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Weingarten Type Surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$

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Abstract. In this article, we study complete surfaces Σ, isometrically immersed in the product spaces $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ having positive extrinsic curvature K_e. Let K_i denote the intrinsic curvature of Σ. Assume that the equation $aK_i + bK_e = c$ holds for some real constants $a \neq 0$, $b > 0$, and c. The main result of this article states that when such a surface is a topological sphere, it is rotational.

1 Introduction

The holomorphic Hopf quadratic differential, defined on a surface having a constant mean curvature in $\mathbb{R}^3$, enabled Hopf to give a proof that topological spheres in $\mathbb{R}^3$ having constant mean curvature are rotational. A few years ago, Abresch and Rosenberg [1,2] discovered a holomorphic quadratic differential on constant mean curvature surfaces in the homogeneous 3-manifolds. With the aid of this quadratic differential, they extended Hopf's result to constant mean curvature topological spheres immersed in such homogeneous spaces.

In the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$, Aledo, Espinar, and Gálvez [3] associated a holomorphic quadratic differential with constant intrinsic curvature (Gaussian curvature) surfaces immersed in such product spaces, which enabled them to extend the classical Liebmann Theorem that in the euclidean space $\mathbb{R}^3$ ensure that the round spheres are the unique complete surfaces of positive constant intrinsic curvature. For complete surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ having positive extrinsic curvature, Gálvez, Espinar, and Rosenberg [8] proved that such surfaces are embedded and homeomorphic to either the euclidean sphere $\mathbb{S}^2$ or to the euclidean plane $\mathbb{R}^2$. Moreover, they constructed a quadratic differential on positive constant extrinsic surfaces that vanishes identically or its zeros are isolated with negative index. As a consequence, they proved that the complete immersions having positive constant extrinsic curvature in the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ are rotational spheres.

In this article, we consider complete surfaces Σ, isometrically immersed in the product spaces $\mathbb{H}^2 \times \mathbb{R}$ or $\mathbb{S}^2 \times \mathbb{R}$ having positive extrinsic curvature (non-constant) such that $aK_i + bK_e = c$, where K_i, K_e are the intrinsic and the extrinsic curvatures, respectively, and $a \neq 0$, $b > 0$, and c are real constants. Our goal is to prove that

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if Σ is a topological sphere, then Σ is rotational. In order to obtain this result, we first construct a quadratic differential Qdz^2 that vanishes identically or its zeros are isolated with negative index. This quadratic differential exists if $a + b \neq 0$, $2a + b \neq 0$ (§4.2). Moreover, we obtain vertical and horizontal height estimates which enable us to realize when Σ is a topological sphere (§4 and §5). In Section 6, we prove the main theorem.

The article is organized as follows: in Section 2 we give the definition of Weingarten type surfaces. Section 3 is devoted to the study of rotational examples. In Section 4 we construct a quadratic differential on a Weingarten type surface which vanishes identically or its zeros are isolated with negative index. We also establish horizontal and vertical height estimates. In Section 5 we study the non-existence of complete non-compact Weingarten type surfaces. In Section 6 we prove our main theorem.

2 Weingarten Type Surfaces Having Positive Extrinsic Curvature

For $\epsilon \in \{-1, 1\}$, we denote by $M^2(\epsilon)$ the complete, connected, simply-connected, two-dimensional space form having sectional curvature ϵ. That is, for $\epsilon = 1$, $M^2(\epsilon)$ denotes the canonical euclidean unit sphere $\mathbb{S}^2$ and for $\epsilon = -1$, $M^2(\epsilon)$ denotes the complete, connected, simply-connected hyperbolic plane $\mathbb{H}^2$ having sectional curvature -1. Also, we denote by $M^2(\epsilon) \times \mathbb{R}$ the product space (where $\mathbb{R}$ is the real line), endowed with the product metric.

Recall that the surface Σ is called a Weingarten surface if its two principal curvatures k_1 and k_2 are not independent one of another or, equivalently, if there exists a relation of the form $W(k_1, k_2) = 0$ for a smooth real function $W: \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ defined on a set $\mathcal{D}$.

In this article, we study complete, connected, surfaces Σ isometrically immersed in the product space $M^2(\epsilon) \times \mathbb{R}$ whose intrinsic and extrinsic curvature are linearly related.

Definition 2.1 Let $\varphi: \Sigma \to M^2(\epsilon) \times \mathbb{R}$ be an isometric immersion from a connected surface having intrinsic curvature K_i and extrinsic curvature K_e. We say that Σ is a Weingarten type surface or simply a W-surface if there exist three real numbers, $a \neq 0$, $b > 0$, and c such that,

$$aK_i + bK_e - c = 0. \tag{2.1}$$

Remark 2.2 The assumption $b > 0$ is not a restriction since we can multiply equation (2.1) by -1 if necessary.

For simplicity, we treat properties of an immersion φ as those of Σ and denote merely by Σ the image $\varphi(\Sigma)$. For example, we call Σ a W-surface in $M^2(\epsilon) \times \mathbb{R}$ instead of saying that the immersion φ is a W-surface in $M^2(\epsilon) \times \mathbb{R}$.

Let $\varphi: \Sigma \to M^2(\epsilon) \times \mathbb{R}$ be an isometric immersion from an orientable surface Σ into the product space $M^2(\epsilon) \times \mathbb{R}$. We choose a global unit normal vector field N and, as usual, we denote by $\nu = \langle N, \frac{\partial}{\partial t} \rangle$ the angle function of Σ. Here $\frac{\partial}{\partial t}$ denotes the tangent vector field to the real line $\mathbb{R}$. From [5], we have that the Gauss equation for such an immersed surface into the product space $M^2(\epsilon)\times\mathbb{R}$ is given by

$$K_i = K_e + \epsilon\nu^2. \tag{2.2}$$

As a consequence of the Gauss equation, we have the following.

Lemma 2.3 *Let $\varphi:\Sigma\to M^2(\epsilon)\times\mathbb{R}$ be an isometric immersion. Assume that Σ is a complete W-surface having positive extrinsic curvature K_e.*

(i) *Suppose $a+b>0$.*
  (a) *If $c\le 0$, then Σ is not compact,*
  (b) *If $\epsilon=-1$ and $c>b$, then Σ is closed.*
  (c) *If $\epsilon=1$ and $c>0$, then Σ is closed.*

(ii) *Suppose $a+b<0$.*
  (a) *If $c\ge 0$, then Σ is not compact.*
  (b) *If $\epsilon=-1$ and $c<0$, then Σ is closed.*
  (c) *If $\epsilon=1$ and $c<-b$, then Σ is closed.*
  (d) *If $\epsilon=1$ and $-b\le c<0$, then Σ cannot be closed.*

(iii) *For $a+b=0$, the angle function is constant.*

Proof As the extrinsic curvature of the W-surface is positive, Σ is orientable and we choose the unit global normal vector field such that the second fundamental form is definite positive.

For $\mathbb{H}^2\times\mathbb{R}$, if $a+b<0$, it is clear that for $c<0$ the intrinsic curvature satisfies $K_i\ge\frac{c}{a+b}>0$. Then from the Bonnet–Myers Theorem, Σ must be compact. On the other hand, for $a+b<0$ and $c\ge 0$, if Σ were compact, there would exist a point $p\in\Sigma$ such that $\nu(p)=0$. It would imply that the extrinsic curvature satisfies $K_e(p)\le 0$, which contradicts our assumption. The proofs of the other cases are similar.

From equations (2.1) and (2.2), we conclude that $a+b=0$ implies that the angle function is constant. ∎

Remark 2.4 Henceforth, since surfaces having constant angle were treated in [6,7], we omit this case.

3 Complete Rotational Surfaces of Weingarten Type in $M^2(\epsilon)\times\mathbb{R}$

In this section, we deal with complete W-surfaces having positive extrinsic curvature, which are invariant under a one-parameter group of rotations of the ambient space $M^2(\epsilon)\times\mathbb{R}$.

