \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \operatorname{cosoc}_{\mathcal{G}} M \cong \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M/\mathfrak{r}(M), \tag{9}$$

as $M/\mathfrak{r}(M) = \operatorname{cosoc}_{\mathcal{G}} M$ by assumption. Lemma 2.1 together with formula (9) implies that by applying $\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]}$ to formula (8), we obtain isomorphisms

$$\begin{aligned}
\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \operatorname{End}_E^{\mathrm{cont}}(M) &\xrightarrow{\cong} \operatorname{Hom}_E^{\mathrm{cont}}(M, \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M)\\
&\xrightarrow{\cong} \operatorname{Hom}_E^{\mathrm{cont}}(M, \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} M/\mathfrak{r}(M))\\
&\xrightarrow{\cong} \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \operatorname{Hom}_E^{\mathrm{cont}}(M, M/\mathfrak{r}(M))\\
&\xrightarrow{\cong} \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \operatorname{End}_E^{\mathrm{cont}}(M/\mathfrak{r}(M)),
\end{aligned} \tag{10}$$

where the $\mathcal{G}$-action on $\operatorname{End}_E^{\mathrm{cont}}(M)$ and $\operatorname{End}_E^{\mathrm{cont}}(M/\mathfrak{r}(M))$ is given by $(g.\varphi)(v) := g(\varphi(v))$.

If the map $\mathcal{O}[\![\mathcal{G}]\!] \to \operatorname{End}_E^{\mathrm{cont}}(M)$ is not surjective, then its cokernel is a nonzero left pseudocompact $\mathcal{O}[\![\mathcal{G}]\!]$-module and thus will have an irreducible quotient m'. If we let $\mathrm{m} = \operatorname{Hom}_k(\mathrm{m}', k)$ with the right $\mathcal{O}[\![\mathcal{G}]\!]$-action, then $\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \mathrm{m}'$ is nonzero, as the evaluation map $\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \mathrm{m}' \twoheadrightarrow k$ is nonzero. By construction, the composition

$$\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \mathcal{O}[\![\mathcal{G}]\!] \to \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \operatorname{End}_E^{\mathrm{cont}}(M) \twoheadrightarrow \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}[\![\mathcal{G}]\!]} \mathrm{m}'$$

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is the zero map. Thus $\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}\llbracket\mathcal{G}\rrbracket}\,\mathcal{O}\llbracket\mathcal{G}\rrbracket \to \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}\llbracket\mathcal{G}\rrbracket}\,\mathrm{End}_E^{\mathrm{cont}}(M)$ cannot be surjective. However, the commutative diagram

$$\begin{array}{ccc}
\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}\llbracket\mathcal{G}\rrbracket}\,\mathcal{O}\llbracket\mathcal{G}\rrbracket & \longrightarrow & \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}\llbracket\mathcal{G}\rrbracket}\,\mathrm{End}_E^{\mathrm{cont}}(M) \\
\Big\downarrow{=} & & (10)\Big\downarrow{\cong} \\
\mathrm{m}\,\widehat{\otimes}_{\mathcal{O}\llbracket\mathcal{G}\rrbracket}\,\mathcal{O}\llbracket\mathcal{G}\rrbracket & \overset{(7)}{\twoheadrightarrow} & \mathrm{m}\,\widehat{\otimes}_{\mathcal{O}\llbracket\mathcal{G}\rrbracket}\,\mathrm{End}_E^{\mathrm{cont}}(M/\mathfrak{r}(M))
\end{array}$$

implies that the top horizontal arrow is surjective, yielding a contradiction. □

We remind the reader that as a consequence of the topological Nakayama's lemma [32, Exposé $\mathrm{VII_B}$, Lemma 0.3.3], the following holds:

Lemma 2.4. *Let N be a pseudocompact left E-module. Then N is projective in* $\mathrm{PC}(E)$ *if and only if the functor* $\mathrm{m} \mapsto \mathrm{m}\,\widehat{\otimes}_E\, N$ *from the category of right pseudocompact E-modules to the category of abelian groups is exact. In this case, $N \twoheadrightarrow N/\mathfrak{r}(N)$ is a projective envelope of $N/\mathfrak{r}(N)$.*

Corollary 2.5. *If in addition to the assumptions of Proposition 2.3 we assume that $M/\mathfrak{r}(M) \cong E/\mathfrak{r}(E)$ as E-modules, then M is a free E-module of rank* 1 *and the action of $\mathcal{O}\llbracket\mathcal{G}\rrbracket$ on M induces a surjection*

$$\mathcal{O}\llbracket\mathcal{G}\rrbracket \twoheadrightarrow E^{\mathrm{op}},$$

which is uniquely determined up to a conjugation by $E^{\times}$.

We will now give a characterisation of the kernel of $\mathcal{O}\llbracket\mathcal{G}\rrbracket \to \mathrm{End}_E^{\mathrm{cont}}(M)$ in favourable settings.

Let m be a finite-dimensional L-vector space with continuous $\mathcal{O}$-linear action of E on the right. The image of E in $\mathrm{End}_L(\mathrm{m})$ is a compact $\mathcal{O}$-module, and thus E stabilises an $\mathcal{O}$-lattice m^0 in m. The action of $\mathcal{O}\llbracket\mathcal{G}\rrbracket$ on M induces a continuous left action of $\mathcal{O}\llbracket\mathcal{G}\rrbracket$ on $\mathrm{m}^0\,\widehat{\otimes}_E\, M$ and hence on

$$\mathrm{m}\otimes_E M = (\mathrm{m}^0 \otimes_E M)[1/p] = (\mathrm{m}^0\,\widehat{\otimes}_E\, M)[1/p].$$

Lemma 2.6. *Let $\{\mathrm{m}_i\}_{i\in I}$ be a family of finite-dimensional L-vector spaces with continuous right $\mathcal{O}$-linear action of E. For each $i \in I$, let $\mathfrak{a}_i$ be the E-annihilator of m_i and let $\mathfrak{b}_i$ be the $\mathcal{O}\llbracket\mathcal{G}\rrbracket$-annihilator of $\mathrm{m}_i \otimes_E M$. If M is a free E-module of finite rank and $\bigcap_{i\in I} \mathfrak{a}_i = 0$, then*

$$\mathrm{Ker}(\mathcal{O}\llbracket\mathcal{G}\rrbracket \to \mathrm{End}_E^{\mathrm{cont}}(M)) = \bigcap_{i\in I} \mathfrak{b}_i.$$

Proof. For each $i \in I$, let $\mathfrak{c}_i$ be the $\mathrm{End}_E^{\mathrm{cont}}(M)$-annihilator of $\mathrm{m}_i \otimes_E M$. Since $\mathfrak{b}_i$ is the preimage of $\mathfrak{c}_i$ in $\mathcal{O}\llbracket\mathcal{G}\rrbracket$, it is enough to show that $\bigcap_{i\in I} \mathfrak{c}_i = 0$.

Let $w_1, \ldots, w_n$ be an E-basis of M. Then we may identify $\mathrm{End}_E^{\mathrm{cont}}(M)$ with $M_n(E)$ by mapping φ to the matrix (a_{kj}), given by

$$\varphi(w_k) = \sum_{j=1}^{n} a_{kj} w_j$$

for all $1 \le k \le n$. If $v \in \mathrm{m}_i$, then

$$v\,\widehat{\otimes}\,\varphi(w_k) = \sum_{j=1}^{n} (v a_{kj})\,\widehat{\otimes}\, w_j.$$

Thus φ annihilates $\mathrm{m}_i \otimes_E M$ if and only if $v a_{kj} = 0$ for all $v \in \mathrm{m}_i$ and all $1 \le k, j \le n$, which is equivalent to $\varphi \in M_n(\mathfrak{a}_i)$. Since $\bigcap_{i\in I} \mathfrak{a}_i = 0$, we have $\bigcap_{i\in I} M_n(\mathfrak{a}_i) = 0$ and thus $\bigcap_{i\in I} \mathfrak{c}_i = 0$. □

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3. Pseudorepresentations

Let $\mathcal{G}$ be a profinite group and let $\bar{\rho}$ be a continuous semisimple representation of $\mathcal{G}$ on a 2-dimensional k-vector space. We fix a continuous group homomorphism $\psi : \mathcal{G} \to \mathcal{O}^\times$ lifting $\det \bar{\rho}$. Let $D^{\mathrm{ps},\psi}$ be the functor from the category of augmented Artinian $\mathcal{O}$-algebras with residue field k to the category of sets, which maps $(A, \mathfrak{m}_A)$ to the set of continuous functions $t : \mathcal{G} \to A$ satisfying the following conditions:

- $t(1) = 2$.
- $t(g) \equiv \operatorname{tr} \bar{\rho}(g) \pmod{\mathfrak{m}_A}, \quad \forall g \in \mathcal{G}$.
- $t(gh) = t(hg), \quad \forall g, h \in \mathcal{G}$.
- $\psi(g)t(g^{-1}h) - t(g)t(h) + t(gh) = 0, \quad \forall g, h \in \mathcal{G}$.

The data (t, ψ) determines a continuous polynomial law of homogeneous degree 2 on $\mathcal{G}$ (see [18, Lemma 1.9]). This deformation problem is prorepresentable by a local $\mathcal{O}$-algebra $R^{\mathrm{ps},\psi}$ with residue field k, complete with respect to profinite topology. We denote by $T : \mathcal{G} \to R^{\mathrm{ps},\psi}$ the universal deformation. We extend it $R^{\mathrm{ps},\psi}$-linearly to a continuous function $T : R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!] \to R^{\mathrm{ps},\psi}$. The homomorphism $\psi : \mathcal{G} \to \mathcal{O}^\times$ induces a continuous $\mathcal{O}$-algebra homomorphism $\psi : \mathcal{O}[\![\mathcal{G}]\!] \to \mathcal{O}$, which we extend $R^{\mathrm{ps},\psi}$-linearly to a continuous $R^{\mathrm{ps},\psi}$-algebra homomorphism $\psi : R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!] \to R^{\mathrm{ps},\psi}$. Let J be the closed two-sided ideal of $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]$ generated by $a^2 - T(a)a + \psi(a)$ for all $a \in R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]$.

Proposition 3.1. *The ring homomorphism $\mathcal{O}[\![\mathcal{G}]\!] \to R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]/J$ is surjective.*

Proof. Let $\overline{R}$ and C be the images of $R^{\mathrm{ps},\psi}$ and $\mathcal{O}[\![\mathcal{G}]\!]$ in $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]/J$, respectively. Since $R^{\mathrm{ps},\psi}$, $\mathcal{O}[\![\mathcal{G}]\!]$ and $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]/J$ are pseudocompact $\mathcal{O}$-modules, $\overline{R}$ and C are closed subrings of $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]/J$. It is enough to show that C contains $\overline{R}$, since in this case we deduce that C contains the image of $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]$, which is equal to $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]/J$.

Let $B = \overline{R} \cap C$ and let $\mathfrak{m}_B$ be the intersection of B with the maximal ideal of $\overline{R}$. Then B is a closed subring of $\overline{R}$. This implies that B is complete for the profinite topology. If $x \in \mathfrak{m}_B$, then $1 + x$ has an inverse in B given by the geometric series. Since B is an $\mathcal{O}$-algebra and the residue field of $\overline{R}$ is k, we conclude that $(B, \mathfrak{m}_B)$ is a local ring with residue field k.

Let $\overline{T}$ be the specialisation of T along $R^{\mathrm{ps},\psi} \twoheadrightarrow \overline{R}$. The relation $g^2 - \overline{T}(g)g + \psi(g) = 0$ in $R^{\mathrm{ps},\psi}[\![\mathcal{G}]\!]/J$ implies that $\overline{T}(g) = g + g^{-1}\psi(g)$. Thus $\overline{T}$ takes values in B. The universal property of $R^{\mathrm{ps},\psi}$ implies that there is a continuous homomorphism of $\mathcal{O}$-algebras $\varphi : R^{\mathrm{ps},\psi} \to B$, such that $\varphi(T(g)) = \overline{T}(g)$ for all $g \in \mathcal{G}$. Using the universal property of $R^{\mathrm{ps},\psi}$ again, we conclude that if we compose φ with the inclusion $B \subset \overline{R}$, we get back the surjection $R^{\mathrm{ps},\psi} \twoheadrightarrow \overline{R}$ that we started with. Thus $B = \overline{R}$. □

Remark 3.2. The proposition does not hold if we do not fix the determinant or consider representations $\bar{\rho}$ of dimension greater than 2. Counterexamples may be obtained with $\mathcal{G} = \mathbb{Z}_p$ and $\bar{\rho}$ trivial representation of $\mathcal{G}$ on an n-dimensional k-vector space, using [18, Examples 1.7(i) and 1.11(i)].

4. Representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

Let G be a p-adic analytic group and let Z be its centre. We let $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ be the category of smooth representations of G on $\mathcal{O}$-torsion modules. Using Pontryagin duality, $\pi \mapsto \pi^\vee := \mathrm{Hom}_{\mathcal{O}}(\pi, L/\mathcal{O})$ equipped with the compact open topology induces an antiequivalence of categories between $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ and the category $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$ of linearly compact $\mathcal{O}$-modules with a continuous G-action [28, Lemma 2.2.7]. The inverse is given by $M \mapsto M^\vee := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L/\mathcal{O})$. In particular, if G is compact, then $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$ is the category of linearly compact $\mathcal{O}[\![G]\!]$-modules, where $\mathcal{O}[\![G]\!]$ is the completed group algebra. We define $\mathrm{Mod}^{\mathrm{sm}}_G(k)$ and $\mathrm{Mod}^{\mathrm{pro}}_G(k)$ the same way, with $\mathcal{O}$ replaced by k. Moreover, for a continuous character $\zeta : Z \to \mathcal{O}^\times$, adding the subscript ζ in any of these categories indicates the corresponding full subcategory of G-representations on which Z acts by ζ. Denote by $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(\mathcal{O})$ consisting of representations in $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(\mathcal{O})$ which are equal to the union of their subrepresentations of finite length. We let $\mathfrak{C}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$ antiequivalent to $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ under the Pontryagin duality.

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4.1. Blocks

From now on, we will assume $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Every irreducible object π of $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ is killed by ϖ and hence is an object of $\mathrm{Mod}^{\mathrm{sm}}_G(k)$.

Let $\mathrm{Irr}_{G,\zeta}$ be the set of irreducible representations in $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(k)$. We write $\pi \leftrightarrow \pi'$ if $\pi \cong \pi'$ or $\mathrm{Ext}^1_{G,\zeta}(\pi,\pi') \neq 0$ or $\mathrm{Ext}^1_{G,\zeta}(\pi',\pi) \neq 0$, where $\mathrm{Ext}^1_{G,\zeta}(\pi,\pi')$ is the Yoneda extension group of π' by π in $\mathrm{Mod}^{\mathrm{sm}}_{G,\zeta}(k)$. We write $\pi \sim \pi'$ if there exist $\pi_1, \dots, \pi_n \in \mathrm{Irr}_{G,\zeta}$ such that $\pi \cong \pi_1$, $\pi' \cong \pi_n$ and $\pi_i \leftrightarrow \pi_{i+1}$ for $1 \leq i \leq n-1$. The relation $\sim$ is an equivalence relation on $\mathrm{Irr}_{G,\zeta}$. A block is an equivalence class of $\sim$.

Barthel and Livné [2] and Breuil [9] have classified the absolutely irreducible smooth representations π admitting a central character. The blocks containing an absolutely irreducible representation have been determined in [40, Corollary 6.2]. We have the following cases:

(i) $\mathfrak{B} = \{\pi\}$ with π supersingular.
(ii) $\mathfrak{B} = \{(\mathrm{Ind}^G_B \chi_1 \otimes \chi_2\omega^{-1})_{\mathrm{sm}}, (\mathrm{Ind}^G_B \chi_2 \otimes \chi_1\omega^{-1})_{\mathrm{sm}}\}$ with $\chi_2\chi_1^{-1} \neq \mathbf{1}, \omega^{\pm 1}$.
(iii) $p > 2$ and $\mathfrak{B} = \{(\mathrm{Ind}^G_B \chi \otimes \chi\omega^{-1})_{\mathrm{sm}}\}$.
(iv) $p \geq 5$ and $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}, (\mathrm{Ind}^G_B \omega \otimes \omega^{-1})_{\mathrm{sm}}\} \otimes \chi \circ \det$.
(v) $p = 3$ and $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}, \omega \circ \det, \mathrm{Sp} \otimes \omega \circ \det\} \otimes \chi \circ \det$.
(vi) $p = 2$ and $\mathfrak{B} = \{\mathbf{1}, \mathrm{Sp}\} \otimes \chi \circ \det$.

In these cases, $\chi, \chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ are smooth characters and $\omega : \mathbb{Q}_p^\times \to k^\times$ is the character $\omega(x) = x|x| \pmod{\varpi}$, and Sp is the Steinberg representation defined by the exact sequence

$$0 \to \mathbf{1} \to (\mathrm{Ind}^G_B \mathbf{1})_{\mathrm{sm}} \to \mathrm{Sp} \to 0.$$

If $\pi \in \mathrm{Irr}_{G,\zeta}$ is not absolutely irreducible then there is a finite extension k' of k, such that $\pi \otimes_k k'$ is a finite direct sum of absolutely irreducible representations, see [39, Proposition 5.11], so no information is lost by working with absolutely irreducible representations.

Given a block $\mathfrak{B}$, we denote by $\pi_{\mathfrak{B}}$ the direct sum of all representations in $\mathfrak{B}$ and let $P_{\mathfrak{B}}$ be a projective envelope of $\pi_{\mathfrak{B}}^\vee$ in $\mathfrak{C}(\mathcal{O})$. Then $E_{\mathfrak{B}} = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}})$ is a pseudocompact $\mathcal{O}$-algebra. We denote the centre of $E_{\mathfrak{B}}$ by $Z_{\mathfrak{B}}$.

By [39, Corollary 5.35], the category $\mathfrak{C}(\mathcal{O})$ decomposes into a direct product of subcategories

$$\mathfrak{C}(\mathcal{O}) \cong \prod_{\mathfrak{B} \in \mathrm{Irr}_{G,\zeta}/\sim} \mathfrak{C}(\mathcal{O})_{\mathfrak{B}}, \tag{11}$$

where the objects of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ are those M in $\mathfrak{C}(\mathcal{O})$ such that for every irreducible subquotient S of M, $S^\vee$ lies in $\mathfrak{B}$. Moreover, the category $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ is equivalent to the category of compact right $E_{\mathfrak{B}}$-modules and the centre of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ is isomorphic to $Z_{\mathfrak{B}}$ [39, Proposition 5.45].

Lemma 4.1. *If $\mathfrak{B}$ contains an absolutely irreducible representation, then $Z_{\mathfrak{B}}$ is a local pseudocompact $\mathcal{O}$-algebra with residue field k.*

Proof. If π is absolutely irreducible, then $\mathrm{End}_G(\pi) = k$, and thus the action of $Z_{\mathfrak{B}}$ on π defines a homomorphism of $\mathcal{O}$-algebras $c_\pi : Z_{\mathfrak{B}} \to k$. If $\pi, \pi' \in \mathfrak{B}$ are distinct and there is a nonsplit extension $0 \to \pi \to \tau \to \pi' \to 0$, then $\mathrm{End}_G(\tau) = k$, and we conclude that $c_\pi = c_\tau = c_{\pi'}$. Using the transitivity property of the relation $\sim$ on $\mathrm{Irr}_{G,\zeta}$, we conclude that $c_\pi = c_{\pi'}$ for all $\pi, \pi' \in \mathfrak{B}$. It follows from the proof of [31, Proposition IV.4.12] that the Jacobson radical of $Z_{\mathfrak{B}}$ consists of elements that kill all the irreducible representations. Thus $\mathrm{Ker}\, c_\pi$ is the maximal ideal of $Z_{\mathfrak{B}}$ with residue field k. The last assertion follows from the fact that $Z_{\mathfrak{B}}$ is closed in $E_{\mathfrak{B}}$ and [31, Proposition IV.4.13]. □

Lemma 4.2. *Let P be projective in $\mathfrak{C}(\mathcal{O})$. Then $P^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$.*

Proof. Let J be the Pontryagin dual of P. Then J is injective in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ and the assertion is equivalent to $J_{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. Let N be the unipotent subgroup $\left(\begin{smallmatrix} 1 & \mathbb{Q}_p \\ 0 & 1 \end{smallmatrix}\right)$. Then by [29, Proposition 3.6.2] and [30, Corollary 3.12], we have $J_N = 0$. Thus $J_{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$ and the lemma follows. Alternatively, one can deduce the statement from Proposition 4.3. □

Proposition 4.3. *Let P and M be in $\mathfrak{C}(\mathcal{O})$, such that P is projective and $M/\varpi M$ is of finite length. Then* $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M, P) = 0$.

Proof. Let K' be a compact open pro-p subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ such that $K' \cap Z = \{1\}$. Then P is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K'}(\mathcal{O})$ by [30, Corollary 3.10], and thus $P \cong \prod_{i \in I} \mathcal{O}[\![K']\!]$ for some index set I. Thus it is enough to show that $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K']\!]}(M, \mathcal{O}[\![K']\!]) = 0$. The topological Nakayama's lemma for compact $\mathcal{O}$-modules implies that it is enough to show that $\mathrm{Hom}^{\mathrm{cont}}_{k[\![K']\!]}(M/\varpi M, k[\![K']\!]) = 0$. Since $M/\varpi M$ is of finite length, it is enough to show that $\mathrm{Hom}_{k[\![K']\!]}(\pi^\vee, k[\![K']\!]) = 0$ for every $\pi \in \mathfrak{B}$. This follows from [42, Lemma 5.16]. Note that if $K_n = 1 + M_2(2p^n\mathbb{Z}_p)$ and $K'_n = K_n \cap \mathrm{SL}_2(\mathbb{Q}_p)$, then $K_n = K'_n(Z \cap K_n)$ and so $\pi^{K_n} = \pi^{K'_n}$ as $Z \cap K_n$ acts trivially on π; so the argument in [42, Lemma 5.16] carries over to the restriction of π to $\mathrm{SL}_2(\mathbb{Q}_p)$. □

We will denote by Ord_B Emerton's functor of ordinary parts [28].

Lemma 4.4. *Let $\pi \hookrightarrow J$ be an injective envelope of $\pi \in \mathfrak{B}$ in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$. If π is supersingular, then $\mathrm{Ord}_B J = 0$. If π is an irreducible subquotient of $(\mathrm{Ind}^G_{\overline{B}}\chi)_{\mathrm{sm}}$ for some character $\chi : T \to k^\times$, where $\overline{B}$ is the subgroup of lower triangular matrices, then $\mathrm{Ord}_B J$ is isomorphic to an injective envelope of χ in $\mathrm{Mod}^{\mathrm{l.fin}}_{T,\zeta}(\mathcal{O})$, and also in $\mathrm{Mod}^{\mathrm{sm}}_{T,\zeta}(\mathcal{O})$.*

Proof. Since J is injective and Ord_B is adjoint to parabolic induction, which is an exact functor, $\mathrm{Ord}_B J$ is injective in $\mathrm{Mod}^{\mathrm{l.fin}}_{T,\zeta}(\mathcal{O})$. Thus it is a direct sum of injective envelopes of characters of T. Moreover,

$$\mathrm{Hom}_T(\chi, \mathrm{Ord}_B J) \cong \mathrm{Hom}_G((\mathrm{Ind}^G_{\overline{B}}\chi)_{\mathrm{sm}}, J). \tag{12}$$

Since J is an injective envelope of π, this group is nonzero if and only if π is a subquotient of $(\mathrm{Ind}^G_{\overline{B}}\chi)_{\mathrm{sm}}$, in which case the dimension of the spaces in formula (12) is equal to the multiplicity with which π occurs as a subquotient of $(\mathrm{Ind}^G_{\overline{B}}\chi)_{\mathrm{sm}}$. If π is supersingular, then it does not occur as a subquotient of principal series, and thus $\mathrm{Ord}_B J = 0$. Otherwise, it follows from [2] that there is a unique character χ such that the multiplicity is nonzero, in which case it is equal to 1. This proves the first assertion. The same (albeit easier) proof as in [39, Proposition 5.16] implies that every injective object in $\mathrm{Mod}^{\mathrm{l.fin}}_{T,\zeta}(\mathcal{O})$ is also injective in $\mathrm{Mod}^{\mathrm{sm}}_{T,\zeta}(\mathcal{O})$. □

Lemma 4.5. *Let $\pi \hookrightarrow J$ be an injective envelope of $\pi \in \mathfrak{B}$ in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$. If π is not a character, then $J^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$; otherwise $(J^{\mathrm{SL}_2(\mathbb{Q}_p)})^\vee$ is nonzero and finitely generated over $\mathcal{O}$.*

Proof. If $J^{\mathrm{SL}_2(\mathbb{Q}_p)}$ is nonzero, then $\pi \cap J^{\mathrm{SL}_2(\mathbb{Q}_p)}$ is nonzero, as $\pi \hookrightarrow J$ is essential. Since π is absolutely irreducible, we deduce that it is a character and is the G-socle of $J^{\mathrm{SL}_2(\mathbb{Q}_p)}$. Note that the action of G on $J^{\mathrm{SL}_2(\mathbb{Q}_p)}$ factors through

$$G/Z\,\mathrm{SL}_2(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } p > 2, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2. \end{cases}$$

It follows that $J^{\mathrm{SL}_2(\mathbb{Q}_p)}[\varpi]$ is a finite-dimensional k-vector space. Dually, this implies that $(J^{\mathrm{SL}_2(\mathbb{Q}_p)})^\vee/\varpi(J^{\mathrm{SL}_2(\mathbb{Q}_p)})^\vee$ is finite-dimensional over k, and the lemma follows from Nakayama's lemma. □

Lemma 4.6. *Let J be injective in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ and $\kappa \in \mathrm{Mod}^{\mathrm{l.fin}}_{T,\zeta}(\mathcal{O})$ be such that $\kappa^\vee$ is a finitely generated $\mathcal{O}$-module. Then* $\mathrm{Hom}_T(\mathrm{Ord}_B J, \kappa) = 0$.

Proof. The following is an analogue of Proposition 4.3 in an easier setting. Let T_0 to be an open torsion-free pro-p subgroup of $T \cap \mathrm{SL}_2(\mathbb{Q}_p)$. Since $\mathrm{Ord}_B J$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_{T,\zeta}(\mathcal{O})$ by Lemma 4.4, it is injective in $\mathrm{Mod}^{\mathrm{sm}}_{T_0}(\mathcal{O})$, as restriction to T_0 is adjoint to $\text{c-Ind}^{T}_{ZT_0}$, which is exact. Thus the Pontryagin dual of $\mathrm{Ord}_B J$ is isomorphic to $\prod_{i\in I}\mathcal{O}[\![T_0]\!]$ for some index set I. Hence, it suffices to show that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![T_0]\!]}(\kappa^\vee, \mathcal{O}[\![T_0]\!]) = 0.$$

This is clear, as $\mathcal{O}[\![T_0]\!]$ is isomorphic to the ring of formal power series in one variable and $\kappa^\vee$ is a finitely generated $\mathcal{O}$-module. □

Lemma 4.7. *Let τ be a smooth representation of G whose irreducible subquotients consist of characters in $\mathfrak{B}$. Then $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on τ and there is an exact sequence*

$$0 \to \tau \to (\mathrm{Ind}^G_B\,\tau)_{\mathrm{sm}} \to Q \to 0. \tag{13}$$

Moreover, the irreducible subquotients of Q are twists of Sp *by a character. In particular,* $\mathrm{Hom}_G(\tau, Q) = 0$.

Proof. Since $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on τ by [47, Lemma 1.2.1], the unipotent radical of B acts trivially on τ. Thus $(\mathrm{Ind}^G_B\,\tau|_B)_{\mathrm{sm}}$ coincides with the parabolic induction. Since the map $\tau \to (\mathrm{Ind}^G_B\,\tau)_{\mathrm{sm}}$ defined by $v \mapsto (g \mapsto gv)$ is G-equivariant and injective, we obtain the first assertion.

To show the second assertion, we choose an increasing and exhaustive filtration $\{R^j\}_{j\geq 0}$ of τ such that R^j/R^{j-1} is a character for each j. Then we have $R^j/R^{j-1} \hookrightarrow (\mathrm{Ind}^G_B\,R^j/R^{j-1})_{\mathrm{sm}}$ with quotient isomorphic to a twist of Sp by a character. Thus the second assertion follows from the exactness of $(\mathrm{Ind}^G_B\,-)_{\mathrm{sm}}$. □

Lemma 4.8. *Let J be injective in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$ and let τ and Q be as in Lemma 4.7. If $\tau^\vee$ is finitely generated over $\mathcal{O}$, then*

$$\mathrm{Hom}_G(J,\tau) = \mathrm{Hom}_G(J, (\mathrm{Ind}^G_B\,\tau)_{\mathrm{sm}}) = \mathrm{Hom}_G(J, Q) = 0.$$

Proof. Since $\tau^\vee$ is finitely generated over $\mathcal{O}$, then $\tau[\varpi]$ is a finite k-vector space, and thus

$$(\mathrm{Ind}^G_B\,\tau)^\vee_{\mathrm{sm}}/\varpi \cong (\mathrm{Ind}^G_B\,\tau[\varpi])^\vee_{\mathrm{sm}}$$

is of finite length in $\mathfrak{C}(\mathcal{O})$ and the assertion follows from Proposition 4.3. □

Proposition 4.9. *Let J and τ be as in Lemma 4.8. Then $\mathrm{Ext}^1_{G,\zeta}(J,\tau) = 0$.*

Proof. Consider an exact sequence $0 \to \tau \to I \to J \to 0$. Applying Ord_B to it, we get an exact sequence

$$0 \to \mathrm{Ord}_B\,\tau \to \mathrm{Ord}_B\,I \to \mathrm{Ord}_B\,J \to \mathbb{R}^1\,\mathrm{Ord}_B\,\tau \to \mathbb{R}^1\,\mathrm{Ord}_B\,I \to \mathbb{R}^1\,\mathrm{Ord}_B\,J$$

of smooth T-representations. It is proved in [30] that the functors $H^i\,\mathrm{Ord}_B$ in [29] coincide with the derived functors $\mathbb{R}^i\,\mathrm{Ord}_B$. Moreover, $H^1\,\mathrm{Ord}_B$ coincides with the N-coinvariants twisted by the character α^{-1}, where $\alpha\left(\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right)\right) = \omega(ad^{-1})$, by [29, Proposition 3.6.2].

Since $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on τ, we have $\mathrm{Ord}_B\,\tau = 0$ and $\mathbb{R}^1\,\mathrm{Ord}_B\,\tau \cong \tau\otimes\alpha^{-1}$. Thus the exact sequence reduces to

$$0 \to \mathrm{Ord}_B\,I \to \mathrm{Ord}_B\,J \to \tau\otimes\alpha^{-1} \to I_N\otimes\alpha^{-1} \to 0. \tag{14}$$

Since the middle map is zero by Lemma 4.6, the map $\tau \hookrightarrow I$ induces an isomorphism $\tau \cong I_N$ of T-representations and hence

$$\mathrm{Hom}_G(I, (\mathrm{Ind}^G_B\,\tau)_{\mathrm{sm}}) \cong \mathrm{Hom}_G(\tau, (\mathrm{Ind}^G_B\,\tau)_{\mathrm{sm}}) \tag{15}$$

by the adjunction formula.

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It follows from Lemma 4.8 that the first three terms in the long exact sequence obtained by applying $\mathrm{Hom}_G(J, -)$ to formula (13) are zero. Thus by applying $\mathrm{Hom}_G(I, -) \to \mathrm{Hom}_G(\tau, -)$ to formula (13), we obtain the following commutative diagram with exact columns:

$$\begin{array}{ccc} \mathrm{Hom}_G(I, \tau) & \hookrightarrow & \mathrm{Hom}_G(\tau, \tau) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_G(I, (\mathrm{Ind}_B^G \tau)_{\mathrm{sm}}) & \xrightarrow[(15)]{\sim} & \mathrm{Hom}_G(\tau, (\mathrm{Ind}_B^G \tau)_{\mathrm{sm}}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_G(I, Q) & \hookrightarrow & \mathrm{Hom}_G(\tau, Q). \end{array}$$

The last part of Lemma 4.7 says that $\mathrm{Hom}_G(\tau, Q) = 0$ and hence $\mathrm{Hom}_G(I, Q) = 0$. Thus all the maps in the top square of the diagram are isomorphisms. The preimage of $\mathrm{id}_\tau \in \mathrm{Hom}_G(\tau, \tau)$ in $\mathrm{Hom}_G(I, \tau)$ splits the exact sequence $0 \to \tau \to I \to J \to 0$. This proves the proposition. □

4.2. Quotient category

Let $\mathfrak{T}(\mathcal{O})$ be the full subcategory of $\mathfrak{C}(\mathcal{O})$ whose objects have trivial $\mathrm{SL}_2(\mathbb{Q}_p)$-action. By [39, Lemma 10.25] (for $p > 2$) and [47, Lemma 1.2.1] (for $p = 2$), $\mathfrak{T}(\mathcal{O})$ is a thick subcategory of $\mathfrak{C}(\mathcal{O})$ and hence we may consider the quotient category $\mathfrak{Q}(\mathcal{O}) := \mathfrak{C}(\mathcal{O})/\mathfrak{T}(\mathcal{O})$. Let $\mathcal{T} : \mathfrak{C}(\mathcal{O}) \to \mathfrak{Q}(\mathcal{O})$ be the quotient functor; we note that $\mathcal{T}$ is the identity on objects. It is shown in [39, Section 10.3] that $\mathfrak{Q}(\mathcal{O})$ is an abelian category with enough projectives and $\mathcal{T}$ is an exact functor.

For a block $\mathfrak{B}$, we denote

$$P'_{\mathfrak{B}} = \bigoplus_{\substack{\pi \in \mathfrak{B} \\ \pi^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0}} P_{\pi^\vee}, \qquad E'_{\mathfrak{B}} = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P'_{\mathfrak{B}}), \qquad Z'_{\mathfrak{B}} = Z(E'_{\mathfrak{B}}),$$

where $P_{\pi^\vee}$ is a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$.

Proposition 4.10. *Let $\mathfrak{B}$ be a block.*

1. *$\mathcal{T} P'_{\mathfrak{B}}$ is a projective object of $\mathfrak{Q}(\mathcal{O})$ and $E'_{\mathfrak{B}} \cong \mathrm{End}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} P'_{\mathfrak{B}})$.*
2. *The functor $M \mapsto \mathrm{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} P'_{\mathfrak{B}}, M)$ defines an equivalence of categories between $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ and the category of pseudocompact right $E'_{\mathfrak{B}}$-modules, with the inverse given by $\mathrm{m} \mapsto \mathrm{m} \widehat{\otimes}_{E'_{\mathfrak{B}}} \mathcal{T} P'_{\mathfrak{B}}$.*

Proof. See [39, Lemma 10.27] for the first assertion and [31, Section IV.4, Theorem 4, Corollaries 1 and 5] for the second assertion. □

4.3. The centre

In this section we will prove key results toward showing that $E_{\mathfrak{B}}$ and $Z_{\mathfrak{B}}$ are finite over $Z'_{\mathfrak{B}}$. If $\mathfrak{B}$ is of type (i), (ii) or (iii), then we have $P_{\mathfrak{B}} = P'_{\mathfrak{B}}$ and thus $Z_{\mathfrak{B}} = Z'_{\mathfrak{B}}$.

Lemma 4.11. *Let E be a ring with centre Z and let m be a finitely generated (right) E-module. Assume that Z is Noetherian and E is (module) finite over Z. Then $\mathrm{End}_E(\mathrm{m})$ and its centre $Z(\mathrm{End}_E(\mathrm{m}))$ are Noetherian and finite over Z.*

Proof. Since m is finitely generated over E, there is a surjection $E^{\oplus n} \twoheadrightarrow \mathrm{m}$ for some n. This induces an injection $\mathrm{End}_E(\mathrm{m}) \hookrightarrow \mathrm{Hom}_E(E^{\oplus n}, \mathrm{m})$ and a surjection $M_n(E^{\mathrm{op}}) \cong \mathrm{End}_E(E^{\oplus n}) \twoheadrightarrow \mathrm{Hom}_E(E^{\oplus n}, \mathrm{m})$ of Z-modules. Since $M_n(E^{\mathrm{op}})$ is finite over E^{op}, it is finite and Noetherian over Z. It follows that $\mathrm{Hom}_E(E^{\oplus n}, \mathrm{m})$ and thus $\mathrm{End}_E(\mathrm{m})$, which can be identified with a Z-submodule of $\mathrm{Hom}_E(E^{\oplus n}, \mathrm{m})$, are finite and Noetherian over Z. This proves the lemma. □

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Lemma 4.12. *Let* $M_{\mathfrak{B}} = \ker(P_{\mathfrak{B}} \to (P_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)})$*. Then the following hold:*

1. $(M_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$.
2. *For all* $\pi \in \mathfrak{B}$, $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, \pi^\vee)$ *is finite-dimensional over* k.

Proof. It suffices to consider the block $\mathfrak{B}$ of type (iv), (v) or (vi); otherwise $M_{\mathfrak{B}} = P_{\mathfrak{B}}$ and the assertion is trivial. Let $J_{\mathfrak{B}}$ be the Pontryagin dual of $P_{\mathfrak{B}}$. Then we have an exact sequence $0 \to J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)} \to J_{\mathfrak{B}} \to M_{\mathfrak{B}}^\vee \to 0$. By applying $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi, -)$, we get a long exact sequence

$$0 \to \mathrm{Hom}_G(\pi, J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)}) \to \mathrm{Hom}_G(\pi, J_{\mathfrak{B}}) \to \mathrm{Hom}_G(\pi, M_{\mathfrak{B}}^\vee)$$
$$\to \mathrm{Ext}^1_{G,\zeta}(\pi, J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)}) \to \mathrm{Ext}^1_{G,\zeta}(\pi, J_{\mathfrak{B}}) = 0,$$

where $\mathrm{Ext}^1_{G,\zeta}$ is the extension group in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$.

If $\pi \in \mathfrak{B}$ is a character, then both $\mathrm{Hom}_G(\pi, J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)})$ and $\mathrm{Hom}_G(\pi, J_{\mathfrak{B}})$ are 1-dimensional over k, and $\mathrm{Ext}^1_{G,\zeta}(\pi, J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)}) = 0$ by [47, Lemma 1.2.1]. Thus $\mathrm{Hom}_G(\pi, M_{\mathfrak{B}}^\vee) = 0$ and the first assertion follows (see [20, Lemma III.40] for another proof).

If $\pi \in \mathfrak{B}$ is not a character, then $\mathrm{Hom}_G(\pi, J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)}) = 0$ and $\mathrm{Hom}_G(\pi, J_{\mathfrak{B}})$ is 1-dimensional over k. Since π is killed by ϖ, for the second assertion it is enough to show that $\mathrm{Ext}^1_{G,\zeta}(\pi, (J_{\mathfrak{B}}[\varpi])^{\mathrm{SL}_2(\mathbb{Q}_p)})$ is finite-dimensional over k. This holds, because $(J_{\mathfrak{B}}[\varpi])^{\mathrm{SL}_2(\mathbb{Q}_p)}$ is of finite length by Lemma 4.5, and if χ is a character, then $\mathrm{Ext}^1_{G,\zeta}(\pi, \chi \circ \det)$ is finite-dimensional. □

Corollary 4.13. *There is a surjection* $(P'_{\mathfrak{B}})^{\oplus n} \twoheadrightarrow M_{\mathfrak{B}}$ *for some* $n \in \mathbb{N}$.

Proof. This follows from Lemma 4.12, which shows that the cosocle of $M_{\mathfrak{B}}$ contains no characters and each irreducible representation in $\mathfrak{B}$ appears with finite multiplicity. □

Lemma 4.14. *Both* $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}/M_{\mathfrak{B}}, P_{\mathfrak{B}})$ *and* $\mathrm{Ext}^1_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}/M_{\mathfrak{B}}, P_{\mathfrak{B}})$ *are equal to zero.*

Proof. The first assertion follows because $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on $P_{\mathfrak{B}}/M_{\mathfrak{B}}$ and $P_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$ (see Lemma 4.2), and the second assertion follows from Proposition 4.9 applied to $J = P_{\mathfrak{B}}^\vee$ and $\tau = J_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)}$. It follows from Lemma 4.5 that $\tau^\vee$ is a finitely generated $\mathcal{O}$-module. □

Proposition 4.15. *There is a natural isomorphism* $E_{\mathfrak{B}} \cong \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})$*. In particular,* $Z_{\mathfrak{B}} \cong Z(\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}))$.

Proof. Set $\phi \in E_{\mathfrak{B}} = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}})$. Then the composition $M_{\mathfrak{B}} \xrightarrow{\phi} P_{\mathfrak{B}} \twoheadrightarrow P_{\mathfrak{B}}/M_{\mathfrak{B}}$ is the zero map, since $\mathrm{SL}_2(\mathbb{Q}_p)$ acts trivially on $P_{\mathfrak{B}}/M_{\mathfrak{B}} \cong (P_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)}$ and $(M_{\mathfrak{B}})_{\mathrm{SL}(\mathbb{Q}_p)} = 0$ by Lemma 4.12. Thus ϕ maps $M_{\mathfrak{B}}$ to $M_{\mathfrak{B}}$, and restriction to $M_{\mathfrak{B}}$ induces a ring homomorphism $E_{\mathfrak{B}} \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})$.