For $\epsilon\in\{-1,1\}$, let us consider the four-dimensional space $\mathbb{R}^3_\epsilon\times\mathbb{R}$, endowed with the metric $ds^2=\epsilon dx_1^2+dx_2^2+dx_3^2+dx_4^2$. And let us identify the product space $M^2(\epsilon)\times\mathbb{R}$ as being the sub-manifold of $\mathbb{R}^3_\epsilon\times\mathbb{R}$, given by

$$M^2(\epsilon)\times\mathbb{R}=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^3_\epsilon\times\mathbb{R}:\epsilon x_1^2+x_2^2+x_3^2=\epsilon,\ \text{and if } \epsilon=-1, x_1>0\}.$$

The rotation in $M^2(\epsilon)\times\mathbb{R}$ is a subgroup of the isometry group of $M^2(\epsilon)\times\mathbb{R}$ which preserves the orientation and fixes an axis $\{p\}\times\mathbb{R}$ with $p\in M^2(\epsilon)\times\{0\}$. This subgroup can be identified with the special orthogonal group $SO(2)$. Up to isometries, we can assume that the axis is given by $\{(1,0,0)\}\times\mathbb{R}$.

We consider the plane $\Pi = \{(x_1, x_2, 0, x_4) \in M^2(\epsilon) \times \mathbb{R}, x_2 \geq 0\}$ and the curve

$$\alpha_\epsilon(u) = \begin{cases} (\cosh k(u), \sinh k(u), 0, h(u)) \subset \Pi & \text{if } \epsilon = -1, \\ (\cos k(u), \sin k(u), 0, h(u)) \subset \Pi & \text{if } \epsilon = 1. \end{cases}$$

Where $k(u) \geq 0$ and u is the arclength of α, that is, $(k'(u))^2 + (h'(u))^2 = 1$. Here $k'(u)$ denotes the derivative with respect to the variable u.

In order to obtain a rotational surface, we apply the one-parameter group of rotational isometries to the curve α_ϵ. Denoting by $\mathcal{S}$ such a generated surface, we can parametrize $\mathcal{S}$ by

$$\varphi_\epsilon(u, v) = \begin{cases} (\cosh k(u), \sinh k(u) \cos v, \sinh k(u) \sin v, h(u)) & \text{if } \epsilon = -1, \\ (\cos k(u), \sin k(u) \cos v, \sin k(u) \sin v, h(u)) & \text{if } \epsilon = 1. \end{cases}$$

In order to simplify the expressions, we define the functions

$$\cos_\epsilon k = \begin{cases} \cosh k & \text{if } \epsilon = -1, \\ \cos k & \text{if } \epsilon = 1, \end{cases} \quad \text{and} \quad \cot_\epsilon k = \begin{cases} \coth k & \text{if } \epsilon = -1, \\ \cot k & \text{if } \epsilon = 1. \end{cases}$$

3.1 The First Integral

The aim of this section is to classify complete rotational W-surfaces having positive extrinsic curvature. A straightforward computation gives us that the intrinsic and extrinsic curvature functions of an isometrically immersed surface which is invariant by rotational isometries in the space $M^2(\epsilon) \times \mathbb{R}$ are given by

$$K_i = \epsilon(k'(u))^2 - k''(u) \cot_\epsilon k(u),$$
$$K_e = -k''(u) \cot_\epsilon k(u).$$

The Weingarten equation is written as

$$(a + b)k''(u) \cot_\epsilon k(u) - \epsilon a(k'(u))^2 = -c, \tag{3.1}$$

for real numbers $a \neq 0$, $b > 0$, and c satisfying $a + b \neq 0$. It is straightforward to check that the first integral of the ordinary differential equation (ODE) (3.1) is

$$(k'(u))^2 = \epsilon \frac{c}{a} + C_1 (\cos_\epsilon k(u))^{-\frac{2a}{a+b}}$$

for some constant C_1. Moreover, we can assume that α cuts the axis orthogonally at $t = 0$. Then $k(0) = 0$ and $k'(0) = 1$. In this case, the first integral is given by

$$(k'(u))^2 = \epsilon \frac{c}{a} + \frac{a - \epsilon c}{a} (\cos_\epsilon k(u))^{-\frac{2a}{a+b}}.$$

Notice that the problem of finding all complete rotational W-surfaces that cut the axis orthogonally consists in determining all the admissible expressions of the profile curve α_ϵ. We mean that we wish to find all the possible compact (and non-compact) integral curves of the ODE system

$$\begin{cases} (k'(u))^2 - \epsilon \frac{c}{a} = \frac{a - \epsilon c}{a} (\cos_\epsilon k(u))^{-\frac{2a}{a+b}}, \\ (k'(u))^2 + (h'(u))^2 = 1. \end{cases} \tag{3.2}$$

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Proposition 3.1 *Let $\mathcal{S}$ be a complete W-surface isometrically immersed into the product space $M^2(\epsilon)\times\mathbb{R}$ having positive extrinsic curvature K_e, which is rotationally invariant and whose generating curve α_ϵ cuts the rotation axis orthogonally. Assume that $a+b\neq 0$.*

(i) *For $a+b>0$, there are two cases.*
  (a) *If $c>0$, then $\mathcal{S}$ is a rotational topological sphere.*
  (b) *If $c\leq 0$, then $\mathcal{S}$ is homeomorphic to $\mathbb{R}^2$.*

(ii) *For $a+b<0$, there are four cases.*
  (a) *If $c\geq 0$, then $\mathcal{S}$ is homeomorphic to $\mathbb{R}^2$.*
  (b) *In $\mathbb{H}^2\times\mathbb{R}$, if $c<0$, then $\mathcal{S}$ is a rotational topological sphere.*
  (c) *In $\mathbb{S}^2\times\mathbb{R}$, if $c<-b$, then $\mathcal{S}$ is a rotational topological sphere.*
  (d) *In $\mathbb{S}^2\times\mathbb{R}$, if $-b\leq c<0$, then there is no rotational surface $\mathcal{S}$ whose generating curve cuts the rotation axis orthogonally.*

Proof It is known that complete surfaces, isometrically immersed in the product spaces $M^2(\epsilon)\times\mathbb{R}$, and having positive extrinsic curvature, are homeomorphic either to a sphere or to the euclidean plane $\mathbb{R}^2$ (see [8, Theorem 3.1], and [9, Theorem 2.4]). By Lemma 2.3, in order to prove the proposition, we just need to consider two cases. The first one is $\epsilon=-1$, $a+b>0$, and $c>0$, and the second is $\epsilon=1$, $a+b<0$, and $-b\leq c<0$.

From (3.2), for $\epsilon=-1$, $a+b>0$, and $c>0$, we have

$$\left(\frac{dh}{dk}\right)^2=\left(\frac{a+c}{c}\right)\frac{(\cosh k)^{\frac{2a}{a+b}}-1}{\left(\sqrt{\frac{a+c}{c}}+(\cosh k)^{\frac{a}{a+b}}\right)\left(\sqrt{\frac{a+c}{c}}-(\cosh k)^{\frac{a}{a+b}}\right)}. \tag{3.3}$$

That is, we consider the function $h=h(k)$ as a function of the variable k; notice that k is the hyperbolic distance to the origin in the slice $\mathbb{H}^2\times\{0\}$. This function is defined on the interval $[0,k_0]$, where k_0 satisfies

$$(\cosh k_0)^{\frac{a}{a+b}}=\sqrt{\frac{a+c}{c}}.$$

The graph of the function h has a vertical tangent line at k_0. In order to obtain a rotational topological sphere, we need to show that the height function $h(k)$ is bounded and it is of class C^2 at $k=k_0$.

Up to isometries of the ambient space, we can assume that $\frac{dh}{dk}\geq 0$. We separate the proof into two cases, depending on the sign of a (recall we are assuming that $a\neq 0$).