Applying the functor $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, -)$ to the exact sequence $0 \to M_{\mathfrak{B}} \to P_{\mathfrak{B}} \to P_{\mathfrak{B}}/M_{\mathfrak{B}} \to 0$, we get the exact sequence

$$0 \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, P_{\mathfrak{B}}) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, P_{\mathfrak{B}}/M_{\mathfrak{B}}).$$

Since the last term is equal to zero by Lemma 4.12, we obtain $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, P_{\mathfrak{B}})$. On the other hand, by applying the functor $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(-, P_{\mathfrak{B}})$ to the same short exact sequence, we get the exact sequence

$$0 \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}/M_{\mathfrak{B}}, P_{\mathfrak{B}}) \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}) \to \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, P_{\mathfrak{B}})$$
$$\to \mathrm{Ext}^1_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}/M_{\mathfrak{B}}, P_{\mathfrak{B}}).$$

Since $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}/M_{\mathfrak{B}}, P_{\mathfrak{B}}) = 0$ and $\mathrm{Ext}^1_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}/M_{\mathfrak{B}}, P_{\mathfrak{B}}) = 0$ by Lemma 4.14, we deduce $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}) \cong \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}, P_{\mathfrak{B}})$ and the proposition follows. □

Corollary 4.16. *There is a natural surjective homomorphism*

$$Z'_{\mathfrak{B}} \to Z(\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})) \cong Z_{\mathfrak{B}}.$$

Proof. Note that we have $(M_{\mathfrak{B}})^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$ by Lemma 4.2 and $(M_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$ by Lemma 4.12. It follows from [39, Lemma 10.26] that the functor $\mathcal{T}$ induces an isomorphism

$$\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}) \cong \mathrm{End}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} M_{\mathfrak{B}}), \quad \varphi \mapsto \mathcal{T}\varphi.$$

Since $Z'_{\mathfrak{B}}$ is the centre of $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$, it acts on $\mathcal{T} M_{\mathfrak{B}}$, and this action induces a homomorphism $Z'_{\mathfrak{B}} \to Z(\mathrm{End}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} M_{\mathfrak{B}}))$, which we may compose with the previous isomorphism to obtain a homomorphism $Z'_{\mathfrak{B}} \to Z(\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}))$. Since $Z_{\mathfrak{B}}$ is the centre of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, a similar argument shows that $Z'_{\mathfrak{B}}$ is a $Z_{\mathfrak{B}}$-algebra and the surjection $(P'_{\mathfrak{B}})^{\oplus n} \twoheadrightarrow M_{\mathfrak{B}}$ in Corollary 4.13 is $Z_{\mathfrak{B}}$-equivariant. It induces a $Z_{\mathfrak{B}}$-equivariant surjection $(\mathcal{T} P'_{\mathfrak{B}})^{\oplus n} \twoheadrightarrow \mathcal{T} M_{\mathfrak{B}}$. We deduce that the map

$$Z'_{\mathfrak{B}} \to Z(\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}))$$

is a homomorphism of $Z_{\mathfrak{B}}$-algebras. Proposition 4.15 implies that the composition

$$Z_{\mathfrak{B}} \to Z'_{\mathfrak{B}} \to Z(\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})) \cong Z_{\mathfrak{B}}$$

is the identity map, which implies that the homomorphism is surjective. □

4.4. Colmez's Montreal functor

Let $\mathcal{G}_{\mathbb{Q}_p}$ be the absolute Galois group of $\mathbb{Q}_p$. We will consider ζ as a character of $\mathcal{G}_{\mathbb{Q}_p}$ via local class field theory, normalised so that the uniformisers correspond to geometric Frobenius. Let $\varepsilon : \mathcal{G}_{\mathbb{Q}_p} \to \mathbb{Z}_p^{\times}$ be the p-adic cyclotomic character.

Colmez [19] has defined an exact and covariant functor $\mathbf{V}$ from the category of smooth, finite-length representations of G on $\mathcal{O}$-torsion modules with a central character to the category of continuous finite-length representations of $\mathcal{G}_{\mathbb{Q}_p}$ on $\mathcal{O}$-torsion modules. This functor is modified in [39, Section 5.7] to an exact covariant functor

$$\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \to \mathrm{Mod}^{\mathrm{pro}}_{\mathcal{G}_{\mathbb{Q}_p}}(\mathcal{O})$$

as follows. Let M be in $\mathfrak{C}(\mathcal{O})$; if it is of finite length, we define $\check{\mathbf{V}}(M) := \mathbf{V}(M^{\vee})^{\vee}(\zeta\varepsilon)$, where $\vee$ denotes the Pontryagin dual. For general $M \in \mathfrak{C}(\mathcal{O})$, we may write $M \cong \varprojlim M_i$, with M_i of finite length in $\mathfrak{C}(\mathcal{O})$, and define $\check{\mathbf{V}}(M) := \varprojlim \check{\mathbf{V}}(M_i)$. With this normalisation, we have the following:

- $\check{\mathbf{V}}(\pi^{\vee}) = 0$ if $\pi \cong \chi \circ \det$.
- $\check{\mathbf{V}}(\pi^{\vee}) = \chi_1$ if $\pi \cong (\mathrm{Ind}_B^G \chi_1 \otimes \chi_2)_{\mathrm{sm}}$.
- $\check{\mathbf{V}}(\pi^{\vee}) = \chi$ if $\pi \cong \mathrm{Sp} \otimes \chi \circ \det$.
- $\check{\mathbf{V}}(\pi^{\vee}) = \mathbf{V}(\pi)$ is a 2-dimensional absolutely irreducible Galois representation if π is supersingular.

The functor $\check{\mathbf{V}}$ induces a bijection $\mathfrak{B} \mapsto \bar{\rho}_{\mathfrak{B}}$ between blocks containing an absolutely irreducible representation and equivalence classes of semisimple representations $\bar{\rho} : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ such that all irreducible summands of $\bar{\rho}$ are absolutely irreducible. The representation $\bar{\rho}_{\mathfrak{B}}$ can be described explicitly according to the classification of blocks given in Section 4.1: in case (i), $\bar{\rho}_{\mathfrak{B}} = \check{\mathbf{V}}(\pi^{\vee})$ is absolutely irreducible; in case (ii), $\bar{\rho}_{\mathfrak{B}} = \chi_1 \oplus \chi_2$; in cases (iii) and (vi), $\bar{\rho}_{\mathfrak{B}} = \chi \oplus \chi$; and in cases (iv) and (v), $\bar{\rho}_{\mathfrak{B}} = \chi \oplus \chi\omega$.

Since the functor $\check{\mathbf{V}} : \mathfrak{C}(\mathcal{O}) \to \mathrm{Mod}^{\mathrm{pro}}_{\mathcal{G}_{\mathbb{Q}_p}}(\mathcal{O})$ kills characters and hence every object in $\mathfrak{T}(\mathcal{O})$, it factors through $\mathcal{T} : \mathfrak{C}(\mathcal{O}) \to \mathfrak{Q}(\mathcal{O})$. We denote $\check{\mathbf{V}} : \mathfrak{Q}(\mathcal{O}) \to \mathrm{Mod}^{\mathrm{pro}}_{\mathcal{G}_{\mathbb{Q}_p}}(\mathcal{O})$ by the same letter.

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Proposition 4.17. *For each block $\mathfrak{B}$, the functor $\check{\mathbf{V}}$ induces an equivalence of categories between $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ and its essential image in $\mathrm{Mod}^{\mathrm{pro}}_{\mathscr{G}_{\mathbb{Q}_p}}(\mathcal{O})$.*

Proof. This is due to [39] for cases (i)–(iv), [46, Proposition 2.8] for case (v) and [47, Proposition 1.3.2] for case (iv). □

Proposition 4.18. *The map $\mathcal{O}[\![\mathscr{G}_{\mathbb{Q}_p}]\!] \to \mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}}))$ is surjective. Moreover, if $\mathfrak{B}$ is supersingular, then $\mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})) \cong M_2((E'_{\mathfrak{B}})^{\mathrm{op}})$; otherwise, $\mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})) \cong (E'_{\mathfrak{B}})^{\mathrm{op}}$.*

Proof. It suffices to show that $M = \check{\mathbf{V}}(P'_{\mathfrak{B}})$ satisfies all four conditions in Proposition 2.3. We note that the functor $\mathrm{m} \mapsto \mathrm{m}\,\widehat{\otimes}_{E'_{\mathfrak{B}}} \check{\mathbf{V}}(P'_{\mathfrak{B}})$ is exact by the equivalence of categories in Proposition 4.10(2), the exactness of $\check{\mathbf{V}}$ and the isomorphism

$$\check{\mathbf{V}}(\mathrm{m}\,\widehat{\otimes}_{E'_{\mathfrak{B}}} P'_{\mathfrak{B}}) \cong \mathrm{m}\,\widehat{\otimes}_{E'_{\mathfrak{B}}} \check{\mathbf{V}}(P'_{\mathfrak{B}})$$

(see the proof of [39, Lemma 5.53]). Thus by Lemma 2.4, $\check{\mathbf{V}}(P'_{\mathfrak{B}})$ is a projective $E'_{\mathfrak{B}}$-module. Let $\mathfrak{r}$ be the Jacobson radical of $E'_{\mathfrak{B}}$. Since

$$(E'_{\mathfrak{B}}/\mathfrak{r})\,\widehat{\otimes}_{E'_{\mathfrak{B}}} \check{\mathbf{V}}\,(P'_{\mathfrak{B}}) \cong \check{\mathbf{V}}(\,(E'_{\mathfrak{B}}/\mathfrak{r})\,\widehat{\otimes}_{E'_{\mathfrak{B}}} P'_{\mathfrak{B}})$$

is either a 1- or a 2-dimensional k-vector space, the topological Nakayama's lemma implies that $\check{\mathbf{V}}(P'_{\mathfrak{B}})$ is a finitely generated $E'_{\mathfrak{B}}$-module, and thus Proposition 2.3(1) holds. Proposition 4.17 implies that Proposition 2.3(2) holds.

If m is a right pseudocompact $E'_{\mathfrak{B}}$-module then [12, Lemma 2.4] implies that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\mathrm{m}\,\widehat{\otimes}_{E'_{\mathfrak{B}}} \check{\mathbf{V}}(P'_{\mathfrak{B}}), k) \cong \mathrm{Hom}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}}), \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\mathrm{m}, k)). \tag{16}$$

Since in our situation $E'_{\mathfrak{B}}$ is a compact $\mathcal{O}$-algebra, the irreducible (left or right) $E'_{\mathfrak{B}}$-modules are finite-dimensional vector spaces with discrete topology, and thus the map $\mathrm{m} \mapsto \mathrm{m}^* := \mathrm{Hom}_k(\mathrm{m}, k)$ induces a bijection between irreducible left and irreducible right $E'_{\mathfrak{B}}$-modules. Moreover, if m is an irreducible right $E'_{\mathfrak{B}}$-module, then it follows from formula (16) that $\rho_{\mathrm{m}^*} \cong (\mathrm{m}\,\widehat{\otimes}_{E'_{\mathfrak{B}}} \check{\mathbf{V}}(P'_{\mathfrak{B}}))^*$; thus Proposition 2.3(3) and (4) follow from Proposition 4.17.

If $\mathfrak{B}$ is supersingular, then it contains only one irreducible π, which is not a character. Thus $P_{\mathfrak{B}} = P'_{\mathfrak{B}}$, $E_{\mathfrak{B}} = E'_{\mathfrak{B}}$ is a local ring and $k\,\widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}} = \pi^{\vee}$. Thus $k\,\widehat{\otimes}_{E_{\mathfrak{B}}} \check{\mathbf{V}}(P_{\mathfrak{B}}) \cong \check{\mathbf{V}}(\pi^{\vee}) \cong \bar{\rho}_{\mathfrak{B}}$ is an absolutely irreducible 2-dimensional representation. Since $E_{\mathfrak{B}}$ is a local ring, we deduce that $\check{\mathbf{V}}(P_{\mathfrak{B}})$ is a free $E_{\mathfrak{B}}$-module of rank 2. Thus $\mathrm{End}^{\mathrm{cont}}_{E_{\mathfrak{B}}}(\check{\mathbf{V}}(P_{\mathfrak{B}})) \cong M_2((E'_{\mathfrak{B}})^{\mathrm{op}})$.

If $\mathfrak{B}$ is of type (iii) or (vi), then the block in the quotient category contains only one irreducible object, and Colmez's functor maps it to a 1-dimensional $\mathscr{G}_{\mathbb{Q}_p}$-representation. The same argument as in the supersingular case shows that $E'_{\mathfrak{B}}$ is a local ring and $\check{\mathbf{V}}(P'_{\mathfrak{B}})$ is a free $E'_{\mathfrak{B}}$-module of rank 1, and thus $\mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})) = (E'_{\mathfrak{B}})^{\mathrm{op}}$.

If $\mathfrak{B}$ is of type (ii), (iv) or (v), then $\mathfrak{Q}(\mathcal{O})_{\mathfrak{B}}$ contains exactly two irreducible objects, and Colmez's functor sends them to distinct 1-dimensional $\mathscr{G}_{\mathbb{Q}_p}$-representations χ_1, χ_2. It follows from Corollary 2.5 that $\check{\mathbf{V}}(P'_{\mathfrak{B}})$ is a free $E'_{\mathfrak{B}}$-module of rank 1, and thus $\mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})) = (E'_{\mathfrak{B}})^{\mathrm{op}}$. □

4.5. Banach-space representations

Let $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ be the category of admissible unitary L-Banach-space representations [44, Section 3] on which Z acts by the character ζ. We note that $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ is an abelian category [44, Theorem 3.5]. Any $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ has an open, bounded and G-invariant lattice Θ, and $\Theta \otimes_{\mathcal{O}} k$ is an admissible smooth

k-representation of G. Let $\Theta^d = \mathrm{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$ be the Schikhof dual of Θ endowed with the topology of pointwise convergence. Then Θ^d is an object of $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$ [39, Lemma 4.4]. If Θ^d is in $\mathfrak{C}(\mathcal{O})$, then Ξ^d is in $\mathfrak{C}(\mathcal{O})$ for every open bounded G-invariant lattice Ξ in Π, since Θ and Ξ are commensurable and $\mathfrak{C}(\mathcal{O})$ is closed under subquotients [39, Lemma 4.6].

If $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$, then we let

$$\check{\mathbf{V}}(\Pi) = \check{\mathbf{V}}(\Theta^d) \otimes_{\mathcal{O}} L,$$

where Θ is any open bounded G-invariant lattice in Π. Then $\check{\mathbf{V}}$ is exact and contravariant on $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$.

5. Density

5.1. Capture

Let $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and $K = \mathrm{GL}_2(\mathbb{Z}_p)$. Write Z for the centre of G and $Z(K)$ for the centre of K. Let $\psi : Z(K) \to \mathcal{O}^\times$ be a continuous character. We identify Z with $\mathbb{Q}_p^\times$ and $Z(K)$ with $\mathbb{Z}_p^\times$ via the map $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \mapsto x$.

Lemma 5.1. *Let $\{V_i\}_{i\in I}$ be a family of continuous representations of K on finite-dimensional L-vector spaces with central character ψ, and let $M \in \mathrm{Mod}^{\mathrm{pro}}_{K,\psi}(\mathcal{O})$ be $\mathcal{O}$-torsion-free. The following conditions are equivalent:*

1. *For all $i \in I$, the smallest quotient $M \twoheadrightarrow Q$ such that $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(Q, V_i^*) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$ is equal to M.*
2. *The intersection of the kernels of all $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(M, V_i^*)$ for each $i \in I$ is equal to zero.*
3. *The image of the evaluation map*

$$\bigoplus_{i\in I} \mathrm{Hom}_K(V_i, \Pi(M)) \otimes_L V_i \to \Pi(M)$$

is a dense subspace, where $\Pi(M) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L)$ is an L-Banach space equipped with the supremum norm.

Proof. See [21, Lemmas 2.7 and 2.10]. □

Definition 5.2. We say that $\{V_i\}_{i\in I}$ *captures* M if it satisfies one of the equivalent conditions in Lemma 5.1.

Since $1 + p\mathbb{Z}_p$ (resp., $1 + 4\mathbb{Z}_2$) is a free pro-p group of rank 1 if $p > 2$ (resp., $p = 2$), there are a smooth nontrivial character $\chi : \mathbb{Z}_p^\times \to L^\times$ and a continuous character $\eta_0 : \mathbb{Z}_p^\times \to L^\times$ such that $\psi = \chi\eta_0^2$. Let e be the smallest integer such that χ is trivial on $1 + p^e\mathbb{Z}_p$. Let

$$J = \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p^e\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$$

and let $\chi \otimes \mathbf{1} : J \to L^\times$ be the character which sends $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $\chi(a)$. Then the representation $\tau = \mathrm{Ind}_J^K(\chi \otimes \mathbf{1})$ is a principal series type. That is, for an irreducible smooth $\overline{L}$-representation π of G, we have $\mathrm{Hom}_{L[K]}(\tau, \pi) \neq 0$ if and only if $\pi \cong (\mathrm{Ind}_B^G \psi_1 \otimes \psi_2)_{\mathrm{sm}}$, where B is a Borel subgroup and $\psi_1|_{\mathbb{Z}_p^\times} = \chi$ and $\psi_2|_{\mathbb{Z}_p^\times} = \mathbf{1}$ [11, Section A2.2].

Proposition 5.3. *The family*

$$\{\mathrm{Ind}_J^K(\chi \otimes \mathbf{1}) \otimes \mathrm{Sym}^{2a} L^2 \otimes (\det)^{-a} \otimes \eta\eta_0 \circ \det\}_{a\in\mathbb{N},\eta},$$

where η runs over all the characters with $\eta^2 = \mathbf{1}$, captures every projective object in $\mathrm{Mod}^{\mathrm{pro}}_{K,\psi^{-1}}(\mathcal{O})$.

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Proof. See [41, Proposition 2.7]. □

We will denote the family of representations in this proposition by $\{V_i\}_{i\in I}$. We note that each V_i is a twist of a locally algebraic representation by a unitary character η_0, which might not be locally algebraic. However, twisting by its inverse will get us to a locally algebraic situation, which is sufficient for all arguments that follow.

5.2. Locally algebraic vectors in $\Pi(P)$

Let $\zeta : Z \to \mathcal{O}^\times$ be a continuous character and $\psi = \zeta|_K$. Let P be a projective object in $\mathfrak{C}(\mathcal{O})$ and $E = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P)$. In particular, P is a torsion-free compact linear-topological $\mathcal{O}$-module. Define

$$\Pi(P) := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(P, L)$$

with the topology induced by the supremum norm. Then we have $E[1/p] \cong \mathrm{End}_G^{\mathrm{cont}}(\Pi(P))$.

If V is a continuous representation of K on a finite-dimensional L-vector space, then

$$\mathrm{Hom}_K(V, \Pi(P)) \cong \mathrm{Hom}_{\mathcal{O}[\![K]\!]}^{\mathrm{cont}}(P, V^*). \tag{17}$$

Since P is projective in $\mathrm{Mod}^{\mathrm{pro}}_{K,\psi^{-1}}(\mathcal{O})$ by [30, Corollary 3.10], the family of finite-dimensional K-representations associated to ψ in Proposition 5.3, which we denote by $\{V_i\}_{i\in I}$, captures P. We view V_i as a representation of KZ by letting $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ act by $\zeta(\varpi)$.

Proposition 5.4. *For each $i \in I$, we define $A_i := \mathrm{End}_G(\text{c-Ind}_{KZ}^G V_i)$. Then*

1. *A_i is isomorphic to $L[T]$ and*
2. *$\text{c-Ind}_{KZ}^G V_i$ is flat over A_i.*

Proof. We may write $V_i = \mathrm{Ind}_J^K(\chi \otimes \mathbf{1}) \otimes_L W_i$, where the action of KZ on W_i extends to an action of G (see Proposition 5.3). Then

$$\text{c-Ind}_{KZ}^G V_i \cong \text{c-Ind}_{KZ}^G(\mathrm{Ind}_J^K(\chi \otimes \mathbf{1})) \otimes_L W_i.$$

Here we view $\mathrm{Ind}_J^K(\chi \otimes \mathbf{1})$ as a representation of KZ by letting $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ act on $\mathrm{Ind}_J^K(\chi \otimes \mathbf{1})$ by $\zeta(\varpi)\zeta_{W_i}^{-1}(\varpi)$, where ζ_{W_i} is the central character of W_i. Since the restriction of W_i to any compact open subgroup of G remains absolutely irreducible, the isomorphism induces an isomorphism of L-algebras

$$\mathrm{End}_G(\text{c-Ind}_{KZ}^G V_i) \cong \mathrm{End}_G(\text{c-Ind}_{KZ}^G(\mathrm{Ind}_J^K(\chi \otimes \mathbf{1}))).$$

Thus we may assume that W_i is the trivial representation.

By [14], the K-type $\mathrm{Ind}_J^K(\chi \otimes \mathbf{1})$ is a G-cover of the K_M-type $\chi \otimes \mathbf{1}$, where $M = \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ and $K_M = \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$. Thus there is an algebra isomorphism

$$j_M : \mathrm{End}_M(\text{c-Ind}_{K_M}^M(\chi \otimes \mathbf{1})) \xrightarrow{\sim} \mathrm{End}_G(\text{c-Ind}_J^G(\chi \otimes \mathbf{1}))$$

such that for each $f \in \mathrm{End}_M(\text{c-Ind}_{K_M}^M \chi \otimes \mathbf{1})$, we have $\mathrm{Supp}(j_M f) = J\,\mathrm{Supp}(f) J$. It follows that

$$L[T] \xrightarrow{\sim} \mathrm{End}_M(\text{c-Ind}_{K_M Z}^M(\chi \otimes \mathbf{1})) \xrightarrow[j_M]{\sim} A_i,$$

where T maps to an element in A_i supported at $JZ\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}JZ$. Here we view $\chi \otimes \mathbf{1}$ as a representation of $K_M Z$ by letting $\left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$ act by $\zeta(\varpi)$. This shows the first assertion.

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To prove the second assertion, it suffices to show that c-$\mathrm{Ind}_{KZ}^G V_i$ is torsion-free, since A_i is a principal ideal domain. After tensoring with $\overline{L}$, this is equivalent to c-$\mathrm{Ind}_{KZ}^G V_i$ having no $T-\lambda$ torsion, which is easily seen using the fact that the functions in c-$\mathrm{Ind}_{KZ}^G V_i$ are compactly supported. □

In particular, Frobenius reciprocity gives

$$\mathrm{Hom}_K(V_i, \Pi(P)) \cong \mathrm{Hom}_G(\text{c-}\mathrm{Ind}_{KZ}^G V_i, \Pi(P)). \tag{18}$$

Hence $\mathrm{Hom}_K(V_i, \Pi(P))$ is naturally an A_i-module and we may transport the action of A_i onto $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, V_i^*)$ via formula (17).

If V is a continuous representation of K on a finite-dimensional L-vector space and if Θ is an open, bounded K-invariant lattice in V, let $|\cdot|$ be the norm on V^* given by $|\ell| := \sup_{v\in\Theta}|\ell(v)|$, so that $\Theta^d = \mathrm{Hom}_{\mathcal{O}}(\Theta, \mathcal{O})$ is the unit ball in V^* with respect to $|\cdot|$. The topology on $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, V^*)$ is given by the norm $\|\phi\| := \sup_{v\in P}|\phi(v)|$, and $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, \Theta^d)$ is the unit ball in this Banach space.

Proposition 5.5. *For all $i \in I$, the submodule*

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, V_i^*)_{\mathrm{l.fin}} := \{\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, V_i^*) : \ell_{A_i}(A_i\phi) < \infty\}$$

is dense in $\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, V_i^*)$*, where $\ell_{A_i}(A_i\phi)$ is the length of $A_i\phi$ as an A_i-module.*

Proof. See [21, Proposition 2.19]. □

Proposition 5.6. *Let $\mathfrak{m}$ be a maximal ideal of A_i and let Π be a completion of $A_i/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$ with respect to a G-invariant norm. Then Π is the universal unitary completion of $A_i/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$. Moreover, the action of A_i on $A_i/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$ extends to a continuous action of A_i on Π.*

Proof. Let Π° be a G-invariant $\mathcal{O}$-lattice of Π. Then

$$\Theta := \Pi^\circ \cap (A_i/\mathfrak{m}^n \otimes_{A_i} \text{c-}\mathrm{Ind}_{KZ}^G V_i)$$

is a G-invariant $\mathcal{O}$-lattice of $A_i/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$. By [27, Proposition 1.17], it suffices to show that Θ is of finite type over $\mathcal{O}[G]$.

By Proposition 5.4(i), we have that $\kappa(\mathfrak{m}) := A_i/\mathfrak{m} \cong L[T]/f(T)$ – where $f(T) \in L[T]$ is an irreducible polynomial – is a finite extension of L. Define a finite, increasing, exhaustive filtration $\{R^j\}_{n\geq j\geq 0}$ of $A_i/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$ by G-invariant A_i-submodules $R^j = \mathfrak{m}^{n-j}/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$. Then we have

$$R^j/R^{j-1} \cong \kappa(\mathfrak{m}) \otimes_{A_i} \text{c-}\mathrm{Ind}_{KZ}^G V_i$$

for each j and $\{\Theta^j := \Theta \cap R^j\}_{n\geq j\geq 0}$ is a finite, increasing, exhaustive filtration of Θ such that Θ^j is a G-invariant $\mathcal{O}$-lattice of R^j for each j. Moreover, Θ^j/Θ^{j-1} gives rise to a G-invariant $\mathcal{O}$-lattice of $R^j/R^{j-1} \cong A_i/\mathfrak{m} \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$, and thus is finitely generated over $\mathcal{O}[G]$ by the proof of [4, Theorem 4.3.1]. This implies that Θ is finitely generated over $\mathcal{O}[G]$, and the first assertion follows.

If $\phi \in A_i$, then $\phi(\Theta) \subset \varpi^n\Theta$ for some $n \in \mathbb{Z}$, as Θ is finitely generated over $\mathcal{O}[G]$. This implies the second assertion. □

We denote the universal unitary completion of $A_i/\mathfrak{m}^n \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$ by $\Pi_{i,\mathfrak{m},n}$. If $n = 1$, then $\Pi_{i,\mathfrak{m},1}$ is the universal unitary completion of $\kappa(\mathfrak{m}) \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$ studied in [4, Theorem 4.3.1] and [10, Proposition 2.2.1].

Corollary 5.7. *If $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(P, V_i^*)_{\mathrm{l.fin}}$ is such that $A_i\phi \cong A_i/\mathfrak{m}^n$ for a maximal ideal $\mathfrak{m}$ of A_i, then ϕ induces an injection $\Pi_{i,\mathfrak{m},n} \hookrightarrow \Pi(P)$. Moreover, $\Pi_{i,\mathfrak{m},n}$ admits a filtration of length n such that each graded piece is isomorphic to the universal unitary completion $\Pi_{i,\mathfrak{m},1}$ of $\kappa(\mathfrak{m}) \otimes_{A_i}$ c-$\mathrm{Ind}_{KZ}^G V_i$.*

Proof. The assumption $A_i\phi \cong A_i/\mathfrak{m}^n$ implies that $A_i/\mathfrak{m}^n \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i$ injects into $\Pi(P)$. The first assertion follows immediately from Proposition 5.6. To show the second assertion, we let $\{R^j\}_{n\geq j\geq 0}$ be the filtration of $A_i/\mathfrak{m}^n \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i$ defined in Proposition 5.6. Let Π^j be the closure of R^j in $\Pi_{i,\mathfrak{m},n}$; then $\Pi^n = \Pi_{i,\mathfrak{m},n}$. Since $\mathfrak{m}R^j = R^{j-1}$, we have $\mathfrak{m}\Pi^j = \Pi^{j-1}$ and hence $\Pi^j = \mathfrak{m}^{n-j}\Pi_{i,\mathfrak{m},n}$. If $\Pi^j = \Pi^{j-1}$, then $\Pi^j = \mathfrak{m}\Pi^j$ and hence $\Pi^j = \mathfrak{m}^j\Pi^j = \mathfrak{m}^n\Pi_{i,\mathfrak{m},n} = 0$. Since $\Pi^j \neq 0$ for $1 \leq j \leq n$, we conclude that $\Pi^j \neq \Pi^{j-1}$ for $1 \leq j \leq n$. Moreover, Π^j/Π^{j-1} contains $R^j/R^{j-1} \cong A_i/\mathfrak{m} \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i$ as a dense subspace, and thus is isomorphic to $\Pi_{i,\mathfrak{m},1}$. This proves the corollary. □

Note that the image of any $\phi \in \text{Hom}_G(A_i/\mathfrak{m}^n \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i, \Pi(P))$ is isomorphic to $A_i/\mathfrak{m}^k \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i$ for some $0 \leq k \leq n$. Hence it induces an injection $\Pi_{i,\mathfrak{m},k} \hookrightarrow \Pi(P)$ by Corollary 5.7.

Proposition 5.8. *Let P be a projective object in $\mathfrak{C}(\mathcal{O})$. Then the image of the evaluation map*

$$\bigoplus_{i\in I}\bigoplus_{\mathfrak{m},n} \text{Hom}_G^{\text{cont}}(\Pi_{i,\mathfrak{m},n}, \Pi(P)) \otimes_L \Pi_{i,\mathfrak{m},n} \to \Pi(P),$$

where $\mathfrak{m}$ runs through maximal ideals of A_i and $n \in \mathbb{N}$ is a dense subspace.

Proof. Let Π be the closure of the image of the evaluation map and M the image of P under $\text{Hom}_L^{\text{cont}}(\Pi(P), L) \twoheadrightarrow \text{Hom}_L^{\text{cont}}(\Pi, L)$. Then we have

$$\begin{aligned}
\text{Hom}_{\mathcal{O}[\![K]\!]}^{\text{cont}}(P, V_i^*)_{\text{l.fin}} &\cong \oplus_{\mathfrak{m}} \text{Hom}_{\mathcal{O}[\![K]\!]}^{\text{cont}}(P, V_i^*)[\mathfrak{m}^\infty]\\
&\cong \oplus_{\mathfrak{m}} \text{Hom}_K(V_i, \Pi(P))[\mathfrak{m}^\infty]\\
&\cong \oplus_{\mathfrak{m}} \text{Hom}_G(\text{c-Ind}_{KZ}^G V_i, \Pi(P))[\mathfrak{m}^\infty]\\
&\cong \oplus_{\mathfrak{m}} \varinjlim_n \text{Hom}_G(\text{c-Ind}_{KZ}^G V_i, \Pi(P))[\mathfrak{m}^n]\\
&\cong \oplus_{\mathfrak{m}} \varinjlim_n \text{Hom}_G(A_i/\mathfrak{m}^n \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i, \Pi(P))\\
&\cong \oplus_{\mathfrak{m}} \varinjlim_n \text{Hom}_G^{\text{cont}}(\Pi_{i,\mathfrak{m},n}, \Pi(P)).
\end{aligned}$$

Here the first isomorphism is due to the fact that any module M over a commutative ring A such that every finitely generated submodule is of finite length admits a decomposition $M \cong \oplus_{\mathfrak{m}} M[\mathfrak{m}^\infty]$ with $\mathfrak{m}$ running through maximal ideals of A. The second isomorphism is given by formula (17), and the last isomorphism is due to Corollary 5.7. Similarly, we have

$$\begin{aligned}
\text{Hom}_{\mathcal{O}[\![K]\!]}^{\text{cont}}(M, V_i^*)_{\text{l.fin}} &\cong \oplus_{\mathfrak{m}} \text{Hom}_{\mathcal{O}[\![K]\!]}^{\text{cont}}(M, V_i^*)[\mathfrak{m}^\infty]\\
&\cong \oplus_{\mathfrak{m}} \text{Hom}_K(V_i, \Pi)[\mathfrak{m}^\infty]\\
&\cong \oplus_{\mathfrak{m}} \text{Hom}_G(\text{c-Ind}_{KZ}^G V_i, \Pi)[\mathfrak{m}^\infty]\\
&\cong \oplus_{\mathfrak{m}} \varinjlim_n \text{Hom}_G(A_i/\mathfrak{m}^n \otimes_{A_i} \text{c-Ind}_{KZ}^G V_i, \Pi)\\
&\cong \oplus_{\mathfrak{m}} \varinjlim_n \text{Hom}_G^{\text{cont}}(\Pi_{i,\mathfrak{m},n}, \Pi)\\
&= \oplus_{\mathfrak{m}} \varinjlim_n \text{Hom}_G^{\text{cont}}(\Pi_{i,\mathfrak{m},n}, \Pi(P)).
\end{aligned}$$

Thus by Proposition 5.5, we have $\text{Hom}_{\mathcal{O}[\![K]\!]}^{\text{cont}}(P, V_i^*) \cong \text{Hom}_{\mathcal{O}[\![K]\!]}^{\text{cont}}(M, V_i^*)$ for each $i \in I$. Combining this with Lemma 5.1, we deduce the proposition. □

Corollary 5.9. *Define $\mathrm{m}_{i,\mathfrak{m},n} := \text{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Theta^d) \otimes_{\mathcal{O}} L$, where Θ is an open bounded G-invariant*

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lattice in $\Pi_{i,\mathfrak{m},n}$. *Then*

$$\bigcap_{i\in I}\bigcap_{\mathfrak{m},n}\mathfrak{a}_{i,\mathfrak{m},n}=0,$$

where $\mathfrak{a}_{i,\mathfrak{m},n} := \mathrm{ann}_E(\mathrm{m}_{i,\mathfrak{m},n})$.

Proof. For $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ with a G-invariant $\mathcal{O}$-lattice Θ, we have

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P,\Theta^d)\otimes_{\mathcal{O}} L \cong \mathrm{Hom}^{\mathrm{cont}}_G(\Pi,\Pi(P)).$$

Thus the evaluation map in Proposition 5.8 induces an $E[1/p]$-homomorphism

$$\bigoplus_{i\in I}\bigoplus_{\mathfrak{m},n}\mathrm{m}_{i,\mathfrak{m},n}\otimes_L \Pi_{i,\mathfrak{m},n}\to \Pi(P)$$

with a dense image. Since $E[1/p]$ acts faithfully on the right-hand side of the map, it acts faithfully on the left-hand side as well. This proves the corollary, since the E-action on the left-hand side factors through the quotient $E/\bigcap_{i\in I}\bigcap_{\mathfrak{m},n}\mathfrak{a}_{i,\mathfrak{m},n}$. □

6. Main results

Given a block $\mathfrak{B}$, we have defined $\pi_{\mathfrak{B}}$, $P_{\mathfrak{B}}$ and $E_{\mathfrak{B}}$ in Section 4.1 and $P'_{\mathfrak{B}}$ and $E'_{\mathfrak{B}}$ in Section 4.2. We assume that all irreducibles in $\mathfrak{B}$ are absolutely irreducible. This can be achieved by replacing k with a finite extension. Let $\bar{\rho}_{\mathfrak{B}}$ be the 2-dimensional semisimple Galois representation of $\mathscr{G}_{\mathbb{Q}_p}$ over k defined by $\check{\mathbf{V}}(\pi^{\vee}_{\mathfrak{B}})$ in cases (i), (ii), (iv) and (v), and by a direct sum of two copies of $\check{\mathbf{V}}(\pi^{\vee}_{\mathfrak{B}})$ in cases (iii) and (vi); see Section 4.4 for an explicit description. We write $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ for the universal pseudodeformation ring of $\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}$ with a fixed determinant $\zeta\varepsilon$. This ring is Noetherian by [18, Proposition F]. We let $T:\mathscr{G}_{\mathbb{Q}_p}\to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ be the universal object (see Section 3).

6.1. Finiteness

Let $\{V_i\}_{i\in I}$ be a family of K-representations defined in Proposition 5.3 and let $A_i = \mathrm{End}_G(\text{c-Ind}^G_{KZ} V_i)$. For each $i\in I$, a maximal ideal $\mathfrak{m}$ of A_i and $n\in\mathbb{N}$, we write $\Pi_{i,\mathfrak{m},n}$ for the universal unitary completion of $A_i/\mathfrak{m}^n\otimes_{A_i}\text{c-Ind}^G_{KZ} V_i$.

Lemma 6.1. *Assume* $\Pi_{i,\mathfrak{m},n}$ *is a subrepresentation of* $\Pi(P'_{\mathfrak{B}})$. *Then* $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ *is a finite free* $A_i/\mathfrak{m}^n$-*module of rank equal to* $\dim_{A_i/\mathfrak{m}}(\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},1}))\le 2$. *Moreover,*

- (i) *if* $\mathrm{rank}_{A_i/\mathfrak{m}^n}\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})=2$, *then* $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ *is a deformation to* $A_i/\mathfrak{m}^n$ *of the absolutely irreducible 2-dimensional L-representation* $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},1})$ *of* $\mathscr{G}_{\mathbb{Q}_p}$; *and*
- (ii) *if* $\mathrm{rank}_{A_i/\mathfrak{m}^n}\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})=1$, *then the action of* $\mathscr{G}_{\mathbb{Q}_p}$ *on* $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ *is given by an* $(A_i/\mathfrak{m}^n)^{\times}$-*valued character lifting* $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},1})$.

Proof. Since V_i is a principal series type, the $\mathscr{G}_{\mathbb{Q}_p}$-module

$$\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})/\mathfrak{m}\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})\xrightarrow{\sim}\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},1})$$

has dimension $r\le 2$ over $\kappa(\mathfrak{m}) := A_i/\mathfrak{m}$ by [4, Theorem 4.3.1] and [10, Proposition 2.2.1]. Nakayama's lemma implies that we have a surjection

$$(A_i/\mathfrak{m}^n)^{\oplus r}\twoheadrightarrow\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n}).$$

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Proposition 5.7 and the exactness of $\check{\mathbf{V}}$ imply that $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ has length nr as an $A_i/\mathfrak{m}^n$-module. Hence the surjection is an isomorphism and the lemma follows. □

Lemma 6.2. *Under the same assumptions as in Lemma 6.1, there is a natural map* $\theta_{i,\mathfrak{m},n} : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}} \to A_i/\mathfrak{m}^n$*, which induces a map*

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J \to \mathrm{End}_{A_i/\mathfrak{m}^n}(\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})).$$

Proof. If $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ is of rank 2 over $A_i/\mathfrak{m}^n$, it follows from Lemma 6.1(i) that $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ is a deformation of the 2-dimensional Galois representation $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},1})$ to $A_i/\mathfrak{m}^n$ with determinant $\zeta\varepsilon$. It follows from [18, Section 4.1, Theorem 3.17] that there is an $\mathcal{O}$-algebra map $\theta_{i,\mathfrak{m},n} : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}} \to A_i/\mathfrak{m}^n$ such that the specialisation of T along $\theta_{i,\mathfrak{m},n}$ is equal to $\mathrm{tr}_{A_i/\mathfrak{m}^n}\,\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$. This map induces a homomorphism of $\mathcal{O}$-algebras

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!] \to \mathrm{End}_{A_i/\mathfrak{m}^n}(\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})),$$

and Cayley–Hamilton for $M_2(A_i/\mathfrak{m}^n)$ implies that J lies in the kernel of this map.

If $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ is given by a character $\chi_{i,\mathfrak{m},n} : \mathcal{G}_{\mathbb{Q}_p} \to (A_i/\mathfrak{m}^n)^\times$, then the same argument applies to the representation $\chi_{i,\mathfrak{m},n} \oplus \chi^{-1}_{i,\mathfrak{m},n}\zeta\varepsilon$, so that we get a map

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J \to \mathrm{End}_{A_i/\mathfrak{m}^n}(\chi_{i,\mathfrak{m},n} \oplus \chi^{-1}_{i,\mathfrak{m},n}\zeta\varepsilon).$$

Its image commutes with the idempotent which projects onto the direct summand $\chi_{i,\mathfrak{m},n}$. Hence, the image is contained in $\mathrm{End}_{A_i/\mathfrak{m}^n}(\chi_{i,\mathfrak{m},n}) \times \mathrm{End}_{A_i/\mathfrak{m}^n}(\chi^{-1}_{i,\mathfrak{m},n}\zeta\varepsilon)$, and we may project to $\mathrm{End}_{A_i/\mathfrak{m}^n}(\chi_{i,\mathfrak{m},n})$ to obtain the required homomorphism. □

In Propositions 3.1 and 4.18, we established surjections

$$\alpha : \mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!] \twoheadrightarrow \mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})), \qquad \beta : \mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!] \twoheadrightarrow R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J.$$

Theorem 6.3. *The maps just given induce a surjection*

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J \twoheadrightarrow \mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})).$$

In particular, $E'_{\mathfrak{B}}$ *and* $Z'_{\mathfrak{B}}$ *are finite over* $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ *and hence Noetherian.*

Proof. For the first part we have to show that $\mathrm{Ker}\,\beta \subset \mathrm{Ker}\,\alpha$. Let $M = \check{\mathbf{V}}(P'_{\mathfrak{B}})$ and define $\mathrm{m}_{i,\mathfrak{m},n} := \mathrm{m}(\Pi_{i,\mathfrak{m},n})$, with $i, \mathfrak{m}, n$ as in Corollary 5.9. Then the assumptions in Lemma 2.6 are satisfied by Corollary 5.9. It follows that the kernel of α is given by $\cap_{i\in I} \cap_{\mathfrak{m},n} \mathfrak{b}_{i,\mathfrak{m},n}$, where $\mathfrak{b}_{i,\mathfrak{m},n}$ is the $\mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]$-annihilator of

$$\mathrm{m}_{i,\mathfrak{m},n} \otimes_{E'_{\mathfrak{B}}} \check{\mathbf{V}}(P'_{\mathfrak{B}}) \cong \check{\mathbf{V}}(\mathrm{m}_{i,\mathfrak{m},n} \otimes_{E'_{\mathfrak{B}}} P'_{\mathfrak{B}}) \cong \check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n}).$$

Since the action of $\mathcal{O}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]$ on $\check{\mathbf{V}}(\Pi_{i,\mathfrak{m},n})$ factors through $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$ by Lemma 6.2, $\mathfrak{b}_{i,\mathfrak{m},n}$ contains the kernel of β and hence $\mathrm{Ker}\,\beta \subset \mathrm{Ker}\,\alpha$.