(i) If $a>0$, then $\frac{2a}{a+b}>0$ and $(\cosh k)^{\frac{2a}{a+b}}-1>0$. For $k>0$, we set

$$A_1(k)=\sqrt{\left(\frac{a+c}{c}\right)\frac{(\cosh k)^{\frac{2a}{a+b}}-1}{\left(\sqrt{\frac{a+c}{c}}+(\cosh k)^{\frac{a}{a+b}}\right)}},$$

thus, equation (3.3) implies

$$\frac{dh}{dk}=A_1(k)\frac{1}{\sqrt{\sqrt{\frac{a+c}{c}}-(\cosh k)^{\frac{a}{a+b}}}}\frac{2\left(\frac{-a}{a+b}\right)(\cosh k)^{\frac{-b}{a+b}}\sinh k}{2\left(\frac{-a}{a+b}\right)(\cosh k)^{\frac{-b}{a+b}}\sinh k}. \tag{3.4}$$

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Moreover, if we consider the function

$$A_2(k) = A_1(k)\frac{2}{(\frac{a}{a+b})(\cosh k)^{\frac{-b}{a+b}}\sinh k},$$

we can write equation (3.4) as

$$\frac{dh}{dk} = A_2(k)\frac{(-1)(\frac{-a}{a+b})(\cosh k)^{\frac{-b}{a+b}}\sinh k}{2\sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}} = A_2(k)\frac{d}{dk}\sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}.$$

Notice that $A_1(k)$ and $A_2(k)$ are bounded functions on the interval $[0, k_0]$. Then, for each $0 < \delta < k_0$, there exist a positive number $M > 0$, such that, for all $k \in [k_0 - \delta, k_0]$, we have

$$\frac{dh}{dk} \leq -M\frac{d}{dk}\sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}. \tag{3.5}$$

Integrating (3.5), there exists a constant C_1 large enough, such that

$$0 < h(k) \leq M\Big(C_1 - \sqrt{\sqrt{\frac{a+c}{c}} - (\cosh k)^{\frac{a}{a+b}}}\Big).$$

The function $h = h(k)$ is bounded in $[k_0 - \delta, k_0]$. From equation (3.3), its graph has a vertical tangent line at $k = k_0$. A straightforward computation gives that the function h is of class C^2 at $k = k_0$, that is, its graph has bounded curvature at $k = k_0$. So after a reflection about the slice $t = h(k_0)$, we obtain a complete rotational topological sphere.

(ii) The proof for the case $a < 0$ is analogous, taking into account that in this case $\frac{2a}{a+b} < 0$ and $(\cosh k)^{\frac{2a}{a+b}} - 1 < 0$.

For the case $\mathbb{S}^2 \times \mathbb{R}$, assume $a + b < 0$ and $-b \leq c < 0$. If $\mathcal{S}$ were a rotational surface whose generating curve cuts the rotation axis orthogonally, there would exist a point $p \in \mathcal{S}$ such that $\nu^2(p) = 1$. Our assumption on a, b, and c implies $a < -b \leq c$, *i.e.*, $c - a > 0$. Thus, in such a point $p \in \mathcal{S}$, we would have $K_e(p) = \frac{c-a}{a+b} < 0$, a contradiction. This completes the proof. ∎

4 Vertical and Horizontal Height Estimates

In this section we consider a W-surface Σ isometrically immersed in $M^2(\epsilon) \times \mathbb{R}$, having positive extrinsic curvature. Once the extrinsic curvature is positive, the surface is orientable and we orient Σ in such way that the second fundamental form is positive definite. Let z be a conformal local parameter for the second fundamental form. In this parameter the first and second fundamental forms of Σ are written as

$$\mathrm{I} = Edz^2 + 2F|dz|^2 + \overline{E}d\overline{z}^2, \tag{4.1}$$

$$\mathrm{II} = 2\rho|dz|^2, \tag{4.2}$$

where ρ is a positive function and $\overline{z}$ denotes the conjugate of z. The extrinsic curvature of Σ is given by $K_e = \rho^2/D$, where $D = F^2 - |E|^2 > 0$, and we denote by K_i the intrinsic curvature of the surface.

4.1 Some Basic Equations

In this subsection we compute some equations that will be necessary to achieve the classification of W-surfaces.

Lemma 4.1 *Let $\varphi\colon \Sigma \to M^2(\epsilon)\times\mathbb{R}$ be an isometric immersion in $M^2(\epsilon)\times\mathbb{R}$. Assume Σ is a W-surface having positive extrinsic curvature and that $a+b$ is different from zero. Let N be the global unit normal vector field such that the second fundamental form of Σ is positive definite and let z be a conformal parameter for the second fundamental form. Then the following equations are satisfied:*

$$K_e = \frac{c-\epsilon a v^2}{a+b} \tag{4.3}$$

$$\frac{\rho_{\bar z}}{\rho} = -\frac{\epsilon v \alpha}{\rho} - (\Gamma_{12}^1 - \Gamma_{22}^2) \qquad \text{(Codazzi equation)} \tag{4.4}$$

$$h_{z\bar z} = v\rho + \Gamma_{12}^1 h_z + \Gamma_{12}^2 h_{\bar z} \tag{4.5}$$

$$h_{zz} = \Gamma_{11}^1 h_z + \Gamma_{11}^2 h_{\bar z} \tag{4.6}$$

$$v_{\bar z} = -\frac{\alpha K_e}{\rho} \tag{4.7}$$

$$|T|^2 = 1 - v^2 = \frac{1}{D}(\alpha h_z + \overline{\alpha} h_{\bar z}) \tag{4.8}$$

$$|h_z|^2 = -\frac{|\alpha|^2}{D} + F|T|^2, \tag{4.9}$$

where Γ_{ij}^k are the Christoffel symbols associated with z for $i, j, k \in \{1,2,3\}$, $E, F,$ and ρ are coefficients of the first and second fundamental forms given by equations (4.1) and (4.2), and

$$\alpha := F h_{\bar z} - \overline{E} h_z \tag{4.10}$$

$$D := F^2 - |E|^2 \tag{4.11}$$

$$T = \frac{1}{D}(\alpha \partial_z + \overline{\alpha}\partial_{\bar z}). \tag{4.12}$$

Proof This lemma is similar to [4, Lemma 3.1]; for completeness we present a proof here. The idea is to write the compatibility equations in terms of the conformal parameter z. The compatibility equations for immersions in $M^2(\epsilon)\times\mathbb{R}$ are described in [5].

Let $\pi_2\colon M^2(\epsilon)\times\mathbb{R}\to\mathbb{R}$, $\pi_2(p,t) = t$ be the projection on the second factor. We write $\frac{\partial}{\partial_t} = T + vN$, where T is a tangent vector field to Σ. Once $\frac{\partial}{\partial_t}$ is the gradient in $M^2(\epsilon)\times\mathbb{R}$ of the function π_2, the vector field T, tangent to Σ, is the gradient of the height function $h := \pi_2|_\Sigma$. Then $T = \frac{1}{D}(\alpha\partial_z + \overline{\alpha}\partial_{\bar z})$.

Observe that $|T|^2 = 1-\nu^2$, and after a direct computation, we obtain equation (4.8). On the other hand, by equation (4.10), $h_z = \frac{1}{D}(E\alpha + F\overline{\alpha})$. Then

$$\begin{aligned} |h_z|^2 &= \frac{1}{D^2}(|\alpha|^2(|E|^2 + F^2) + F(E\alpha^2 + \overline{E}\,\overline{\alpha}^2)) \\ &= -\frac{|\alpha|^2}{D} + \frac{F}{D^2}(E\alpha^2 + 2F|\alpha|^2 + \overline{E}\,\overline{\alpha}^2) = -\frac{|\alpha|^2}{D} + F|T|^2, \end{aligned}$$

which proves equation (4.9).