The second assertion is a consequence of the first assertion and the finiteness of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$ over $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ [48, Proposition 3.6]. □

Corollary 6.4. $E_{\mathfrak{B}}$ *and* $Z_{\mathfrak{B}}$ *are finite over* $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ *and hence Noetherian.*

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Proof. By [39, Lemma 10.26], we have

$$\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}) \cong \mathrm{End}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} M_{\mathfrak{B}}),$$
$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P'_{\mathfrak{B}}, M_{\mathfrak{B}}) \cong \mathrm{Hom}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} P'_{\mathfrak{B}}, \mathcal{T} M_{\mathfrak{B}}),$$

since $M_{\mathfrak{B}}^{\mathrm{SL}_2(\mathbb{Q}_p)} = (M_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)} = (P'_{\mathfrak{B}})_{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. Define $\mathrm{m} := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P'_{\mathfrak{B}}, M_{\mathfrak{B}})$, which is a finitely generated right $E'_{\mathfrak{B}}$-module by Corollary 4.13. Moreover, we have

$$\mathrm{End}_{E'_{\mathfrak{B}}}(\mathrm{m}) \cong \mathrm{End}_{\mathfrak{Q}(\mathcal{O})}(\mathcal{T} M_{\mathfrak{B}}) \cong \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})$$

by Proposition 4.10(2) and the isomorphism above. Theorem 6.3 implies that the conditions of Lemma 4.11 are satisfied with $E = E'_{\mathfrak{B}}$ and $Z = Z'_{\mathfrak{B}}$. Thus $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})$ and its centre are finite over $Z'_{\mathfrak{B}}$, and thus finite over $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}$ by Theorem 6.3, and hence Noetherian. This implies the corollary, since $\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}}) \cong E_{\mathfrak{B}}$ and $Z(\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M_{\mathfrak{B}})) \cong Z_{\mathfrak{B}}$ by Proposition 4.15. □

Remark 6.5. Since $Z_{\mathfrak{B}}$ is Noetherian, the $\mathfrak{m}$-adic topology coincides with the linearly compact topology in Lemma 4.1.

Let us spell out the properties of the map $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}} \to Z_{\mathfrak{B}}$ constructed in Corollary 6.4. Since $Z_{\mathfrak{B}}$ acts functorially on every object in $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, the homomorphism $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}} \to Z_{\mathfrak{B}}$ induces a functorial ring homomorphism

$$c_M : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}} \to \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M)$$

for every object M in $\mathfrak{C}(\mathcal{O})$. Since $\check{\mathbf{V}}$ is a functor, it induces a ring homomorphism

$$\mathrm{End}_{\mathfrak{C}(\mathcal{O})}(M) \to \mathrm{End}^{\mathrm{cont}}_{\mathcal{G}_{\mathbb{Q}_p}}(\check{\mathbf{V}}(M)), \quad \varphi \mapsto \check{\mathbf{V}}(\varphi).$$

We denote the action of $\mathcal{G}_{\mathbb{Q}_p}$ on $\check{\mathbf{V}}(M)$ by $\rho_{\check{\mathbf{V}}(M)}$. Finally, for all $g \in \mathcal{G}_{\mathbb{Q}_p}$ we may evaluate the universal pseudorepresentation $T : \mathcal{G}_{\mathbb{Q}_p} \to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}$ at $g \in \mathcal{G}_{\mathbb{Q}_p}$ to obtain an element $T(g) \in R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}$.

Proposition 6.6. *For each $M \in \mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ and each $g \in \mathcal{G}_{\mathbb{Q}_p}$,*

$$\check{\mathbf{V}}(c_M(T(g))) = \rho_{\check{\mathbf{V}}(M)}(g) + \rho_{\check{\mathbf{V}}(M)}(g^{-1})\zeta\varepsilon(g)$$

in $\mathrm{End}^{\mathrm{cont}}_{\mathcal{G}_{\mathbb{Q}_p}}(\check{\mathbf{V}}(M))$.

Proof. Since $g^2 - T(g)g + \zeta\varepsilon(g) = 0$ in $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[\![\mathcal{G}_{\mathbb{Q}_p}]\!]/J$, the equality $T(g)\,\mathrm{id} = g + \zeta\varepsilon(g)g^{-1}$ holds in that ring. The rest is just unravelling the definitions. □

Since $P_{\mathfrak{B}}$ is a projective generator for $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, the functor

$$N \mapsto \mathrm{m}(N) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}, N)$$

induces an equivalence of categories between $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ and the category of right pseudocompact $E_{\mathfrak{B}}$-modules. The inverse functor is given by $\mathrm{m} \mapsto \mathrm{m}\,\widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}}$.

Corollary 6.7. *For N in $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ the following assertions are equivalent:*

1. *There is a surjection $P_{\mathfrak{B}}^{\oplus n} \twoheadrightarrow N$ for some $n \geq 1$.*
2. $\mathrm{m}(N)$ *is a finitely generated $E_{\mathfrak{B}}$-module.*
3. $\mathrm{m}(N)$ *is a finitely generated $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}$-module.*

4. *$k \widehat{\otimes}_{R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}} N$ is of finite length in $\mathfrak{C}(\mathcal{O})$.*
5. *The cosocle of N in $\mathfrak{C}(\mathcal{O})$ is of finite length.*

The equivalent conditions hold if N is finitely generated over $\mathcal{O}[\![H]\!]$ for a compact open subgroup H of G.

Proof. (1) implies (2), since m is exact. (2) implies (3) by Corollary 6.4. Since

$$k \widehat{\otimes}_{R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}} \mathrm{m}(N) \cong \mathrm{m}(k \widehat{\otimes}_{R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}} N),$$

and the functor m is an antiequivalence, (3) implies (4). Let $N \twoheadrightarrow \mathrm{cosoc}(N)$ be the cosocle of N. Since the maximal ideal of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}$ acts trivially on every semisimple object, the surjection factors through

$$k \widehat{\otimes}_{R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}} N \twoheadrightarrow \mathrm{cosoc}(N),$$

and so (4) implies (5). If $\mathrm{cosoc}(N)$ is of finite length, then there is a surjection $\pi_{\mathfrak{B}}^{\oplus n} \twoheadrightarrow \mathrm{cosoc}(N)$ for some $n \geq 1$. Since $P_{\mathfrak{B}}$ is projective, there is a map $\varphi : P_{\mathfrak{B}}^{\oplus n} \longrightarrow N$ lifting $(P_{\mathfrak{B}})^{\oplus n} \twoheadrightarrow \pi_{\mathfrak{B}}^{\oplus n} \twoheadrightarrow \mathrm{cosoc}(N)$. The cokernel of φ will have zero cosocle and hence φ is surjective, so that (5) implies (1).

If N is finitely generated over $\mathcal{O}[\![H]\!]$, which we may assume to be pro-p, then $((N/\varpi N)^\vee)^H$ is a finite-dimensional k-vector space, and hence the G-socle of $N^\vee$ is of finite length, which dually implies that (5) holds. □

Since every irreducible in $\mathfrak{B}$ is admissible, its Pontryagin dual is finitely generated over $\mathcal{O}[\![H]\!]$ for any compact open subgroup H of G. It follows from Corollary 6.7(4) that $k \widehat{\otimes}_{R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}} P_{\mathfrak{B}}$ is also a finitely generated $\mathcal{O}[\![H]\!]$-module. This implies that the assumptions made in [39, Section 4] are satisfied with the category $\mathfrak{C}(\mathcal{O})$ there equal to $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, and we will record some consequences.

6.2. Banach-space representations

The category $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ decomposes into a direct sum of categories [39, Proposition 5.36]:

$$\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L) \cong \bigoplus_{\mathfrak{B} \in \mathrm{Irr}_{G,\zeta}/\sim} \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathfrak{B}}, \tag{19}$$

where the objects of $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathfrak{B}}$ are those Π in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ such that for every open bounded G-invariant lattice Θ in Π, the irreducible subquotients of $\Theta \otimes_{\mathcal{O}} k$ lie in $\mathfrak{B}$. This condition is equivalent to requiring that Θ^d be an object of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$.

As in the previous subsection, we fix a block $\mathfrak{B}$ consisting of absolutely irreducible representations. Let $\mathrm{Mod}^{\mathrm{fg}}_{E_{\mathfrak{B}}[1/p]}$ be the category of finitely generated right $E_{\mathfrak{B}}[1/p]$-modules. The functor

$$\mathrm{m} : \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathfrak{B}} \longrightarrow \mathrm{Mod}^{\mathrm{fg}}_{E_{\mathfrak{B}}[1/p]}, \quad \Pi \mapsto \mathrm{m}(\Pi) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}, \Theta^d) \otimes_{\mathcal{O}} L,$$

where Θ is any open bounded G-invariant lattice in Π and is exact, contravariant and fully faithful by [39, Lemma 4.45]. Moreover, it induces an antiequivalence of categories

$$\mathrm{m} : \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B}} \xrightarrow{\cong} \mathrm{Mod}^{\mathrm{fl}}_{E_{\mathfrak{B}}[1/p]}, \tag{20}$$

where the superscript 'fl' indicates the subcategories of objects of finite length in the respective categories [39, Theorem 4.34]. We write *antiequivalence* instead of *equivalence* to indicate that m is contravariant.

If $\mathfrak{m}$ is a maximal ideal of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}[1/p]$, then we let $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}}$ be the full subcategory of $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ consisting of finite-length Banach-space representations, which are killed by some power

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of $\mathfrak{m}$. The functor m induces an antiequivalence between this category and the category of $E_{\mathfrak{B}}[1/p]$-modules of finite length, which are killed by a power of $\mathfrak{m}$. The Chinese remainder theorem [39, Theorem 4.36] implies that we have an equivalence of categories

$$\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B}} \cong \bigoplus_{\mathfrak{m}\in \text{m-Spec}\, R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[1/p]} \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}}. \tag{21}$$

Corollary 6.8. *If* $\Pi_1 \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}_1}$ *and* $\Pi_2 \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}_2}$ *for distinct maximal ideals* $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *of* $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[1/p]$*, then the Yoneda* $\mathrm{Ext}^i(\Pi_1,\Pi_2)$ *computed in* $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ *vanish for all* $i \geq 0$.

Proof. It follows from formula (21) that the assertion holds for the Yoneda Ext groups computed in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)^{\mathrm{fl}}_{\mathfrak{B}}$. It follows from [39, Proposition 4.46, Corollary 4.48] that these coincide with Yoneda Ext groups computed in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathfrak{B}}$, which is a direct summand of $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ (see formaul (19)). □

We will determine the set of isomorphism classes $\mathrm{Irr}(\mathfrak{m}, L')$ of irreducible objects in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}}$ for a sufficiently large finite extension L' of L. Recall that $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$ is *absolutely irreducible* if $\Pi \otimes_L L'$ is irreducible in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')$ for all finite extensions L' of L. It follows from formula (20) that for such Π, Schur's lemma holds, so that $\mathrm{End}^{\mathrm{cont}}_G(\Pi) = L$. This result is also proved in [26] in a much more general setting. It follows from formula (20) that irreducibles in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)_{\mathfrak{B},\mathfrak{m}}$ correspond to irreducible modules of the algebra $E_{\mathfrak{B}} \otimes_{R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}} \kappa(\mathfrak{m})$. Corollary 6.4 implies that this algebra is finite-dimensional over $\kappa(\mathfrak{m})$, and thus $\mathrm{Irr}(\mathfrak{m}, L')$ is finite for every finite extension L' of L, and there is a finite extension L' of L such that all Π in $\mathrm{Irr}(\mathfrak{m}, L')$ are absolutely irreducible.

Proposition 6.9. *Let* L' *be a finite extension of* L *and let* Π *be absolutely irreducible in* $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')_{\mathfrak{B}}$. *Let* $c_\Pi : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}} \to L'$ *be the composition*

$$c_\Pi : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}} \to Z_{\mathfrak{B}} \to \mathrm{End}^{\mathrm{cont}}_G(\Pi) = L'.$$

Then one of the following holds:

1. *If* Π *is a subquotient of* $(\mathrm{Ind}^G_B\, \psi_1 \otimes \psi_2\varepsilon^{-1})_{\mathrm{cont}}$ *for some unitary characters* $\psi_1, \psi_2 : \mathbb{Q}_p^\times \to (L')^\times$, *then* $T_{c_\Pi} = \psi_1 + \psi_2$;
2. *otherwise,* $\check{\mathbf{V}}(\Pi)$ *is a* 2*-dimensional absolutely irreducible* L'*-representation of* $\mathscr{G}_{\mathbb{Q}_p}$, $\det \check{\mathbf{V}}(\Pi) = \zeta\varepsilon$ *and* $T_{c_\Pi} = \mathrm{tr}\, \check{\mathbf{V}}(\Pi)$,

where T_{c_Π} *is the specialisation of the universal pseudorepresentation* $T : \mathscr{G}_{\mathbb{Q}_p} \to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ *along* c_Π.

Proof. Let Ψ be the unitary principal series representation in (1). If $\psi_1 \neq \psi_2\varepsilon^{-1}$, then $\Psi^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$, and by looking at its reduction modulo p one may conclude that Ψ is absolutely irreducible. If $\psi_1 = \psi_2\varepsilon$, then Ψ is a nonsplit extension

$$0 \to \psi_1 \circ \det \to \Psi \to \widehat{\mathrm{Sp}} \otimes \psi_1 \circ \det \to 0,$$

where $\widehat{\mathrm{Sp}}$ is the universal unitary completion of the smooth Steinberg representation. This representation is absolutely irreducible, since its mod p reduction is. In both cases, $\mathrm{End}^{\mathrm{cont}}_G(\Psi) = L'$, and thus $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$ acts on all irreducible subquotients of Ψ via the same homomorphism c_Ψ. Since $\check{\mathbf{V}}(\Psi) = \psi_2$, regarded as a representation of $\mathscr{G}_{\mathbb{Q}_p}$ via the class field theory, $g + \varepsilon\zeta(g)g^{-1}$ acts on it via the scalar $\psi_2(g) + \psi_2(g^{-1})\varepsilon\zeta(g) = \psi_2(g) + \psi_1(g)$ for all $g \in \mathscr{G}_{\mathbb{Q}_p}$. Proposition 6.6 implies that the specialisation of T at c_Ψ is the character $\psi_1 + \psi_2$.

If we are not in part (1), then [21, Corollary 1.2, Theorem 1.9] imply that $\check{\mathbf{V}}(\Pi)$ is absolutely irreducible 2-dimensional and $\det \check{\mathbf{V}}(\Pi) = \zeta\varepsilon$. A calculation with 2×2 matrices implies that $g + \zeta\varepsilon(g)g^{-1}$

acts on $\check{\mathbf{V}}(\Pi)$ by a scalar $(\operatorname{tr}\check{\mathbf{V}}(\Pi))(g)$. Proposition 6.6 implies that the specialisation of T at c_Ψ is the character $\operatorname{tr}\check{\mathbf{V}}(\Pi)$. □

Corollary 6.10. *Let L' be a finite extension of L and let $x : R^{\mathrm{ps},\zeta\varepsilon}_{\operatorname{tr}\bar{\rho}_{\mathfrak{B}}} \to L'$ be an $\mathcal{O}$-algebra homomorphism. If $T_x = \psi_1 + \psi_2$ for characters $\psi_1, \psi_2 : \mathcal{G}_{\mathbb{Q}_p} \to (L')^\times$, then one of the following holds:*

1. *If $\psi_1\psi_2^{-1} = \mathbf{1}$, then* $\operatorname{Irr}(\mathfrak{m}_x, L') = \{(\operatorname{Ind}_B^G \mathbf{1} \otimes \varepsilon^{-1})_{\mathrm{cont}} \otimes \psi_1 \circ \det\}$.
2. *If $\psi_1\psi_2^{-1} = \varepsilon^{\pm 1}$, then* $\operatorname{Irr}(\mathfrak{m}_x, L') = \{\mathbf{1}, \widehat{\mathrm{Sp}}, (\operatorname{Ind}_B^G \varepsilon \otimes \varepsilon^{-1})_{\mathrm{cont}}\} \otimes \psi \circ \det$.
3. *If $\psi_1\psi_2^{-1} \neq \varepsilon^{\pm 1}, \mathbf{1}$, then*

$$\operatorname{Irr}(\mathfrak{m}_x, L') = \{(\operatorname{Ind}_B^G \psi_1 \otimes \psi_2\varepsilon^{-1})_{\mathrm{cont}}, (\operatorname{Ind}_B^G \psi_2 \otimes \psi_1\varepsilon^{-1})_{\mathrm{cont}}\},$$

where we consider ψ_1 and ψ_2 as unitary characters of $\mathbb{Q}_p^\times$ via the class field theory and ψ in (2) is either ψ_1 or ψ_2.

Proof. We have explained in the course of the proof of Proposition 6.9 that the representations listed in this corollary are absolutely irreducible and are contained in $\operatorname{Irr}(\mathfrak{m}_x, L')$. Moreover, using the functor of ordinary parts one may show that they are pairwise nonisomorphic.

We will show that the list is exhaustive. We may enlarge L' so that all $\Pi \in \operatorname{Irr}(\mathfrak{m}_x, L')$ are absolutely irreducible. Since $c_\Pi = x$, we cannot be in part (2) of Proposition 6.9, and thus we must be in part (1), and hence Π is already in our list. □

Proposition 6.11. *Let L' be a finite extension of L and let $x : R^{\mathrm{ps},\zeta\varepsilon}_{\operatorname{tr}\bar{\rho}_{\mathfrak{B}}} \to L'$ be an $\mathcal{O}$-algebra homomorphism. If $T_x = \operatorname{tr}\rho$, where $\rho : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(L')$ is absolutely irreducible, then $\operatorname{Irr}(\mathfrak{m}_x, L') = \{\Pi\}$, with Π absolutely irreducible nonordinary and $\check{\mathbf{V}}(\Pi) \cong \rho$.*

Proof. It follows from [21, Theorem 1.1] that such Π exists. We may enlarge L' so that all $\Pi' \in \operatorname{Irr}(\mathfrak{m}_x, L')$ are absolutely irreducible. Since $c_{\Pi'} = x$, we cannot be in part (1) of Proposition 6.9; thus we must be in part (2), and $\operatorname{tr}\check{\mathbf{V}}(\Pi) = \operatorname{tr}\check{\mathbf{V}}(\Pi')$. Since both $\check{\mathbf{V}}(\Pi')$ and $\check{\mathbf{V}}(\Pi)$ are absolutely irreducible, we deduce that $\check{\mathbf{V}}(\Pi) \cong \check{\mathbf{V}}(\Pi')$, and [21, Theorem 1.8] implies that $\Pi \cong \Pi'$. □

6.3. The centre

We fix a block $\mathfrak{B}$ as in the previous section and explore the relation between $R^{\mathrm{ps},\zeta\varepsilon}_{\operatorname{tr}\bar{\rho}_{\mathfrak{B}}}$ and $Z_{\mathfrak{B}}$. So far we have constructed a finite map

$$R^{\mathrm{ps},\zeta\varepsilon}_{\operatorname{tr}\bar{\rho}_{\mathfrak{B}}} \to Z'_{\mathfrak{B}} \twoheadrightarrow Z_{\mathfrak{B}} \tag{22}$$

(Theorem 6.3 and Corollary 4.16). We show in Corollary A.14 that $R^{\mathrm{ps},\zeta\varepsilon}_{\operatorname{tr}\bar{\rho}_{\mathfrak{B}}}[1/p]$ is normal, and we know by [17, Theorem 2.1] that it is equidimensional, and the locus corresponding to absolutely irreducible pseudorepresentations is Zariski dense in $\operatorname{Spec} R^{\mathrm{ps},\zeta\varepsilon}_{\operatorname{tr}\bar{\rho}_{\mathfrak{B}}}[1/p]$.

Proposition 6.12. *Let $R^{\square,\zeta\varepsilon}_{\bar{\rho}_{\mathfrak{B}}}$ be the universal framed deformation ring of $\bar{\rho}_{\mathfrak{B}}$ with fixed determinant $\zeta\varepsilon$, let S be its maximal $\mathcal{O}$-torsion-free quotient and let V_S be a free S-module of rank 2 with $\mathcal{G}_{\mathbb{Q}_p}$-action induced by the universal deformation $\mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(R^{\square,\zeta\varepsilon}_{\bar{\rho}_{\mathfrak{B}}}) \twoheadrightarrow \mathrm{GL}_2(S)$. There is N in $\mathfrak{C}(\mathcal{O})$ with a continuous action of S, which commutes with the action of G, such that we have an isomorphism of $S[\mathcal{G}_{\mathbb{Q}_p}]$-modules $\check{\mathbf{V}}(N) \cong V_S$.*

Proof. If $x \in \text{m-Spec}\, S[1/p]$, then the specialisation of V_S at x lies in the image of $\check{\mathbf{V}}$ by [22, Theorem 10.1]. Since $S[1/p]$ is reduced (Propositions A.9 and A.13) and Jacobson, such points will be dense, and the existence of such N follows from [19, Theorem II.3.3]. □

The subscript ‘tf’ will indicate the maximal $\mathcal{O}$-torsion-free quotient.

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Theorem 6.13. *The surjection* $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J \twoheadrightarrow \mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}}))$ *in Theorem 6.3 identifies* $\mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}}))$ *with* $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}}$. *In particular, map (22) induces an isomorphism*

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[1/p] \xrightarrow{\cong} Z'_{\mathfrak{B}}[1/p]. \tag{23}$$

Moreover, if $p \neq 2$, *then* $Z'_{\mathfrak{B}} = (R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}})_{\mathrm{tf}}$, *and if* $p = 2$, *then the cokernel of map (22) is killed by* 2.

Proof. As already explained in the proof of Proposition 4.3, projective objects in $\mathfrak{C}(\mathcal{O})$ are also projective in the category of compact $\mathcal{O}[\![K']\!]$-modules, where K' is an open pro-p subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ intersecting Z trivially, and thus are $\mathcal{O}$-torsion-free. Hence, $P'_{\mathfrak{B}}$ is $\mathcal{O}$-torsion-free. Since $E'_{\mathfrak{B}}$ and $Z'_{\mathfrak{B}}$ act faithfully on $P'_{\mathfrak{B}}$, we deduce that both rings are $\mathcal{O}$-torsion-free. Since $\mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}}))$ is either $(E'_{\mathfrak{B}})^{\mathrm{op}}$ or $M_2(E'_{\mathfrak{B}})$ by Proposition 4.18, we deduce that the map in Theorem 6.3 factors through

$$(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}} \twoheadrightarrow \mathrm{End}^{\mathrm{cont}}_{E'_{\mathfrak{B}}}(\check{\mathbf{V}}(P'_{\mathfrak{B}})). \tag{24}$$

If a lies in the kernel of this map, then it will kill $\check{\mathbf{V}}(P'_{\mathfrak{B}})$ and hence $\mathrm{m}\,\widehat{\otimes}_{E'_{\mathfrak{B}}}\check{\mathbf{V}}(P'_{\mathfrak{B}})$ for all compact right $E'_{\mathfrak{B}}$-modules. Thus a will kill all objects in the essential image of $\check{\mathbf{V}}$, and it will therefore also kill the representation V_S defined in Proposition 6.12.

It follows from Propositions A.9 and A.13 that the ring $R^{\square,\zeta\varepsilon}_{\bar\rho_{\mathfrak{B}}}[1/p]$ is normal and the absolutely irreducible locus is dense in $\mathrm{Spec}\,R^{\square,\zeta\varepsilon}_{\bar\rho_{\mathfrak{B}}}[1/p]$. Corollary A.7 implies that $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J)_{\mathrm{tf}}$ acts faithfully on V_S, hence $a = 0$ and formula (24) is injective.

The assertions about the centre follow from Proposition A.11. □

We immediately obtain the following:

Corollary 6.14. $Z'_{\mathfrak{B}}$ *is a complete local Noetherian* $\mathcal{O}$*-algebra with residue field k. It is* $\mathcal{O}$*-torsion-free and* $Z'_{\mathfrak{B}}[1/p]$ *is normal.*

Corollary 6.15. $Z_{\mathfrak{B}} = Z'_{\mathfrak{B}}$.

Proof. Since $Z_{\mathfrak{B}}$ acts faithfully on $P_{\mathfrak{B}}$ it is $\mathcal{O}$-torsion-free. Thus it is enough to show that the surjection $Z'_{\mathfrak{B}} \twoheadrightarrow Z_{\mathfrak{B}}$ (see Corollary 4.16) induces an isomorphism after inversion of p. Since $Z'_{\mathfrak{B}}[1/p]$ is reduced by Corollary 6.14, it is enough to show that m-Spec $Z_{\mathfrak{B}}[1/p]$ contains a subset Σ of m-Spec $Z'_{\mathfrak{B}}[1/p]$, which is dense in Spec $Z'_{\mathfrak{B}}[1/p]$. We may take Σ to be the absolutely irreducible locus in m-Spec $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[1/p]$, as it is dense in Spec $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[1/p]$ by [17, Theorem 2.1] and lies in m-Spec $Z_{\mathfrak{B}}[1/p]$ by the main result of [21]. □

Corollary 6.16. *Let* L' *be a finite extension of* L *and let* $x : R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}} \to L'$ *be an* $\mathcal{O}$*-algebra homomorphism. If the specialisation of the universal pseudodeformation* $T : \mathscr{G}_{\mathbb{Q}_p} \to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}$ *at* x *is not of the form* $\psi + \psi\varepsilon$, *then* $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}_x}$ *is equivalent to the category of modules of finite length over the completion of* $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar\rho_{\mathfrak{B}}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J) \otimes_{\mathcal{O}} L'$ *at* $\mathfrak{m}_x$.

Moreover, if $T_x = \mathrm{tr}\,\rho$, *where* $\rho : \mathscr{G}_{\mathbb{Q}_p} \to \mathrm{GL}_2(L')$ *is absolutely irreducible, then* $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}_x}$ *is equivalent to the category of modules of finite length over the deformation ring* $R^{\zeta\varepsilon}_{\rho}$, *which parameterises the deformations of* ρ *with determinant* $\zeta\varepsilon$ *to local Artinian* L'*-algebras.*

In particular, if $\Pi' \in \mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')$ *is killed by* $\mathfrak{m}_x$ *then* Π' *is isomorphic to a direct sum of finitely many copies of* Π *in Proposition 6.11.*

Proof. After extending scalars, we may assume that $L = L'$. If $T_x \neq \psi + \psi\varepsilon$ for any character ψ, then it follows from Corollaries 6.10 and 6.11 that $\mathrm{Irr}(\mathfrak{m}_x, L')$ does not contain characters. We may apply [39, Theorem 4.36] to deduce that $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}_x}$ is antiequivalent to the category of $E'_{\mathfrak{B}} \otimes_R \widehat{R}_{\mathfrak{m}_x}$-modules

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of finite length, where $R = R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$. Theorem 6.13 implies that this ring coincides with the completion of $(R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J)[1/p]$ at $\mathfrak{m}_x$.

Let us assume that $T_x = \mathrm{tr}\,\rho$ with ρ absolutely irreducible. Then $\mathrm{Irr}(\mathfrak{m}_x, L') = \{\Pi\}$, with Π absolutely irreducible by Proposition 6.11 and $\check{\mathbf{V}}(\Pi) \cong \rho$. It follows from [18, Sections 4.1 and 4.2] that $(R[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J) \otimes_R \widehat{R}_{\mathfrak{m}_x}$ is an Azumaya algebra over $\widehat{R}_{\mathfrak{m}_x}$. Since ρ is an absolutely irreducible 2-dimensional module of $(R[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J) \otimes_R \kappa(x)$, we conclude that $(R[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J) \otimes_R \kappa(x) = M_2(\kappa(x))$, and thus $(R[\![\mathscr{G}_{\mathbb{Q}_p}]\!]/J) \otimes_R \widehat{R}_{\mathfrak{m}_x}$ is isomorphic to the ring of 2×2 matrices over $\widehat{R}_{\mathfrak{m}_x}$. Since $M_2(\widehat{R}_{\mathfrak{m}_x})$ is Morita equivalent to $\widehat{R}_{\mathfrak{m}_x}$, which is isomorphic to $R^{\zeta\varepsilon}_{\rho}$ by [35, Lemma 2.3.3, Proposition 2.3.5], we obtain the first assertion.

In particular, the full subcategory of $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L')^{\mathrm{fl}}_{\mathfrak{B},\mathfrak{m}_x}$ consisting of representations killed by $\mathfrak{m}_x$ is equivalent to the category of finite-dimensional vector spaces over L', and hence the last assertion follows. □

6.4. Complements

We will prove Theorem 1.1. Let $\mathfrak{B}$ be an arbitrary block, so that we do not assume that it contains an absolutely irreducible representation.

If $\pi_1, \pi_2 \in \mathrm{Irr}_{G,\zeta}(k)$, then it follows from [39, Proposition 5.11] that there is a finite extension k' of k such that $\pi_1 \otimes_k k'$ is a finite direct sum of absolutely irreducible representations; then $\pi_2 \otimes_k k'$ is a finite direct sum of irreducible representations, each of them occurring with multiplicity 1. It is implied by [39, Proposition 5.33] that

$$\mathrm{Ext}^1_{k[G],\zeta}(\pi_1, \pi_2) \otimes_k k' \cong \mathrm{Ext}^1_{k'[G],\zeta}(\pi_1 \otimes_k k', \pi_2 \otimes_k k'). \tag{25}$$

If $\mathrm{Ext}^1_{k[G],\zeta}(\pi_1, \pi_2) \neq 0$, then it follows from this formula that there are irreducible summands π_1' of $\pi_1 \otimes_k k'$ and π_2' of $\pi_2 \otimes_k k'$ such that $\mathrm{Ext}^1_{k'[G],\zeta}(\pi_1', \pi_2') \neq 0$. Since π_1' is absolutely irreducible, we conclude by inspecting the list of blocks in Section 4.1 that π_2' is absolutely irreducible, and thus if $\mathfrak{B}$ is the block containing π_1, then $\pi \otimes_k k'$ is a finite direct sum of absolutely irreducible representations for all $\pi \in \mathfrak{B}$.

Let L' be a finite extension of L with ring of integers $\mathcal{O}'$ and residue field k'. If $\pi_1', \pi_2' \in \mathrm{Irr}_{G,\zeta}(k')$ are absolutely irreducible, then it follows from [39, Proposition 5.11] that there exist unique $\pi_1, \pi_2 \in \mathrm{Irr}_{G,\zeta}(k)$ such that π_1' is a direct summand of $\pi_1 \otimes_k k'$ and π_2' is a direct summand of $\pi_2 \otimes_k k'$. It follows from formula (25) that if π_1' and π_2' lie in the same block in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O}')$, then π_1 and π_2 lie in the same block in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O})$.

Thus if $\pi_1 \in \mathfrak{B}$ and we let $\mathfrak{B}_1, \dots, \mathfrak{B}_r$ be the blocks of irreducible subquotients of $\pi_1 \otimes_k k'$ in $\mathrm{Mod}^{\mathrm{l.fin}}_{G,\zeta}(\mathcal{O}')$ and $\mathfrak{B} \otimes_k k'$ be the set of isomorphism classes of irreducible subquotients of $\pi \otimes_k k'$ for all $\pi \in \mathfrak{B}$, then

$$\mathfrak{B} \otimes_k k' = \bigcup_{i=1}^{r} \mathfrak{B}_i.$$

It follows from [39, Corollary 5.40] that $P_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}' \cong \prod_{i=1}^{r} P_{\mathfrak{B}_i}$ and

$$E_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}' \cong \mathrm{End}_{\mathfrak{C}(\mathcal{O}')}(P_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}') \cong \prod_{i=1}^{r} E_{\mathfrak{B}_i}.$$

Since the blocks $\mathfrak{B}_i$ contain only absolutely irreducible representations, it follows from Corollary 6.4 that $E_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}'$ is a finite module over its centre $Z(E_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}')$ and

$$Z(E_{\mathfrak{B}}) \otimes_{\mathcal{O}} \mathcal{O}' \cong Z(E_{\mathfrak{B}} \otimes_{\mathcal{O}} \mathcal{O}') \cong \prod_{i=1}^{r} Z_{\mathfrak{B}_i}$$

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is Noetherian (see the argument in the proof of [25, Lemma 4.14] for the first isomorphism). Since $\mathcal{O}'$ is a finite free $\mathcal{O}$-module, this implies that $Z(E_{\mathfrak{B}})$ is Noetherian, and $E_{\mathfrak{B}}$ is a finitely generated $Z_{\mathfrak{B}}$-module, which finishes the proof of Theorem 1.1.

7. Application to Hecke eigenspaces

Let R be a linearly compact local $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}$-algebra with residue field k; we do not assume that R is Noetherian. If $x : R \to \overline{\mathbb{Q}}_p$ is an $\mathcal{O}$-algebra homomorphism, then we denote by T_x the specialisation of the universal pseudorepresentation $T : \mathcal{G}_{\mathbb{Q}_p} \to R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}$ along $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}} \to R \xrightarrow{x} \overline{\mathbb{Q}}_p$.

Let M be an object of $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, which we assume to be $\mathcal{O}$-torsion-free. Then $\Pi(M) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(M, L)$ is a unitary L-Banach-space representation of G.

We assume that we are given a continuous action of R on M, which commutes with the action of G, such that the following hold:

- the action of R on M is faithful;
- the two actions of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}$ on M induced by the maps

$$R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}} \to R, \qquad R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}} \to Z_{\mathfrak{B}}$$

  coincide; and
- M is a finitely generated $R[\![K]\!]$-module.

Theorem 7.1. *Let $x : R \to \overline{\mathbb{Q}}_p$ be an $\mathcal{O}$-algebra homomorphism and let $\Pi(M)[\mathfrak{m}_x]$ be the subspace of $\Pi(M)$ annihilated by the kernel of x. Then under the foregoing assumptions, $\Pi(M)[\mathfrak{m}_x]$ is nonzero and is of finite length in $\mathrm{Ban}^{\mathrm{adm}}_{G,\zeta}(L)$. Moreover,*

- *if T_x is the trace of an absolutely irreducible Galois representation defined over $\kappa(x)$, then*

$$\Pi(M)[\mathfrak{m}_x] \cong \Pi^{\oplus m}$$

  for some multiplicity $m > 0$, where Π is an absolutely irreducible nonordinary $\kappa(x)$-Banach-space representation of G satisfying $\mathrm{tr}\,\check{\mathbf{V}}(\Pi) = T_x$; and
- *if T_x is the trace of a reducible Galois representation, then (after a possible extension of scalars) all the irreducible subquotients of $\Pi(M)[\mathfrak{m}_x]$ occur as subquotients of a direct sum of unitary parabolic induction*

$$(\mathrm{Ind}_B^G \psi_1 \otimes \psi_2\varepsilon^{-1})_{\mathrm{cont}} \oplus (\mathrm{Ind}_B^G \psi_2 \otimes \psi_1\varepsilon^{-1})_{\mathrm{cont}},$$

  where $\psi_1, \psi_2 : \mathcal{G}_{\mathbb{Q}_p} \to \kappa(x)^\times$ are characters such that $T_x = \psi_1 + \psi_2$.

Proof. Since $P_{\mathfrak{B}}$ is a projective generator for $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$, the functor $N \mapsto \mathrm{m}(N) := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P_{\mathfrak{B}}, N)$ induces an equivalence of categories between $\mathfrak{C}(\mathcal{O})_{\mathfrak{B}}$ and the category of right pseudocompact $E_{\mathfrak{B}}$-modules. The inverse functor is given by $\mathrm{m} \mapsto \mathrm{m}\,\widehat{\otimes}_{E_{\mathfrak{B}}} P_{\mathfrak{B}}$. In particular, the assumption that R acts faithfully on M implies that R acts faithfully on $\mathrm{m}(M)$.

We claim that $\mathrm{m}(M)$ is a finitely generated R-module. The topological Nakayama's lemma implies that it is enough to show that $k\,\widehat{\otimes}_R\,\mathrm{m}(M)$ is a finite-dimensional k-vector space. Since $k\,\widehat{\otimes}_R\,\mathrm{m}(M) \cong \mathrm{m}(k\,\widehat{\otimes}_R\, M)$, it is enough to show that $k\,\widehat{\otimes}_R\, M$ is of finite length in $\mathfrak{C}(\mathcal{O})$. Since by assumption M is a finitely generated $R[\![K]\!]$-module, $k\,\widehat{\otimes}_R\, M$ is a finitely generated $k[\![K]\!]$-module. Since by assumption the actions of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}$ on M induced by $Z_{\mathfrak{B}}$ and by R, coincide we deduce that the maximal ideal of $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\bar{\rho}_{\mathfrak{B}}}$ annihilates $k\,\widehat{\otimes}_R\, M$. Corollary 6.7(4) applied to $N = k\,\widehat{\otimes}_R\, M$ implies the claim.

Since $\mathrm{m}(M)$ is a finitely generated and faithful R-module, its localisation $\mathrm{m}(M)_{\mathfrak{m}_x}$ is a finitely generated faithful $R_{\mathfrak{m}_x}$-module. If $\mathrm{m}(M) \otimes_R \kappa(x) = 0$, then $\mathrm{m}(M)_{\mathfrak{m}_x} = 0$ by Nakayama's lemma, and since

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$R_{\mathfrak{m}_x}$ acts faithfully, $R_{\mathfrak{m}_x} = 0$ and hence $\kappa(x) = 0$, giving a contradiction. In particular, $\mathrm{m}(M) \otimes_R \kappa(x)$ is a nonzero, finite-dimensional $\kappa(x)$-vector space. Since R is a compact $\mathcal{O}$-module and $\kappa(x)$ is a subfield of $\overline{\mathbb{Q}}_p$, we have that $\kappa(x)$ is a finite extension of L and the image of R is contained in the ring of integers of $\kappa(x)$.

Let Q be the maximal $\mathcal{O}$-torsion-free Hausdorff quotient of $M/\mathfrak{m}_x M$. It follows from [44, Proposition 1.3] that $\Pi(Q)$ is a closed subspace of $\Pi(M)$, which then implies that $\Pi(Q) = \Pi(M)[\mathfrak{m}_x]$. It follows from the equivalence of categories already explained that $\mathrm{m}(Q)$ is isomorphic to the image of $\mathrm{m}(M)$ in $\mathrm{m}(M) \otimes_R \kappa(x)$. In particular, Q and thus $\Pi(Q)$ are nonzero.

The last two assertions follow from the antiequivalence (20) and Corollaries 6.10 and 6.16. □

Remark 7.2. If M is finitely generated as a $\mathcal{O}[\![K]\!]$-module, then the argument in the proof of Theorem 7.1 shows that $\mathrm{m}(M)$ is a finitely generated $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$-module, and since R acts faithfully on $\mathrm{m}(M)$, then R is a finitely generated $R^{\mathrm{ps},\zeta\varepsilon}_{\mathrm{tr}\,\bar{\rho}_{\mathfrak{B}}}$-module and hence is Noetherian.

The result allows us to remove the restrictions imposed on the Galois representation $\bar{\rho}_{\mathfrak{m},p}$ in [38, Corollary 6.3.6], by taking M to be the Pontryagin dual of the representation denoted by $\tilde{H}^1(K^p, E/\mathcal{O})_{\mathfrak{m},\zeta'}$ in [38, Theorem 6.3.5] and taking R to be the closure of the subring generated by the Hecke operators in $\mathrm{End}^{\mathrm{cont}}_{\mathcal{O}}(M)$. Since [38, Corollary 6.3.6] is the only place where the restriction on p is used, the proof of [38, Theorem 6.4.7] goes through without a change to give the following result:

Theorem 7.3 (Lue Pan + ε). *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(L)$ be promodular and absolutely irreducible. If ρ is unramified outside finitely many places and $\rho|_{\mathcal{G}_{\mathbb{Q}_p}}$ is Hodge–Tate with weights $0, 0$, then ρ is associated to a weight 1 modular form.*

The promodular condition means that the Hecke eigenvalues associated to ρ appear in completed cohomology; see [38, Definition 6.1.2] for the precise definition. The original theorem in Lue Pan's paper had to additionally assume that if p is 2 or 3, then $(\bar{\rho}|_{\mathcal{G}_{\mathbb{Q}_p}})^{\mathrm{ss}}$ is not isomorphic to $\chi \oplus \chi\omega$ for any character $\chi : \mathcal{G}_{\mathbb{Q}_p} \to k^\times$.