Using the Gauss equation $K_i = K_e + \epsilon\nu^2$, the Weingarten equation $aK_i + bK_e = c$ becomes

$$K_e = \frac{c - \epsilon a \nu^2}{a+b}.$$

The Codazzi equation is $\nabla_X \mathcal{A}Y - \nabla_Y \mathcal{A}X - \mathcal{A}[X,Y] = \epsilon\nu(\langle Y,T\rangle X - \langle X,T\rangle Y)$, where $\mathcal{A}$ is the shape operator of Σ and X, Y are tangent vector fields to Σ. For $X = \partial_{\overline{z}}$, $Y = \partial_z$, the Codazzi equation is $\nabla_{\partial_{\overline{z}}}\mathcal{A}\partial_z - \nabla_{\partial_z}\mathcal{A}\partial_{\overline{z}} = \epsilon\nu(h_z\partial_{\overline{z}} - h_{\overline{z}}\partial_z)$. The scalar product of this equation with $\partial_{\overline{z}}$ gives

$$\begin{aligned} \langle \nabla_{\partial_{\overline{z}}}\mathcal{A}\partial_z, \partial_{\overline{z}}\rangle - \langle \nabla_{\partial_z}\mathcal{A}\partial_{\overline{z}}, \partial_{\overline{z}}\rangle &= \epsilon\nu(h_z\overline{E} - h_{\overline{z}}F) = -\epsilon\nu\alpha \\ \frac{\rho_{\overline{z}}}{\rho} + (\Gamma^1_{12} - \Gamma^2_{22}) &= -\frac{\epsilon\nu\alpha}{\rho}, \end{aligned}$$

which is equation (4.4).

Taking the scalar product of the compatibility equation $\nabla_X T = \nu\mathcal{A}X$ with $\partial_{\overline{z}}$, for $X = \partial_z$, we get

$$\begin{aligned} \langle \nabla_{\partial_z} T, \partial_{\overline{z}}\rangle &= \nu\langle \mathcal{A}\partial_z, \partial_{\overline{z}}\rangle \\ h_{z\overline{z}} - \langle T, \nabla_{\partial_z}\partial_{\overline{z}}\rangle &= \nu\rho. \end{aligned}$$

Then we obtain equation (4.5) $h_{z\overline{z}} = \nu\rho + \Gamma^1_{12}h_z + \Gamma^2_{12}h_{\overline{z}}$. Similarly, taking the scalar product of the compatibility equation $\nabla_X T = \nu\mathcal{A}X$ with ∂_z, for $X = \partial_z$, we get equation (4.6).

From the compatibility equation $d\nu(X) = -\langle \mathcal{A}X, T\rangle$, for $X = \partial_{\overline{z}}$, we have

$$\nu_{\overline{z}} = -\Big\langle \mathcal{A}\partial_{\overline{z}}, \frac{\alpha\partial_z + \overline{\alpha}\partial_{\overline{z}}}{D}\Big\rangle = -\frac{\alpha K_e}{\rho}. \qquad \blacksquare$$

The equations in Lemma 4.1 enable us to rewrite $h_{z\overline{z}}$ and $\nu_{z\overline{z}}$ in a more suitable form in the following proposition.

Proposition 4.2 *Let $\varphi\colon \Sigma \to M^2(\epsilon)\times\mathbb{R}$ be an isometric immersion in $M^2(\epsilon)\times\mathbb{R}$. Assume Σ is a W-surface having positive extrinsic curvature and that $a+b$ is different from zero. Let N be the global unit normal vector field to Σ such that the second fundamental form is positive definite and let z be a conformal parameter for the second fundamental form. Then*

$$h_{z\overline{z}} = \frac{\nu\rho}{2K_e(a+b)}(2K_e(a+b) - \epsilon(2a+b)(1-\nu^2)), \tag{4.13}$$

$$\nu_{z\overline{z}} = -\frac{\epsilon a\nu|\alpha|^2}{(a+b)D} - F\nu K_e, \tag{4.14}$$

where α and D are defined in (4.10) and (4.11), respectively.

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Proof We start by proving equation (4.13). Since Σ is a W-surface, taking the derivative of equation (4.3) with respect to $\overline{z}$ and using equation (4.7), we obtain

$$\frac{(K_e)_{\overline{z}}}{2K_e} = \frac{\epsilon a\alpha\nu}{(a+b)\rho}. \tag{4.15}$$

On the other hand, we have $K_e = \frac{\rho^2}{D}$. Therefore,

$$\frac{(K_e)_{\overline{z}}}{2K_e} = \frac{\rho_{\overline{z}}}{\rho} - \frac{D_{\overline{z}}}{2D}. \tag{4.16}$$

As a consequence of (4.15) and (4.16), we have

$$\frac{\rho_{\overline{z}}}{\rho} - \frac{D_{\overline{z}}}{2D} = \frac{\epsilon\, a\alpha\nu}{(a+b)\rho}. \tag{4.17}$$

A direct computation, see [12, Lemma 8], produces

$$\Gamma_{12}^1 + \Gamma_{22}^2 = \frac{D_{\overline{z}}}{2D}. \tag{4.18}$$

The Codazzi equation (4.4) is equivalent to

$$\frac{\rho_{\overline{z}}}{\rho} - (\Gamma_{22}^2 + \Gamma_{12}^1) + 2\Gamma_{12}^1 = -\frac{\epsilon\nu\alpha}{\rho} \tag{4.19}$$

by (4.18), which implies

$$\frac{\rho_{\overline{z}}}{\rho} - \frac{D_{\overline{z}}}{2D} + 2\Gamma_{12}^1 = -\frac{\epsilon\nu\alpha}{\rho}$$

by (4.17), which in turn implies

$$\Gamma_{12}^1 = -\frac{\epsilon\alpha\nu(2a+b)}{2\rho(a+b)}.$$

Since $\Gamma_{12}^1 = \overline{\Gamma_{12}^2}$, using equation (4.5), we have

$$h_{z\overline{z}} = -\frac{\epsilon\nu(2a+b)}{2\rho(a+b)}(\alpha h_z + \overline{\alpha}h_{\overline{z}}) + \nu\rho$$

which implies

$$\begin{aligned} h_{z\overline{z}} &= -\frac{\epsilon\nu(2a+b)}{2\rho\,(a+b)}(1-\nu^2)D + \nu\rho = \nu\rho\Big(1 - \frac{\epsilon\,(2a+b)(1-\nu^2)}{2\,K_e\,(a+b)}\Big) \\ &= \frac{\nu\rho}{2K_e(a+b)}(2K_e(a+b) - \epsilon(2a+b)(1-\nu^2)), \end{aligned}$$

which proves equation (4.13).

In order to prove equation (4.14), observe that by equations (4.7), (4.15), and (4.17) we have

$$\begin{aligned} \nu_{z\overline{z}} &= -\alpha_z\frac{K_e}{\rho} - \frac{2\epsilon a\nu|\alpha|^2K_e}{\rho^2(a+b)} + \frac{\alpha K_e}{\rho}\Big(\frac{\epsilon a\overline{\alpha}\nu}{\rho(a+b)} + \frac{D_z}{2D}\Big) \\ &= -\alpha_z\frac{K_e}{\rho} - \frac{\epsilon a\nu|\alpha|^2}{D(a+b)} + \frac{\alpha K_e}{\rho}\frac{D_z}{2D}. \end{aligned} \tag{4.20}$$

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We claim that

$$\alpha_z = \alpha\frac{D_z}{D} + F\nu\rho. \tag{4.21}$$

Let us assume this equation for a moment. Then a direct computation using equations (4.20) and (4.21) gives equation (4.14), as desired. So, in order to finish the proof of the proposition, we need to prove that equation (4.21) holds. Recall $\alpha = Fh_{\overline{z}} - \overline{E}h_z$. Then, using equations (4.5) and (4.6), we obtain

$$\begin{aligned}\alpha_z &= \langle\nabla_{\partial_z}\partial_z, \partial_{\overline{z}}\rangle h_{\overline{z}} + \langle\nabla_{\partial_z}\partial_{\overline{z}}, \partial_z\rangle h_{\overline{z}} - 2\langle\nabla_{\partial_z}\partial_{\overline{z}}, \partial_{\overline{z}}\rangle h_z + Fh_{z\overline{z}} - \overline{E}h_{zz}\\ &= \Gamma_{11}^1(Fh_{\overline{z}} - \overline{E}h_z) + \Gamma_{12}^1(Eh_{\overline{z}} - Fh_z) + 2\Gamma_{12}^2(Fh_{\overline{z}} - \overline{E}h_z) + F\nu\rho,\\ &= \Gamma_{11}^1\alpha - \Gamma_{12}^1\overline{\alpha} + 2\Gamma_{12}^2\alpha + F\nu\rho.\end{aligned}$$

A direct computation using equation (4.19) shows that $\Gamma_{12}^2\alpha - \Gamma_{12}^1\overline{\alpha} = 0$. Moreover, conjugating equation (4.18), we obtain $\alpha_z = \alpha(\Gamma_{11}^1 + \Gamma_{12}^2) + F\nu\rho = \alpha(\frac{D_z}{D}) + F\nu\rho$, as claimed. ■

4.2 A Quadratic Differential on Σ

In this section, we will define a quadratic differential Qdz^2 on Σ having the property that Q vanishes identically or its zeros are isolated with negative index.