A. Normality of $R^{\mathrm{ps}}[1/p]$

Let $\mathcal{G}$ be a profinite group satisfying Mazur's finiteness condition at p: The group of continuous group homomorphisms $\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{grp}}(\mathcal{G}', \mathbb{F}_p)$ is finite for every open subgroup $\mathcal{G}'$ of $\mathcal{G}$. Let $\bar{\rho} : \mathcal{G} \to \mathrm{GL}_d(k)$ be a continuous semisimple representation such that all the irreducible summands of $\bar{\rho}$ are absolutely irreducible. Let $\psi : \mathcal{G} \to \mathcal{O}^\times$ be a character lifting $\det \bar{\rho}$. Let $\bar{D} : k[\mathcal{G}] \to k$ be the pseudorepresentation associated to $\bar{\rho}$ in [18], so that $\bar{D}(1 + tg) = \det(1 + t\bar{\rho}(g))$ for all $g \in \mathcal{G}$. We may consider the framed deformation ring $R^{\square}_{\bar{\rho}}$, its quotient $R^{\square,\psi}_{\bar{\rho}}$ parameterising framed deformations of $\bar{\rho}$ with determinant equal to ψ, the universal deformation ring R^{ps} of $\bar{D}$, and its quotient $R^{\mathrm{ps},\psi}$ parameterising deformations of $\bar{D}$ with determinant ψ. This last ring is constructed as follows: If $D^u : \mathcal{G} \to R^{\mathrm{ps}}$ is the universal deformation of $\bar{D}$, then for each $g \in \mathcal{G}$, we have that $D^u(1 + tg) = a_0(g) + \cdots + a_d(g)t^d$, with $a_i(g) \in R^{\mathrm{ps}}$, and $R^{\mathrm{ps},\psi}$ is the quotient of R^{ps} by the ideal generated $\psi(g)a_d(g) - 1$ for all $g \in \mathcal{G}$. The finiteness condition on $\mathcal{G}$ ensures that all these rings are Noetherian. The characteristic polynomial of the universal framed deformations of $\bar{\rho}$ induces maps $R^{\mathrm{ps}} \to R^{\square}_{\bar{\rho}}$ and $R^{\mathrm{ps},\psi} \to R^{\square,\psi}_{\bar{\rho}}$.

Theorem A.1. *If $R^{\square}_{\bar{\rho}}[1/p]$ is normal, then both $R^{\mathrm{ps}}[1/p]$ and the associated rigid space $(\mathrm{Spf}\, R^{\mathrm{ps}})^{\mathrm{rig}}$ are normal.*

We also prove a version of the theorem with a fixed determinant. We apply this theorem to $\mathcal{G} = \mathcal{G}_{\mathbb{Q}_p}$ to prove that the rings $R^{\mathrm{ps},\psi}$, R^{ps} and associated rigid analytic spaces are normal for all 2-dimensional $\bar{\rho}$. There is essentially one case that we need to handle, namely $\bar{\rho} = \mathbf{1} \oplus \omega$, where ω is the cyclotomic character modulo p; in the other cases, all the rings are regular. The trickiest cases are when $p = 2$ and $p = 3$. The case $p = 2$ is treated in [22]. We deal with the case $p = 3$ using the work of Böckle [7].

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The argument of [22] has been extended by Iyengar in [33], to the case when $\bar{\rho}$ is the trivial d-dimensional representation of a Galois group of a p-adic field F, under the assumption that F contains a primitive 4th root of unity if $p = 2$. Thus our theorem applies in that setting.[3]

We will split the proof into several steps. We start with commutative algebra lemmas and recall that all excellent rings are G-rings [45, Tag 07QS].

Lemma A.2. *Let A be a G-ring and set $\mathfrak{p} \in \operatorname{Spec} A$. Then $A_{\mathfrak{p}}$ satisfies Serre's condition (R_i) (resp., (S_i)) if and only if the completion $A_{\mathfrak{p}}$ at $\mathfrak{p}$ does.*

Proof. Let $B = A_{\mathfrak{p}}$ and let $\hat{B}$ be the completion of $A_{\mathfrak{p}}$ at $\mathfrak{p}$. Since A is a G-ring, the fibre rings $\kappa(\mathfrak{q}) \otimes_B \hat{B}$ are regular for all $\mathfrak{q} \in \operatorname{Spec} B$. The assertion follows from [36, Theorem 23.9]. □

Lemma A.3. *Let A be a complete local Noetherian $\mathcal{O}$-algebra with residue field k and $B = A[\![x_1, \dots, x_r]\!]$, set $\mathfrak{q} \in \operatorname{Spec} B[1/p]$, and let $\mathfrak{p}$ be the image of $\mathfrak{q}$ in $\operatorname{Spec} A[1/p]$. Then $A_{\mathfrak{p}}$ satisfies Serre's condition (R_i) (resp., (S_i)) if and only if $B_{\mathfrak{q}}$ does. In particular, $A[1/p]$ is normal if and only if $B[1/p]$ is normal.*

Proof. The proof is a variation on [15, Appendix A]. We may assume that A and hence B are $\mathcal{O}$-torsion-free. Set $\mathfrak{p}' \in \operatorname{Spec} A_{\mathfrak{p}} \subset \operatorname{Spec} A$. We claim that the ring $\kappa(\mathfrak{p}') \otimes_A B$ is regular. By Cohen's structure theorem, there is a subring $C \subset A/\mathfrak{p}'$ such that C is formally smooth over $\mathcal{O}$ and $A/\mathfrak{p}'$ is finite over C. Then

$$\kappa(\mathfrak{p}') \otimes_A B \cong \kappa(\mathfrak{p}') \otimes_{A/\mathfrak{p}'} B/\mathfrak{p}'B.$$

Since $A/\mathfrak{p}'$ is finite over C, we have

$$B/\mathfrak{p}'B = (A/\mathfrak{p}')[\![x_1, \dots, x_r]\!] \cong A/\mathfrak{p}' \otimes_C C[\![x_1, \dots, x_r]\!].$$

Thus

$$\kappa(\mathfrak{p}') \otimes_A B \cong \kappa(\mathfrak{p}) \otimes_{Q(C)} Q(C) \otimes_C C[\![x_1, \dots, x_r]\!],$$

where $Q(C)$ is the quotient field of C. Since C is formally smooth over $\mathcal{O}$, the ring $C[\![x_1, \dots, x_r]\!]$ is isomorphic to a ring of formal power series over $\mathcal{O}$, and thus is regular. Tensoring with $Q(C)$ over C is just localisation with respect to the multiplicative set $C \setminus \{0\}$, and thus $Q(C) \otimes_C C[\![x_1, \dots, x_r]\!]$ is regular. Since $Q(C)$ is of characteristic 0, the extension $\kappa(\mathfrak{p})/Q(C)$ is separable, and it follows from [15, Lemma A.3] that $\kappa(\mathfrak{p}) \otimes_{Q(C)} Q(C) \otimes_C C[\![x_1, \dots, x_r]\!]$ is regular. We deduce that $\kappa(\mathfrak{p}') \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}$ is regular, since it is a localisation of $\kappa(\mathfrak{p}') \otimes_A B$ at $\mathfrak{q}$.

It follows from [36, Theorem 23.9] that $A_{\mathfrak{p}}$ satisfies (R_i) (resp., (S_i)) if and only if $B_{\mathfrak{q}}$ does. To conclude that $A[1/p]$ is normal if and only if $B[1/p]$ is, we only have to show that the map $\operatorname{Spec} B[1/p] \to \operatorname{Spec} A[1/p]$ is surjective, and this is clear because $(\mathfrak{p}, x_1, \dots, x_r)$ maps to $\mathfrak{p}$. □

Lemma A.4. *Let $A \to B$ be a finite étale map of local rings. Then A satisfies Serre's condition (R_i) (resp., (S_i)) if and only if B does.*

Proof. If $\mathfrak{p} \in \operatorname{Spec} A$, then the fibre ring $\kappa(\mathfrak{p}) \otimes_A B$ is a finite étale $\kappa(\mathfrak{p})$-algebra and hence a product of fields, and thus is regular. The assertion follows from [36, Theorem 23.9]. □

Proposition A.5. *Let L' be a finite extension of L and let $\rho : \mathcal{G} \to \mathrm{GL}_n(L')$ be a continuous representation with mod p semisimplification isomorphic to $\bar{\rho}$. If $R^{\square}_{\bar{\rho}}[1/p]$ is normal, then the ring $R^{\square}_{\rho}$, representing the framed deformations of ρ to Artinian L'-algebras, is also normal.*

Proof. We may choose a finite extension L'' of L' with the ring of integers $\mathcal{O}''$ and residue field k'' such that $\rho \otimes_{L'} L''$ has a $\mathcal{G}$-invariant $\mathcal{O}''$-lattice Θ with $\Theta \otimes_{\mathcal{O}''} k'' \cong \bar{\rho} \otimes_k k''$ (see the proof of [22, Lemma 9.5]). Thus Θ is a deformation of $\bar{\rho} \otimes_k k''$ to $\mathcal{O}''$.

[3] This has been further generalised in [8] for all p-adic fields F and all $\bar{\rho}$.

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It follows from Lemma A.4 that $R^{\square}_{\rho}$ is normal if and only if $L'' \otimes_{L'} R^{\square}_{\rho}$ is normal. The same argument shows that $L'' \otimes_L R^{\square}_{\bar\rho}[1/p]$ is normal. Moreover, we may identify $L'' \otimes_{L'} R^{\square}_{\rho}$ with the framed deformation ring of $\rho \otimes_{L'} L''$ to local Artinian L''-algebras, and $\mathcal{O}'' \otimes_{\mathcal{O}} R^{\square}_{\bar\rho}$ with the framed deformation ring of $\bar\rho \otimes_k k''$ to local Artinian $\mathcal{O}''$-algebras. After these identifications, we may assume that $L = L' = L''$, and so Θ is a deformation of $\bar\rho$ to $\mathcal{O}$ and hence induces an $\mathcal{O}$-algebra homomorphism $x : R^{\square}_{\bar\rho} \to \mathcal{O}$.

It follows from [35, Lemma 2.3.3, Proposition 2.3.5] that $R^{\square}_{\rho}$ is isomorphic to the completion of $(R^{\square}_{\bar\rho})_{\mathfrak{p}}$ at $\mathfrak{p} = \operatorname{Ker} x$. Since $R^{\square}_{\bar\rho}[1/p]$ is normal, $(R^{\square}_{\bar\rho})_{\mathfrak{p}}$ will satisfy (R_1) and (S_2). Lemma A.2 implies that the same holds for the completion. Thus $R^{\square}_{\rho}$ is normal by Serre's criterion for normality [36, Theorem 23.8]. □

Lemma A.6. *Let A be a normal Noetherian ring and let G be a group acting on A by ring automorphisms. Then the subring of G-invariants A^G is normal.*

Proof. If A is a domain, then the assertion is proved in [13, Proposition 6.4.1]. The same proof works in our setting, as we will explain for the lack of a reference. Since A is Noetherian and normal, it is a finite product of normal domains. Let $\operatorname{Frac}(A)$ denote its total ring of fractions. Then $\operatorname{Frac}(A)$ is a finite product of fields. The group G acts on $\operatorname{Frac}(A)$ and we have

$$A^G = A \cap \operatorname{Frac}(A)^G. \tag{26}$$

We claim that $\operatorname{Frac}(A)^G$ is a finite product of fields. The claim implies that $\operatorname{Frac}(A)^G$ is its own ring of fractions. Since A is normal, equation (26) implies that A^G is reduced and integrally closed in its ring of fractions and has only finitely many minimal prime ideals – and hence is normal by [45, Tag 037B, Lemma 10.37.16].

To prove the claim, we note that $\operatorname{Spec}\operatorname{Frac}(A)$ consists of finitely many primes and is in bijection with the set $\mathscr{E}$ of idempotents $e \in \operatorname{Frac}(A)$, such that $e\operatorname{Frac}(A)e$ is a field. We have $1 = \sum_{e\in\mathscr{E}} e$ and $ee' = 0$ if $e \neq e'$. If $e \in \operatorname{Frac}(A)$ is a G-invariant idempotent, then

$$\operatorname{Frac}(A)^G = (e\operatorname{Frac}(A)e)^G \times ((1-e)\operatorname{Frac}(A)(1-e))^G,$$

and thus we may assume that the action of G on $\mathscr{E}$ is transitive. If $x \in \operatorname{Frac}(A)^G$ and $ex = 0$ for some $e \in \mathscr{E}$, then using the transitivity of the action we obtain that $ex = 0$ for all $e \in \mathscr{E}$, and so $x = \sum_{e\in\mathscr{E}} ex = 0$. Hence, if $x \in \operatorname{Frac}(A)^G$ is nonzero, then ex is nonzero, and we denote its inverse in the field $e\operatorname{Frac}(A)e$ by $(xe)^{-1}$. If we let $y = \sum_{e\in\mathscr{E}} (xe)^{-1}e \in \operatorname{Frac}(A)$, then $xy = 1$. Thus x is a unit in $\operatorname{Frac}(A)$ and its inverse y is unique. Uniqueness implies that y is G-invariant. Thus if G acts transitively on $\mathscr{E}$, then $\operatorname{Frac}(A)^G$ is a field. □

Proof of Theorem A.1. Let $D^u : \mathscr{G} \to R^{\mathrm{ps}}$ be the universal pseudorepresentation lifting $\overline{D}$. Let $\mathrm{CH}(D^u)$ be the closed two-sided ideal of $R^{\mathrm{ps}}[\![\mathscr{G}]\!]$ defined in [18, Section 1.17], so that $E := R^{\mathrm{ps}}[\![\mathscr{G}]\!]/\mathrm{CH}(D^u)$ is the largest quotient of $R^{\mathrm{ps}}[\![\mathscr{G}]\!]$ where the Cayley–Hamilton theorem for D^u holds. Following [18, Section 1.17], we will call such an algebra a Cayley–Hamilton R^{ps}-algebra of degree d. Then E is a finitely generated R^{ps}-module [48, Proposition 3.6]. If $f : E \to M_d(B)$ is a homomorphism of R^{ps}-algebras for a commutative R^{ps}-algebra B, then we say f is a homomorphism of Cayley–Hamilton algebras if $\det \circ f : E \to B$ is equal to the specialisation of D^u along $R^{\mathrm{ps}} \to B$.

There is a commutative R^{ps}-algebra A^{gen} together with a homomorphism of R^{ps}-algebras $j : E \to M_d(A^{\mathrm{gen}})$ satisfying the following universal property: If $f : E \to M_d(B)$ is a map of Cayley–Hamilton R^{ps}-algebras for a commutative R^{ps}-algebra B, then there is a unique map $\tilde f : A^{\mathrm{gen}} \to B$ of R^{ps}-algebras such that $f = M_d(\tilde f) \circ j$ (see, for example, [48, Theorem 3.8] or [8, Lemma 3.1]). Since E is finitely generated as an R^{ps}-module, A^{gen} is of finite type over R^{ps}.

Let $\Lambda_i : E \to R^{\mathrm{ps}}$, $0 \le i \le d$, be the coefficients of the characteristic polynomial of D^u; these are homogeneous polynomial laws satisfying $D^u(t-a) = \sum_{i=0}^{n} (-1)^i \Lambda_i(a) t^{d-i}$ in $R^{\mathrm{ps}}[t]$ for all $a \in E$ [18, Section 1.10]. Now $E[1/p]$ is a $\mathbb{Q}$-algebra and the pair $(E[1/p], \Lambda_1)$ is a trace algebra satisfying the

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d-dimensional Cayley–Hamilton identity in the sense of [43, Definition 2.6] (see [16, Footnote 10]). Moreover, for $\mathbb{Q}$-algebras, the homomorphisms of Cayley–Hamilton algebras coincide with the notion of maps of algebras with traces in [43, Section 2.5]. Thus $j : E[1/p] \to M_d(A^{\mathrm{gen}}[1/p])$ is injective and its image is equal to the GL_d-invariants [43, Theorem 2.6]. Moreover, $R^{\mathrm{ps}}[1/p] = (A^{\mathrm{gen}}[1/p])^{\mathrm{GL}_d}$ (see [16, Proposition 2.3] and [48, Theorem 2.20]). By Lemma A.6 it is enough to show that $A^{\mathrm{gen}}[1/p]$ is normal. Further, it is enough to show that the localisation of $A^{\mathrm{gen}}[1/p]$ at every maximal ideal is normal (see [45, Tag 037B, Lemma 10.37.10]). (The superscript 'gen' in A^{gen} stands for generic matrices in [43, Section 1.1].)

Let $\mathfrak{m}$ be a maximal ideal of $A^{\mathrm{gen}}[1/p]$. Its residue field $\kappa(\mathfrak{m})$ is a finite extension of L, as $A^{\mathrm{gen}}[1/p]$ is finitely generated over $R^{\mathrm{ps}}[1/p]$. By specialising j at $\mathfrak{m}$ we obtain a continuous representation $\rho : \mathcal{G} \to \mathrm{GL}_d(\kappa(\mathfrak{m}))$ such that

$$\det(1 + t\rho(g)) = D^u \otimes_{R^{\mathrm{ps}}} \kappa(\mathfrak{m})(1 + tg), \quad \forall g \in \mathcal{G}.$$

This implies that if we choose a $\mathcal{G}$-invariant $\mathcal{O}_{\kappa(\mathfrak{m})}$-lattice Θ in ρ, then the semisimplification of $\Theta/\varpi_{\kappa(\mathfrak{m})}\Theta$ is isomorphic to $\bar{\rho}$, so that we are in the setup of Proposition A.5. The universal property of A^{gen} implies that the completion of $A^{\mathrm{gen}}[1/p]$ at $\mathfrak{m}$ is the universal framed deformation ring $R^{\square}_{\rho}$, which is normal by Proposition A.5. Since A^{gen} is finitely generated over R^{ps}, which is a complete local Noetherian ring, A^{gen} and hence its localisation $(A^{\mathrm{gen}}[1/p])_{\mathfrak{m}}$ are excellent, and thus a G-ring. Lemma A.2 implies that $(A^{\mathrm{gen}}[1/p])_{\mathfrak{m}}$ satisfies (R_1) and (S_2) and hence is normal.

Let $\tilde{R}$ be the normalisation of R^{ps} in $R^{\mathrm{ps}}[1/p]$. Then $(\mathrm{Spf}\, R^{\mathrm{ps}})^{\mathrm{rig}} = (\mathrm{Spf}\, \tilde{R})^{\mathrm{rig}}$ [24, Lemma 7.2.2]. Since $R^{\mathrm{ps}}[1/p]$ is normal, so is $\tilde{R}$, and thus $(\mathrm{Spf}\, \tilde{R})^{\mathrm{rig}}$ is normal [24, Proposition 7.2.4(c)]. Alternatively, one could use the fact that the local rings of $(\mathrm{Spf}\, R^{\mathrm{ps}})^{\mathrm{rig}}$ are excellent [23, Theorem 1.1.3] and [24, Lemma 7.1.9] together with Lemma A.2. □

The following is a corollary to the proof; it does not require the assumption that $R^{\square}_{\bar{\rho}}[1/p]$ is normal:

Corollary A.7. *Let V be a free $R^{\square}_{\bar{\rho}}[1/p]$-module of rank d with $\mathcal{G}$-action given by $\rho^{\square} : \mathcal{G} \to \mathrm{GL}_d(R^{\square}_{\bar{\rho}})$. Then $(R^{\mathrm{ps}}[\![\mathcal{G}]\!]/\mathrm{CH}(D^u))[1/p]$ acts faithfully on V. The same holds with the fixed determinant.*

Proof. We use the notation of the proof of Theorem A.1, so that $E = R^{\mathrm{ps}}[\![\mathcal{G}]\!]/\mathrm{CH}(D^u)$ and there is a map $j : E \to M_d(A^{\mathrm{gen}})$ satisfying a universal property. This map is an injection after inverting p. Let V^{gen} be a free A^{gen}-module of rank d, with E-action given by j. Thus $E[1/p]$ acts faithfully on $V^{\mathrm{gen}}[1/p]$.

Suppose that $a \in E[1/p]$ kills off V. Since $E[1/p]$ acts faithfully on $V^{\mathrm{gen}}[1/p]$, there is a maximal ideal $\mathfrak{m}$ of $A^{\mathrm{gen}}[1/p]$ such that a acts nontrivially on $V^{\mathrm{gen}} \otimes_{A^{\mathrm{gen}}} A^{\mathrm{gen}}_{\mathfrak{m}}$. Let $\hat{A}^{\mathrm{gen}}_{\mathfrak{m}}$ be the completion of $A^{\mathrm{gen}}_{\mathfrak{m}}$ with respect to the maximal ideal. Since $\hat{A}^{\mathrm{gen}}_{\mathfrak{m}}$ is faithfully flat over $A^{\mathrm{gen}}_{\mathfrak{m}}$, a acts nontrivially on the completion of $V^{\mathrm{gen}}[1/p]$ at $\mathfrak{m}$, which we denote by $\hat{V}^{\mathrm{gen}}_{\mathfrak{m}}$. However, as explained in the proof of Theorem A.1, $\hat{V}^{\mathrm{gen}}_{\mathfrak{m}}$ is isomorphic as an E-module to the completion of V at a maximal ideal of $R^{\square}_{\bar{\rho}}[1/p]$. Since a annihilates V, it will also annihilate the completion, giving a contradiction.