Let Σ be a W-surface isometrically immersed in $M^2(\epsilon)\times\mathbb{R}$ having positive extrinsic curvature. Assume that $a+b$ and $2a+b$ are different from zero. For such a W-surface we introduce the quadratic differential

$$A := \mathrm{I} + f(1-\nu^2)dh^2, \tag{4.22}$$

$$Qdz^2 := (E + f(1-\nu^2)h_z^2)dz^2, \tag{4.23}$$

where I is the first fundamental form of Σ given in equations in Lemma 4.1 and $f\colon[0,1]\to\mathbb{R}$ is the real analytic function given by

$$f(t) = \frac{-\epsilon(2a+b)(c-\epsilon a)t - (c-\epsilon a)^2 + (c-\epsilon a(1-t))^{\frac{2a+b}{a}}(c-\epsilon a)^{-\frac{b}{a}}}{\epsilon(a+b)(c-\epsilon a)\,t^2}. \tag{4.24}$$

Remark 4.3 We point out that

1 The quadratic differential Qdz^2 is the $(2,0)$-part of A.
2 The Taylor series near zero of f is $f(t) = \sum_{n=0}^{n=\infty} a_n t^n$, where

$$a_n = \frac{\epsilon^{n+1}}{(a+b)(c-\epsilon a)^{(1+n)}(n+2)!}\prod_{j=0}^{n+1}(2a+b-ja).$$

The convergence radius of this series is $\frac{|c-\epsilon a|}{|a|} > 0$. So, f is real analytic on $[0,1]$.

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The extrinsic curvature of the pair (II, A) is (see [12])

$$K(\mathrm{II}, A) = \frac{(F + f(1-\nu^2)\,|h_z|^2)^2 - |E + f(1-\nu^2)\,h_z^2|^2}{\rho^2} \tag{4.25}$$
$$= \frac{F^2 - |E|^2}{\rho^2} - \frac{f(1-\nu^2)\,(\overline{E}h_z^2 + -2F|h_z|^2 + Eh_{\overline{z}}^2)}{\rho^2}$$
$$= \frac{1}{K_e} + \frac{f(1-\nu^2)\,D(\alpha h_z + \overline{\alpha} h_{\overline{z}})}{\rho^2} \qquad \text{(by (4.10))}$$
$$= \frac{1}{K_e}(1 + f(1-\nu^2)\,|T|^2) \qquad \text{(by (4.8))}.$$

In particular, once $|Q|^2 = |E + f(1-\nu^2)\,h_z^2|^2$, using the first and fourth lines of (4.25), we have

$$|Q|^2 = (F + f(1-\nu^2)\,|h_z|^2)^2 - D(1 + f(1-\nu^2)\,|T|^2). \tag{4.26}$$

The next result is the key lemma, which gives an estimate of $|Q_{\overline{z}}|$ in terms of the function $|Q|$.

Lemma 4.4 *Let $\varphi\colon \Sigma \to M^2(\epsilon) \times \mathbb{R}$ be an isometric immersion in $M^2(\epsilon) \times \mathbb{R}$. We assume Σ is a W-surface having positive extrinsic curvature. We also suppose that $a + b$ and $2a + b$ are different from zero. Let z be a conformal parameter for the second fundamental form. Then*

$$|Q_{\overline{z}}| \le \frac{2|\nu\rho h_z^3 f'(1-\nu^2)|}{D}|Q|,$$

where Q and D are defined in (4.23) and (4.11), respectively, and $f'(t)$ is the derivative of f at t.

Proof The derivative of the function Q with respect to $\overline{z}$ is

$$Q_{\overline{z}} = E_{\overline{z}} + 2f(1-\nu^2)h_z h_{z\overline{z}} - 2\nu\nu_{\overline{z}} f'(1-\nu^2)h_z^2. \tag{4.27}$$

Let us determine the expression of $E_{\overline{z}}$. Observe that the Christoffel symbols with respect to the conformal parameter z satisfies, $\Gamma_{12}^1 = \overline{\Gamma_{12}^2}$. Using equations (4.19) and (4.10), we have

$$E_{\overline{z}} = \partial_{\overline{z}}\langle \partial_z, \partial_z\rangle = 2(\Gamma_{12}^1 E + \Gamma_{12}^2 F) = -\frac{\epsilon\,\nu(2a+b)}{\rho(a+b)}(\alpha E + \overline{\alpha} F)$$
$$= -\frac{\epsilon\,\nu(2a+b)}{\rho(a+b)} D\,h_z.$$

Then, since $K_e = \frac{\rho^2}{D}$, we obtain

$$E_{\overline{z}} = -\frac{\epsilon\,\nu(2a+b)\rho\,h_z}{K_e(a+b)}. \tag{4.28}$$

By equations (4.13), (4.7), (4.27), and (4.28), we have

$$Q_{\overline{z}} = \nu\rho h_z\Bigg(-\frac{\epsilon(2a+b)}{K_e(a+b)} + f(1-\nu^2)\Big(\frac{2K_e(a+b)-\epsilon(2a+b)(1-\nu^2)}{K_e(a+b)}\Big) + f'(1-\nu^2)\frac{2\alpha K_e h_z}{\rho^2}\Bigg).$$

A direct computation shows that for $2a+b \neq 0$,

$$-\frac{\epsilon(2a+b)}{K_e(a+b)} + f(1-\nu^2)\left(\frac{2K_e(a+b)-\epsilon(2a+b)(1-\nu^2)}{K_e(a+b)}\right) = -(1-\nu^2)f'(1-\nu^2).$$

Using this, we obtain

$$\begin{aligned}
Q_{\overline{z}} &= \nu\,\rho\,h_z\,f'(1-\nu^2)\Big(-(1-\nu^2)+\frac{2\alpha\,h_z}{D}\Big)\\
&= \nu\,\rho\,h_z\,f'(1-\nu^2)\Big(\frac{\alpha\,h_z-\overline{\alpha}\,h_{\overline{z}}}{D}\Big) \qquad \text{(by (4.8))}\\
&= \frac{\nu\,\rho\,h_z\,f'(1-\nu^2)}{D}(Eh_{\overline{z}}^2-\overline{E}h_z^2) \qquad \text{(by (4.10))}\\
&= \frac{\nu\,\rho\,h_z\,f'(1-\nu^2)}{D}\big((Q-f(1-\nu^2)\,h_z^2)\,h_{\overline{z}}^2-(\overline{Q}-f(1-\nu^2)\,h_{\overline{z}}^2)\,h_z^2\big)\\
&\qquad \text{(by (4.23))}\\
&= \frac{\nu\,\rho\,h_z\,f'(1-\nu^2)}{D}(Qh_{\overline{z}}^2-\overline{Q}h_z^2).
\end{aligned}$$

Then

$$|Q_{\overline{z}}| \leq \frac{2|\nu\,\rho\,h_z^3\,f'(1-\nu^2)|}{D}|Q|,$$

as desired. ∎

Lemma 4.4 is used to apply [11, Lemma 2.7.1] and obtain an important property of the function Q.

Proposition 4.5 *Let $\varphi\colon\Sigma\to M^2(\epsilon)\times\mathbb{R}$ be an isometric immersion in $M^2(\epsilon)\times\mathbb{R}$. Assume Σ is a W-surface having positive extrinsic curvature. Moreover, we suppose $a+b$ and $2a+b$ are different from zero. Consider Σ as a Riemann surface with the conformal structure induced by its second fundamental form. Then the quadratic differential $Q\,dz^2$, where $Q\colon\Sigma\to\mathbb{C}$ is defined in (4.23), vanishes identically or its zeros are isolated with negative index.*

A direct consequence of Proposition 4.5 is the following.

Proposition 4.6 *Let $\varphi\colon\Sigma\to M^2(\epsilon)\times\mathbb{R}$ be an isometric immersion in $M^2(\epsilon)\times\mathbb{R}$. Assume Σ is a W-surface having positive extrinsic curvature. Moreover, we suppose $a+b$ and $2a+b$ are different from zero. Consider Σ as a Riemann surface with the conformal structure induced by its second fundamental form. If Σ is a topological sphere, then the function Q is identically null on Σ.*

4.3 Vertical Height Estimates

This section is devoted to giving vertical height estimates for some W-surfaces.

Theorem 4.7 (Vertical height estimates) *Let $\varphi:\Sigma\to\mathbb{H}^2\times\mathbb{R}$ be a compact graph on a domain $\Omega\subset\mathbb{H}^2$ whose boundary is contained in the slice $\mathbb{H}^2\times\{0\}$. Assume Σ is a W-surface having positive extrinsic curvature. Moreover, suppose $2a+b$ is different from zero, $a+b>0$, and $c>0$. Then there exists a constant C_0 which depends only on K_e, a, b, c such that the height function h satisfies $h(p)\le C_0$ for all p in Σ.*

Proof The idea of this proof is to construct a sub-harmonic function $\phi=h+g(\nu)$ on Σ having non-positive boundary values where $g:[-1,0]\to\mathbb{R}$ is to be determined. In order to compute $\phi_{z\bar z}$, we calculated $(g(\nu))_{z\bar z}$. Taking into account equations (4.3), (4.14), (4.7), and (4.9), we obtain

$$
\begin{aligned}
(g(\nu))_{z\bar z} &= (\nu_z g'(\nu))_{\bar z} = \nu_{z\bar z} g'(\nu) + |\nu_z|^2 g''(\nu)\\
&= \frac{|\alpha|^2}{D}\Big(\frac{a\nu}{(a+b)}g'(\nu) + K_e\, g''(\nu)\Big) - F\nu K_e g'(\nu)\\
&= F\Big(\Big(\frac{a\nu(1-\nu^2)}{(a+b)} - \nu K_e\Big)\, g'(\nu) + K_e(1-\nu^2)g''(\nu)\Big)\\
&\qquad + |h_z|^2\Big(-\frac{a\nu}{(a+b)}g'(\nu) - K_e\, g''(\nu)\Big)\\
&= K_e\bigg(F\Big(\Big(\frac{a\nu(1-\nu^2)}{(c+a\nu^2)} - \nu\Big)\, g'(\nu) + (1-\nu^2)g''(\nu)\Big)\\
&\qquad + |h_z|^2\Big(-\frac{a\nu}{(c+a\nu^2)}g'(\nu) - g''(\nu)\Big)\bigg).
\end{aligned}
$$

Let $g:[-1,0]\to\mathbb{R}$ be a real function whose derivative is given by

$$
g'(t) = M\sqrt{\frac{\big(\frac{c+a}{c+at^2}\big)^{\frac{a+b}{a}} - 1}{(1-t^2)(c+a\,t^2)}},
$$

where M is a constant depending only on a, b, c defined by

$$
M = \begin{cases}
\dfrac{\max_{\nu\in[-1,0]}\Big(1+\dfrac{(2a+b)(1-\nu^2)}{2(c+a\,\nu^2)}\Big)}{\min_{\nu\in[-1,0]}\Big(\sqrt{\dfrac{1}{c+a\,\nu^2}\Big(\dfrac{c+a}{c+a\nu^2}\Big)^{\frac{a+b}{a}}}\Big)} & \text{if } \max_{\nu\in[-1,0]}\Big(1+\dfrac{(2a+b)(1-\nu^2)}{2(c+a\,\nu^2)}\Big) > 0;\\[2ex]
1 & \text{if } \max_{\nu\in[-1,0]}\Big(1+\dfrac{(2a+b)(1-\nu^2)}{2(c+a\,\nu^2)}\Big) \le 0.
\end{cases}
$$

Here $\max_{s\in[s_0,s_1]}(u(s))$ and $\min_{s\in[s_0,s_1]}(u(s))$ are the maximum and minimum of the function $u(s)$ for s in $[s_0,s_1]$.

A direct computation shows that

$$
\Big(\frac{a\nu(1-\nu^2)}{(c+a\nu^2)} - \nu\Big)g'(\nu) + (1-\nu^2)g''(\nu) = -\frac{\nu g'(\nu)}{1+(1-\nu^2)\,f(1-\nu^2)}, \tag{4.29}
$$

where the real function $f(t)$ is defined in (4.24). We also have

$$-\frac{av}{(c+av^2)}g'(v)-g''(v)=f(1-v^2)\Big(-\frac{vg'(v)}{1+(1-v^2)f(1-v^2)}\Big). \tag{4.30}$$

Then by (4.29) and (4.30),

$$g_{z\bar z}=-\frac{v\,K_e\;g'(v)}{1+(1-v^2)f(1-v^2)}(F+|h_z|^2f(1-v^2)). \tag{4.31}$$

Observe that $(F+|h_z|^2f(1-v^2))$ is positive on Σ. In fact, since the extrinsic curvature $K(II,A)$ on equation (4.25) of the pair (II,A) is positive, the quadratic differential A is positive definite or negative definite which implies that either $(F+|h_z|^2f(1-v^2))$ is positive in Σ or it is negative everywhere. At the highest point $h_z=0$ and we have that $(F+|h_z|^2f(1-v^2))=F$ is positive, so we conclude that $(F+|h_z|^2f(1-v^2))$ is positive in Σ.

In order to compute $\phi_{z\bar z}$, it is worth writing $g'(t)$ as

$$g'(t)=M\sqrt{\frac{1+(1-t^2)\,f(1-t^2)}{Ke((c+a\,t^2)-(a+b)(1-t^2)(1+(1-t^2)f(1-t^2)))}}.$$

Keeping this in mind and using equations (4.13), (4.31), and (4.26), we obtain

(4.32)

$$\begin{aligned}(\phi)_{z\bar z}&=v\Big(\rho+\frac{\rho(2a+b)(1-v^2)}{2K_e(a+b)}-\frac{K_eg'(v)}{1+(1-v^2)f(1-v^2)}(F+|h_z|^2f(1-v^2))\Big)\\&=v\Bigg(\rho+\frac{\rho(2a+b)(1-v^2)}{2K_e(a+b)}\\&\qquad-\frac{K_e\,g'(v)}{1+(1-v^2)f(1-v^2)}\sqrt{|Q|^2+\frac{\rho^2(1+(1-v^2)f(1-v^2))}{K_e}}\Bigg)\\&\ge v\rho\Bigg(1+\frac{(2a+b)(1-v^2)}{2K_e(a+b)}-\frac{K_e\,g'(v)}{1+(1-v^2)f(1-v^2)}\sqrt{\frac{1+(1-v^2)f(1-v^2)}{K_e}}\Bigg)\\&=v\rho\Bigg(1+\frac{(2a+b)(1-v^2)}{2K_e(a+b)}\\&\qquad-M\sqrt{\frac{1}{(c+a\,v^2)-(a+b)(1-v^2)(1+(1-v^2)f(1-v^2))}}\Bigg)\\&=v\rho\Bigg(1+\frac{(2a+b)(1-v^2)}{2(c+\,a\,v^2)}-M\sqrt{\frac{1}{c+av^2}\Big(\frac{c+a}{c+a\,v^2}\Big)^{\frac{a+b}{a}}}\Bigg).\end{aligned}$$

So the definition of M implies that $\phi_{z\bar z}\ge 0$. Taking $g(v)=\int_0^v g'(t)\,dt$, we have $\Delta^{\rm II}(h+g(v))=\frac{2}{\rho}(h+g(v))_{z\bar z}\ge 0$ in Σ, where $\Delta^{\rm II}$ is the laplacian with respect to the second fundamental form. Moreover, $h+g(v)$ is non-positive on the boundary of Σ. Once $g'(v)$ is non-negative and $v\le 0$, then we have that $h+g(v)$ is non-positive everywhere. In particular, the maximum of the function h is $C_0=\int_{-1}^0 g'(t)\,dt$. ∎