Define $E^{\psi} := E \otimes_{R^{\mathrm{ps}}} R^{\mathrm{ps},\psi}$ and $A^{\mathrm{gen},\psi} := A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}}} R^{\mathrm{ps},\psi}$. Then $j : E \to M_d(A^{\mathrm{gen}})$ induces a map $j : E^{\psi} \to M_d(A^{\mathrm{gen},\psi})$ which satisfies the same universal property as j. Then the same proof works with $A^{\mathrm{gen},\psi}$ instead of A^{gen}. □

Lemma A.8. *Let R be a complete local Noetherian $\mathcal{O}$-algebra with residue field k, and let $\rho : \mathcal{G} \to \mathrm{GL}_d(R)$ be a continuous representation. Assume that R is $\mathcal{O}$-torsion-free and reduced, and the set of $x \in$ m-Spec $R[1/p]$ such that ρ_x is absolutely irreducible is dense in* Spec $R[1/p]$. *Then*

$$\mathcal{C} := \{X \in M_d(R) : X\rho(g) = \rho(g)X, \quad \forall g \in \mathcal{G}\}$$

consists of scalar matrices.

Proof. Set $X \in \mathcal{C}$ with matrix entries x_{ij}. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R. Since R is reduced, it embeds into $\prod_{s=1}^{n} \kappa(\mathfrak{p}_s)$. It is enough to show that the image of X in $M_d(\kappa(\mathfrak{p}_s))$ for $1 \le s \le n$ is scalar, since if the images of x_{ij} and $x_{ii} - x_{jj}$ for $i \neq j$ are zero in $\kappa(\mathfrak{p}_s)$ for $1 \le s \le n$, then they are

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zero in A and so X is scalar. If $x \in$ m-Spec $R[1/p]$ is such that $\rho_x : \mathcal{G} \to \mathrm{GL}_d(\kappa(x))$ is absolutely irreducible, then $\mathcal{C} \otimes_R \kappa(x)$ is 1-dimensional. Since $\mathfrak{q} \mapsto \dim_{\kappa(\mathfrak{q})} \mathcal{C} \otimes_R \kappa(\mathfrak{q})$ is upper semicontinuous and such x are dense, we deduce that $\dim_{\kappa(\mathfrak{p}_s)} \mathcal{C} \otimes_R \kappa(\mathfrak{p}_s) = 1$ for all minimal primes $\mathfrak{p}_s$. This implies that $\mathcal{C} \otimes_R \kappa(\mathfrak{p}_s)$ consists of scalar matrices. □

Proposition A.9. *Assume that $R^{\square,\psi}_{\bar\rho}[1/p]$ is nonzero. Then $R^{\square}_{\bar\rho}[1/p]$ is normal (resp., reduced) if and only if $R^{\square,\psi}_{\bar\rho}[1/p]$ is normal (resp., reduced).*

Proof. Let Γ be the pro-p completion of the abelianisation of $\mathcal{G}$. Because $\mathrm{Hom}^{\mathrm{cont}}_{\mathrm{grp}}(\mathcal{G}, \mathbb{F}_p)$ is finite, $\Gamma \cong \Delta \times \mathbb{Z}_p^r$, where Δ is a finite p-group. The map $\mathcal{G} \to (R^{\square}_{\bar\rho})^{\times}$, $g \mapsto \psi(g)^{-1} \det \rho^{\square}(g)$, factors through Γ and thus induces an $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\![\Gamma]\!] \to R^{\square}_{\bar\rho}$; and $R^{\square,\psi}_{\bar\rho}$ is equal to the quotient of $R^{\square}_{\bar\rho}$ by the augmentation ideal in $\mathcal{O}[\![\Gamma]\!]$.

Let $\mathcal{X}(\Gamma)$ be the functor which sends a local Artinian $\mathcal{O}$-algebra $(A, \mathfrak{m}_A)$ to the group of continuous characters $\chi : \Gamma \to 1 + \mathfrak{m}_A$. This functor is represented by Spf $\mathcal{O}[\![\Gamma]\!]$. For such $(A, \mathfrak{m}_A)$, the group $\mathcal{X}(\Gamma)(A)$ acts on $D^{\square}(A)$ by twisting. The action induces a homomorphism of local $\mathcal{O}$-algebras $\gamma : R^{\square}_{\bar\rho} \to R^{\square}_{\bar\rho} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Gamma]\!]$. Let $R^{\mathrm{inv}} = \{a \in R^{\square}_{\bar\rho} : \gamma(a) = a \otimes 1\}$ be the subring of $\mathcal{X}(\Gamma)$-invariants in $R^{\square}_{\bar\rho}$. Analogously, $\mathcal{X}(\Delta)$ acts on $D^{\square}$, the action induces the map $\delta : R^{\square}_{\bar\rho} \to R^{\square}_{\bar\rho} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\Delta]$ and we let $R^{\mathrm{inv.t}} = \{a \in R^{\square}_{\bar\rho} : \delta(a) = a \otimes 1\}$ be the subring of $\mathcal{X}(\Delta)$-invariants in $R^{\square}_{\bar\rho}$.

The action of $\mathcal{X}(\Gamma)$ and $\mathcal{X}(\Delta)$ on $D^{\square}$ is free, since if $\rho_A : \mathcal{G}_{\mathbb{Q}_p} \to \mathrm{GL}_d(A)$ is a framed deformation of $\bar\rho$, then for each $g \in \mathcal{G}_{\mathbb{Q}_p}$ at least one matrix entry of $\rho_A(g)$ will not lie in $\mathfrak{m}_A$ and thus is a unit. Hence, $\rho_A(g) = \rho_A(g)\chi_A(g)$ for all $g \in G$ implies that χ_A is the trivial character.

The map $R^{\mathrm{inv.t}} \to R^{\square}_{\bar\rho}$ is finite and becomes étale after inversion of p by [1, Proposition 1.1.11(2)]. Thus $R^{\square}_{\bar\rho}[1/p]$ is normal if and only if $R^{\mathrm{inv.t}}[1/p]$ is normal, by Lemma A.4. Since R^{inv} is the subring of $\mathcal{X}(\Gamma/\Delta)$-invariants in $R^{\mathrm{inv.t}}$ and $\Gamma/\Delta \cong \mathbb{Z}_p^r$, we have $R^{\mathrm{inv.t}} \cong R^{\mathrm{inv}}[\![x_1, \dots, x_r]\!]$ by [1, Proposition 1.1.11(2)]. Thus $R^{\mathrm{inv.t}}[1/p]$ is normal if and only if $R^{\mathrm{inv}}[1/p]$ is normal, by Lemma A.3. The map $R^{\mathrm{inv}} \to R^{\square,\psi}_{\bar\rho}$ is finite and becomes étale after inversion of p by [1, Proposition 1.1.11(3)]. Lemma A.4 implies that $R^{\square,\psi}_{\bar\rho}[1/p]$ is normal if and only if $R^{\mathrm{inv}}[1/p]$ is normal. Putting all the equivalences together proves the assertion.

Since reducedness is equivalent to (R_0) and (S_1), the same proof works. □

Corollary A.10. *If $R^{\square}_{\bar\rho}[1/p]$ is normal, then $R^{\mathrm{ps},\psi}[1/p]$ and the associated rigid analytic space* (Spf $R^{\mathrm{ps},\psi})^{\mathrm{rig}}$ *are normal.*

Proof. Proposition A.9 implies that $R^{\square,\psi}_{\bar\rho}[1/p]$ is normal. The proof of Theorem A.1, with $R^{\square}_{\bar\rho}[1/p]$ replaced by $R^{\square,\psi}_{\bar\rho}[1/p]$, implies the assertion. □

Proposition A.11. *Let $E = R^{\mathrm{ps}}[\![\mathcal{G}]\!]/\mathrm{CH}(D^u)$, E_{tf} the maximal $\mathcal{O}$-torsion-free quotient of E, $Z(E_{\mathrm{tf}})$ the centre of E_{tf} and $R^{\mathrm{ps}}_{\mathrm{tf}}$ the maximal $\mathcal{O}$-torsion-free quotient of R^{ps}. Then $R^{\mathrm{ps}}_{\mathrm{tf}}$ is a subring of $Z(E_{\mathrm{tf}})$.*

If $R^{\square}_{\bar\rho}[1/p]$ is reduced and the set $x \in$ m-Spec *$R^{\square}_{\bar\rho}[1/p]$, such that $\rho^{\square}_x$ is absolutely irreducible, is dense in* Spec *$R^{\square}_{\bar\rho}[1/p]$, then $d \cdot Z(E_{\mathrm{tf}}) \subset R^{\mathrm{ps}}_{\mathrm{tf}}$. In particular, if $p \nmid d$, then $R^{\mathrm{ps}}_{\mathrm{tf}} = Z(E_{\mathrm{tf}})$. Moreover, the same holds for rings with fixed determinant.*

Proof. As in the proof of Theorem A.1, there is an injection

$$j : E[1/p] \hookrightarrow M_d(A^{\mathrm{gen}}[1/p]).$$

Moreover, $\mathrm{tr} \circ j$ induces a surjection $E[1/p] \twoheadrightarrow R^{\mathrm{ps}}[1/p]$. Thus $R^{\mathrm{ps}}_{\mathrm{tf}}$ is a subring of $Z(E_{\mathrm{tf}})$. Corollary A.7 gives us an injection $E_{\mathrm{tf}} \hookrightarrow M_d(R^{\square}_{\bar\rho}[1/p])$. If $a \in E_{\mathrm{tf}}$, then the characteristic polynomial of $j(a)$ has coefficients in $R^{\mathrm{ps}}_{\mathrm{tf}}$. Moreover, $Z(E_{\mathrm{tf}})$ is contained in the centraliser of $\rho^{\square}(\mathcal{G})$ in $M_d(R^{\square}_{\bar\rho}[1/p])$. According to Lemma A.8, the centraliser is equal to scalar matrices. Since $j(z)$ is a scalar matrix, we deduce that $dz \in R^{\mathrm{ps}}_{\mathrm{tf}}$.

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It follows from Proposition A.9 that $R^{\square,\psi}_{\bar\rho}[1/p]$ is reduced. Since twisting by characters does not change the property of being absolutely irreducible, the proof of Proposition A.9 shows that the absolutely irreducible locus is dense in Spec $R^{\square,\psi}_{\bar\rho}[1/p]$. Then the same proof goes through. □

Proposition A.12. *If $p=3$, $\mathscr{G}=\mathscr{G}_{\mathbb{Q}_3}$ and $\bar\rho=\mathbf{1}\oplus\omega$, then $R^{\square}_{\bar\rho}[1/p]$ is normal.*[4]

Proof. We will first relate the framed deformation ring $R^{\square}_{\bar\rho}$ to the ring studied in [7]. Let μ_3 be the group of 3rd roots of unity in $\overline{\mathbb{Q}}_3$, let $E=\mathbb{Q}_3(\mu_3)$ and let $E(3)$ be the compositum of all extensions $E\subset E'\subset\overline{\mathbb{Q}}_3$ such that $[E':E]$ is a power of 3. Then the Galois group $\mathrm{Gal}(E(3)/E)$ is the maximal pro-3 quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}_3/E)$, and thus the map $\rho^{\square}:\mathscr{G}_{\mathbb{Q}_3}\to\mathrm{GL}_2(R^{\square}_{\bar\rho})$ factors through the surjection $\mathscr{G}_{\mathbb{Q}_3}\twoheadrightarrow\mathrm{Gal}(E(3)/\mathbb{Q}_3)$. Since $\mathrm{Gal}(E/\mathbb{Q}_3)$ has order 2, Schur–Zassenhaus implies that the surjection $\mathrm{Gal}(E(3)/\mathbb{Q}_3)\twoheadrightarrow\mathrm{Gal}(E/\mathbb{Q}_3)$ has a splitting, which gives us an isomorphism

$$\mathrm{Gal}(E(3)/\mathbb{Q}_3)\cong\mathrm{Gal}(E(3)/E)\rtimes G,$$

where $G=\{1,\sigma\}$ is a subgroup of $\mathrm{Gal}(E(3)/\mathbb{Q}_3)$. One may define a closed subfunctor, denoted by EH_1 in [7], of the framed deformation functor $D^{\square}$ such that $\mathrm{EH}_1(A)$ consists of pairs (V_A,β_A), where V_A is a deformation of $\omega\oplus 1$ to A and $\beta_A=(v_1,v_2)$ is an A-basis of V_A lifting a fixed basis β_k of $\omega\oplus 1$, such that σ acts by -1 on v_1 and by 1 on v_2. It follows from the Iwahori decomposition for the group $1+M_2(\mathfrak{m}_A)$ that a framed deformation $(V_A,\beta_A)\in D^{\square}(A)$ can be conjugated to a framed deformation in $\mathrm{EH}_1(A)$ by a unique element of the form $\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$, with $b,c\in\mathfrak{m}_A$. Hence if EH_1 is represented by R, then

$$R^{\square}_{\bar\rho}\cong R[\![x,y]\!].$$

Now R is a complete intersection by [7, Theorem 1.1], thus so is $R^{\square}_{\bar\rho}$; and to show the normality of $R^{\square}_{\bar\rho}[1/p]$, it is enough to show that the singular locus in $R[1/p]$ has codimension at least 2.

Let $\rho:\mathrm{Gal}(E(3)/\mathbb{Q}_3)\to\mathrm{GL}_2(R)$ be the representation obtained for the action of the Galois group on V_R with respect to the basis β_R such that $\rho(\sigma)=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$. It follows from the argument of [22, Lemma 4.1] that $x\in$ m-Spec $R[1/p]$ is singular if and only if there is an exact sequence $0\to\delta\to\rho_x\to\delta\varepsilon\to 0$ for some character $\delta:\mathrm{Gal}(E(3)/\mathbb{Q}_3)\to\kappa(x)^{\times}$. Thus the singular locus is contained in the reducible locus, and it is enough to show that it has positive codimension inside the reducible locus: We know that R is a domain by [7, Theorem 1.1], and there are absolutely irreducible lifts of $\bar\rho$, so that the reducible locus has codimension 1 inside Spec R.

We will now describe the ring R as computed in [7] and compute the reducible locus. We know from [7, Lemma 3.2] that the representation $\rho:\mathrm{Gal}(E(3)/\mathbb{Q}_3)\to\mathrm{GL}_2(R)$ factors through a quotient $\mathrm{Gal}(E(3)/\mathbb{Q}_3)\twoheadrightarrow P\rtimes G$, where P is a pro-p group with generators x_1,x_2,x_3,x_4 and one relation

$$r=x_1^3[x_1,x_2][x_3,x_4][x_4,x_3^{-1}][x_2,x_1^{-1}]x_1^3,$$

where $[g,h]=ghg^{-1}h^{-1}$. The action of $\sigma\in G$ on the generators is given by

$$\sigma(x_1)=x_1^{-1},\qquad\sigma(x_2)=x_2,\qquad\sigma(x_3)=x_3^{-1},\qquad\sigma(x_4)=x_4.$$

Let $S=\mathscr{O}[\![a,a',b,b',c,c',d,d']\!]$ and let $A_i\in\mathrm{GL}_2(S)$ be the matrices

$$A_1=\begin{pmatrix}\sqrt{1+bc}&b\\c&\sqrt{1+bc}\end{pmatrix},\qquad A_2=\sqrt{1+a}\begin{pmatrix}\sqrt{1+d}&0\\0&\sqrt{1+d}^{-1}\end{pmatrix},$$
$$A_3=\begin{pmatrix}\sqrt{1+b'c'}&b'\\c'&\sqrt{1+b'c'}\end{pmatrix},\qquad A_4=\sqrt{1+a'}\begin{pmatrix}\sqrt{1+d'}&0\\0&\sqrt{1+d'}^{-1}\end{pmatrix},$$

[4]The statement is proved in [8, Corollary 4.22] without computing the equations for the deformation ring.

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and let

$$B = A_1^3[A_1, A_2][A_3, A_4][A_4, A_3^{-1}][A_2, A_1^{-1}]A_1^3.$$

Then [7, Theorem 4.1(c)] asserts that $R = S/I$, where I is the ideal of S generated by the matrix entries of $B - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\rho : P \rtimes G \to \mathrm{GL}_2(R)$ is obtained by mapping $\sigma \mapsto \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $x_i \mapsto A_i$ for $1 \le i \le 4$.

It follows from this description that the locus in $\operatorname{Spec} R$ parameterising reducible representations, where σ acts on the rank 1 subrepresentation by -1 (resp., 1), is equal to $V(c, c')$ (resp., $V(b, b')$).

The images of A_1 and A_3 in $\mathrm{GL}_2(R/(c, c'))$ are unipotent upper-triangular matrices. It is easy to compute the commutator of a unipotent upper-triangular matrix with a diagonal matrix. One obtains that the image of B in $\mathrm{GL}_2(S/(c, c'))$ is the matrix $\left(\begin{smallmatrix} 1 & 6b-2bd-2b'd' \\ 0 & 1 \end{smallmatrix}\right)$. Thus

$$R/(c, c') = S/(c, c', 3b - bd - b'd')$$

is an integral domain, as $S/(c, c') \cong \mathcal{O}[\![a, a', b, b', d, d']\!]$ is factorial and $3b - bd - b'd'$ is an irreducible element in $S/(c, c')$.

Let X^{sing} be the singular locus in $\operatorname{Spec} R[1/p]$. The point $x \in \operatorname{Spec} R/(c, c')$ corresponding to the representation $\left(\begin{smallmatrix} \varepsilon^3 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ will not lie in X^{sing}, since this representation is not an extension of $\delta\varepsilon$ by δ. Thus $X^{\mathrm{sing}} \cap \operatorname{Spec} R/(c, c')[1/p]$ is of codimension at least 1. In the same way, we obtain that $X^{\mathrm{sing}} \cap \operatorname{Spec} R/(b, b')[1/p]$ is of codimension at least 1 in $\operatorname{Spec} R/(b, b')[1/p]$. Thus X^{sing} is of codimension at least 1 in the reducible locus in $\operatorname{Spec} R[1/p]$ and of codimension at least 2 in $\operatorname{Spec} R[1/p]$. □

Proposition A.13. *If $\mathcal{G} = \mathcal{G}_{\mathbb{Q}_p}$, then $R^{\square}_{\bar\rho}[1/p]$ is normal and the absolutely irreducible locus is dense in* $\operatorname{Spec} R^{\square}_{\bar\rho}[1/p]$ *for all semisimple* 2*-dimensional $\bar\rho$.*

Proof. Since $R^{\square}_{\bar\rho}[1/p]$ is excellent, the singular locus is closed in $R^{\square}_{\bar\rho}[1/p]$. If it is nonempty, then it will contain a maximal ideal x such that

$$\operatorname{Hom}_{\mathcal{G}_{\mathbb{Q}_p}}(\rho^{\square}_x, \rho^{\square}_x(1)) \neq 0$$

(see [22, Lemma 4.1]). Thus $\bar\rho$ is of the form $\bar\chi \oplus \bar\chi\omega$. After twisting by a character, we may assume that $\bar\rho = \mathbf{1} \oplus \omega$. If $p = 2$ or $p = 3$, then $R^{\square}_{\bar\rho}[1/p]$ is normal by [22, Proposition 4.3] or Proposition A.12, respectively. If $p \ge 5$, it follows from the proof of [39, Proposition B2, Theorem B.3], based on the work of Böckle [6], that $R^{\square}_{\bar\rho}$ is formally smooth over $\mathcal{O}[\![x, y, z, w]\!]/(xy - zw)$. (The only change is that because in our setting $\bar\rho$ is split, the generator x_{p-2} maps to the matrix $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. This adds an extra variable but does not change the relation coming from [39, equation (261)].) Thus $R^{\square}_{\bar\rho}[1/p]$ is normal.

Hence, $R^{\square}_{\bar\rho}[1/p]$ is a product of normal domains, and if the absolutely irreducible locus were not dense, there would be a component without absolutely irreducible points. (Let I be the ideal of $R^{\square}_{\bar\rho}$ generated by the matrix entries of $(\rho^{\square}(gh) - \rho^{\square}(hg))^2$ for all $g, h \in \mathcal{G}_{\mathbb{Q}_p}$. Then a specialisation of $\rho^{\square}$ at $x \in \text{m-Spec}\, R^{\square}_{\bar\rho}[1/p]$ is absolutely irreducible over $\kappa(x)$ if and only if $x \notin V(I)$. Thus if an irreducible component of $R^{\square}_{\bar\rho}[1/p]$ contains an absolutely irreducible point, then such points are dense in the component.) In the course of the proof of Proposition A.11, we have shown that $R^{\mathrm{ps}}[1/p]$ is a subring of $R^{\square}_{\bar\rho}[1/p]$. Thus there would exist an irreducible component of $R^{\mathrm{ps}}[1/p]$ without absolutely irreducible points. This would contradict [17, Theorem 2.1]. □

Corollary A.14. *If $\mathcal{G} = \mathcal{G}_{\mathbb{Q}_p}$, then $R^{\mathrm{ps},\psi}[1/p]$, $R^{\mathrm{ps}}[1/p]$ and the corresponding rigid analytic spaces are normal for all semisimple* 2*-dimensional $\bar\rho$.*

Proof. The assertion follows from Proposition A.13, Theorem A.1 and Corollary A.10. □

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