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Remark 4.8 Observe that the height estimate C_0 is not reached. In fact, let $\varphi\colon \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be a compact graph on a domain $\Omega \subset \mathbb{H}^2$ whose boundary is contained in the slice $\mathbb{H}^2 \times \{0\}$. If there exists a point p_0 on the interior of Ω such that $h(p_0) = C_0$, the maximum principle applied to the subharmonic map ϕ would imply that $\phi \equiv 0$ which would imply that $\phi_{z\bar{z}} \equiv 0$. On the other hand, we observe that the function on ν^2

$$1 + \frac{(2a+b)(1-\nu^2)}{2(c-\epsilon a \nu^2)} - M\sqrt{\frac{1}{c+a\nu^2}\Big(\frac{c+a}{c+a\nu^2}\Big)^{\frac{a+b}{a}}}$$

is not constant, and cannot be identically null. So using the inequality (4.32), we conclude that $\phi_{z\bar{z}}$ cannot be identically zero. So such a point p_0 does not exist.

4.4 Horizontal Height Estimates

In this section we will see that the horizontal height for a class of compact embedded W-surfaces in $\mathbb{H}^2 \times \mathbb{R}$ with boundary on a vertical plane is bounded.

Let $\varphi\colon \Sigma \to M^2(\epsilon) \times \mathbb{R}$ be an isometric immersion from a oriented surface Σ. Recall that Σ is a W-surface if the Weingarten function $K_e - \frac{c - a\epsilon\nu^2}{a+b}$ vanishes identically. Observe that we may regard $K_e - \frac{c-a\epsilon\nu^2}{a+b} = 0$ as a second order partial differential equation. From this point of view, once ν depends only on the first derivative of the immersion, it can be shown that the partial differential equation $K_e - \frac{c-a\epsilon\nu^2}{a+b} = 0$ is absolutely elliptic if $K_e > 0$. So if Σ is a W-surface having positive extrinsic curvature, the interior and the boundary maximum principle, in the sense of Hopf, hold.

Let $\varphi_j\colon \Sigma_j \to M^2(\epsilon) \times \mathbb{R}$, $j = 1, 2$, be two isometric immersions. Assume Σ_j is a W-surface having positive extrinsic curvature. Let N_j be the global unit normal vector field to Σ_j such that the second fundamental form is positive definite. Let $p \in \Sigma_1 \cap \Sigma_2$ and assume that $N_1(p) = N_2(p)$. Once $N_1(p) = N_2(p)$, there is a neighbourhood $U_j \subset \Sigma_j$ of p such that U_j is a graph in exponential coordinates of a function f_j defined in a neighbourhood $\mathcal{D}$ of the origin of $T_p\Sigma_1 = T_p\Sigma_2$ ($T_p\Sigma_j$ is the tangent plane of Σ_j at p). Since the extrinsic curvature of Σ_j is positive, f_j is a positive function (for $\mathcal{D}$ small enough). We say that Σ_1 is above Σ_2, which we denote by $\Sigma_1 \geq \Sigma_2$, in a neighborhood of p if $f_1 \geq f_2$ in $\mathcal{D}$.

Under this notation, we have the following important theorem.

Theorem 4.9 (Hopf Maximum Principle, [10]) *Let $\varphi_j\colon \Sigma_j \to M^2(\epsilon) \times \mathbb{R}$, $j = 1, 2$, be two isometric immersions. Assume Σ_j is a W-surface having positive extrinsic curvature. Let N_j be the global unit normal vector field to Σ_j such that the second fundamental form associated with Σ_j is positive definite.*

Suppose that

(i) *Σ_1 and Σ_2 are tangent at an interior point $p \in \Sigma_1 \cap \Sigma_2$, or*
(ii) *there exists $p \in \partial\Sigma_1 \cap \partial\Sigma_2$ such that both $T_p\Sigma_1 = T_p\Sigma_2$ and $T_p\partial\Sigma_1 = T_p\partial\Sigma_2$.*

Furthermore, suppose that the unit normal vector fields of Σ_1 and Σ_2 coincide at p. If $\Sigma_1 \geq \Sigma_2$ in a neighbourhood $U_j \subset \Sigma_j$ of p, then $\Sigma_1 = \Sigma_2$ in $U_1 = U_2$.

In order to state the horizontal height estimate, recall that a vertical plane in $\mathbb{H}^2 \times \mathbb{R}$ is the product $\gamma \times \mathbb{R}$ of a complete geodesic $\gamma \subset \mathbb{H}^2$ with the real line $\mathbb{R}$.

Theorem 4.10 (Horizontal height estimates) *Let $\varphi\colon\Sigma\to\mathbb{H}^2\times\mathbb{R}$ be an isometric immersion. Suppose Σ is a compact embedded W-surface having positive extrinsic curvature whose boundary is contained in a vertical plane P. Moreover, assume $a+b>0$ and $c>0$. Then the distance from Σ to P is bounded,* i.e., *there exists a constant c_0 depending on a, b, c, independent of Σ, such that* $\operatorname{dist}(q,P)\leq c_0$, *for all $q\in\Sigma$.*

Once the interior and boundary maximum principle hold for W-surfaces Σ isometrically immersed in $M^2(\epsilon)\times\mathbb{R}$ having positive extrinsic curvature, the proof of [8, Theorem 6.2] applies to our setting with the exception that the proof used the maximum principle to compare Σ to a surface Σ_0 that in our case is the rotational topological sphere presented in Section 3.1.

5 Properly Embedded W-surfaces With Finite Topology and One Top End

We begin this section with the following definition.

Definition 5.1 Let $\varphi\colon\Sigma\to M^2(\epsilon)\times\mathbb{R}$ be an isometric immersion from a complete surface.

- We say that Σ has a top end $\mathcal{E}$ (respectively, a bottom end) if for any divergent sequence $\{q_j\}\subset\mathcal{E}$, the height function goes to $+\infty$ (respectively, $-\infty$).
- For the case $\mathbb{H}^2\times\mathbb{R}$, we say that Σ has a simple end if the boundary at infinity of the projection on the first factor $\pi_1(\Sigma)\subset\mathbb{H}^2\times\{0\}$ is a unique point θ_0 and in addition, for each vertical plane P whose boundary at infinity does not contain θ_0, the intersection of P and Σ is either empty or a compact set. Here we are denoting by $\pi_1\colon M^2(\epsilon)\times\mathbb{R}\to M^2(\epsilon)$, $\pi_1(p,t)=p$, the projection on the first factor, and, as usual, we identify the base space $M^2(\epsilon)$ with its horizontal lift $M^2(\epsilon)\times\{0\}$.

Recall that there is no properly embedded complete surface in $\mathbb{H}^2\times\mathbb{R}$ having positive constant extrinsic curvature with finite topology and one top (or bottom) end [8, Theorem 7.2]. In this section we extend this result to some W-surfaces.

For fixed real numbers a, b, and c such that $a+b>0$ and $c>0$, we denote by $\mathcal{S}_c(a,b)$ the rotational topological sphere in $\mathbb{H}^2\times\mathbb{R}$ whose intrinsic and positive extrinsic curvatures satisfy the equation $aK_i+bK_e=c$ (such rotational topological sphere was given in §3.1). We denote by $c_1=2\kappa_0$ the horizontal diameter of $\mathcal{S}_c(a,b)$, where κ_0 satisfies $(\cosh\kappa_0)^{\frac{a}{a+b}}=\sqrt{\frac{a+c}{c}}$.

The following lemma extends the Plane Separation Lemma given in [13, Lemma 2.4] to a properly embedded W-surface having positive extrinsic curvature. Using the Maximum Principle (Theorem 4.9), the proof of Lemma 5.2 is similar to the one of [13, Lemma 2.4], so we will not present a proof here.

Lemma 5.2 (Plane Separation Lemma) *Let $\varphi\colon\Sigma\to\mathbb{H}^2\times\mathbb{R}$ be an isometric properly embedded W-surface having positive extrinsic curvature. Assume Σ has finite topology and a top (or bottom) end. Moreover, suppose $a+b>0$ and $c>0$. Let P_1^+ and P_2^+ be two disjoint half-spaces determined by vertical planes P_1 and P_2, respectively. If the distance* *between P_1 and P_2 is larger than the horizontal diameter c_1 of the rotational topological sphere $\mathcal{S}_c(a,b)$. Then either $\Sigma \cap P_1^+$ or $\Sigma \cap P_2^+$ consists entirely of compact components.*

As a consequence of the Plane Separation Lemma and horizontal height estimates, we have the following theorem.

Theorem 5.3 *Let $\varphi: \Sigma \to \mathbb{H}^2 \times \mathbb{R}$ be an isometric immersion. Assume Σ is a complete W-surface having positive extrinsic curvature and finite topology with a top (or a bottom) end. Moreover, suppose $a + b > 0$ and $c > 0$. Then Σ is contained in a vertical cylinder $\alpha \times \mathbb{R}$ in $\mathbb{H}^2 \times \mathbb{R}$, where $\alpha \subset \mathbb{H}^2 \times \{0\}$ is a circle.*

Proof First, observe that, since Σ has positive extrinsic curvature, Σ is properly embedded [8, Theorem 3.1]. We take the disk model for the hyperbolic plane $\mathbb{H}^2$. Up to an isometry of the ambient space, we can assume that the point $\mathcal{O} = (\mathbf{0}, 0)$ belongs to Σ. Here $\mathbf{0}$ denotes the origin of $\mathbb{H}^2$. Let $\gamma: [0, +\infty) \to \mathbb{H}^2 \times \mathbb{R}$ be any horizontal geodesic starting at $\mathcal{O}$ parameterized by arc length. We denote by $P(s)$, $s \in [0, +\infty)$, the vertical plane passing through $\gamma(s)$ orthogonal to γ.

Claim *There exists a constant c_2, independent of γ, such that, if $s_0 > c_2$, then the half-space determined by $P(s_0)$ that does not contain the point $\mathcal{O}$ is disjoint from Σ.*

To prove this, we choose $R > \max\{c_0, c_1\}$, where c_0 and c_1 are the constant given by Theorem 4.10 and Lemma 5.2, respectively. Denote by $P^+(R)$ the half-space determined by $P(R)$ containing the point $\mathcal{O}$ and by $P^-(2R)$ the half-space determined by $P(2R)$ which does not contain the point $\mathcal{O}$. By Lemma 5.2 applied to Σ, we have one of

(a) $\Sigma \cap P^+(R)$ has only compact components,
(b) $\Sigma \cap P^-(2R)$ has only compact components.

By Theorem 4.10, if (a) were true, the distance between the plane $P(R)$ and the point $\mathcal{O} \in \Sigma \cap P^+(R)$ would be at most c_0. Once this horizontal distance is $R > c_0$, (a) cannot occur. So (b) holds. Again, by Theorem 4.10, the maximum distance between $\Sigma \cap P^-(2R)$ and the plane $P(2R)$ is at most c_0; hence Σ is disjoint from the half-space determined by $P(2R + c_0)$, which does not contain the point $\mathcal{O}$. Choosing the constant $c_2 = 2\max\{c_0, c_1\} + c_0$, the claim is proved.

The claim guarantees that Σ is contained in the vertical cylinder $\alpha \times \mathbb{R}$. Here α is a circle centered at the origin of $\mathbb{H}^2 \times \{0\}$ having radius c_2. ■

We finalize this section with a non-existence theorem.

Theorem 5.4 *There is no complete non-compact W-surface of positive extrinsic curvature in $\mathbb{H}^2 \times \mathbb{R}$ with $a + b > 0$ and $c > 0$.*

Proof Let us assume that such a surface Σ exists and we will obtain a contradiction. By hypothesis Σ is complete and has positive extrinsic curvature, so by [8, Theorem 3.1] Σ is topologically either a sphere or a plane and in this case Σ has a horizontal end or it is a vertical graph.

We first observe that Σ is non-compact so it is not topologically a sphere. On the other hand, by horizontal estimates in Theorem 4.10 we conclude that Σ does not have a simple end. So Σ is a vertical graph. Theorem 3.1 in [8] says that Σ bounds a strictly convex domain, so the vertical projection on the first factor $\pi(\Sigma) := \Omega \subset \mathbb{H} \times \{0\}$ is a strictly convex domain.

Let Π be a vertical plane which intersects Σ transversally and let $\gamma = \Sigma \cap \Pi$. Note that γ is a strictly convex curve in Π [8, Proposition 3.1]. We parametrize γ by $s \mapsto (\beta(s), h(\beta(s))) \subset \Sigma \cap \Pi$ and we assume β is parametrized by arc-length. Theorem 5.3 guarantees that Σ is contained in a vertical cylinder, so if $\gamma \subset \Sigma \cap \Pi$ is bounded by two vertical lines of Π, then γ must go up or down. That is, the height function restricted to γ, $h(\beta(s))$ is a convex function that goes to $+\infty$ and is bounded from below or goes to $-\infty$ and is bounded from above. We assume that $h(\beta(s))$ is bounded from below. Once γ is bounded by two parallel lines in Π and $h(\beta(s))$ goes to $+\infty$, the normal vector to Σ becomes uniformly horizontal when h tends to $+\infty$. This implies that the angle function converges uniformly to zero when h tends to $+\infty$. By the Weingarten equation (2.1) and the Gauss equation (2.2), when $h(\beta(s))$ converges to $+\infty$, we have $K_i = \frac{c-b\nu^2}{a+b} \to \frac{c}{a+b} > 0$. Then the intrinsic curvature is bounded from below by a positive constant as long as the angle function is small enough. It means that, outside a sufficiently large compact set $\mathbf{B} \subset \Sigma$, the intrinsic curvature is larger than a positive constant. The completeness of Σ enables us to apply the Bonnet–Myers Theorem to $\Sigma \setminus \mathbf{B}$ and conclude that the intrinsic distance from any point $p \in \Sigma \setminus \mathbf{B}$ to the boundary of $\Sigma \setminus \mathbf{B}$ is uniformly bounded, which contradicts that Σ is a proper graph homeomorphic to $\mathbb{R}^2$. ∎

6 The Main Theorem

Theorem 6.1 *Let $\varphi: \Sigma \to M^2(\epsilon) \times \mathbb{R}$ be an isometric immersion. Assume Σ is a complete W-surface having positive extrinsic curvature. We suppose that $2a + b$ is different from zero. Then Σ is a topological rotational sphere described in Section 3.1 if*

(i) *either $a + b > 0$ and $c > 0$,*
(ii) *or for $a + b < 0$,*
  (a) $\epsilon = 1$ *and* $c < -b$;
  (b) $\epsilon = -1$ *and* $c < 0$.

Proof From [8, Theorem 3.1] and [9, Theorem 2.4], once Σ has positive extrinsic curvature, Σ is either homeomorphic to $\mathbb{R}^2$ or homeomorphic to $\mathbb{S}^2$. So, using Lemma 2.3, Theorem 4.10, and Theorem 5.4, if (i) or (ii) is satisfied, then Σ is a topological sphere. As a consequence, Proposition 4.6 says that the quadratic differential Q vanishes identically over Σ.

Let (u, v) be local doubly orthogonal coordinates for the first and second fundamental form. In these coordinates $\mathrm{I} = \mathbf{E}du^2 + \mathbf{G}dv^2$ and $\mathrm{II} = \kappa_1\mathbf{E}du^2 + \kappa_2\mathbf{G}dv^2$, where κ_1, κ_2 are the principal curvatures of Σ. These coordinates are available on the interior of the set of umbilical points and also on a neighborhood of non umbilical points. So, the set of points where the coordinates (u, v) are available is dense on Σ. Thus, properties obtained on this set are extended to Σ by continuity.

Since Q vanishes on Σ, the quadratic differential A, defined in (4.22), is conformal to the second fundamental form II. It implies that $h_u h_v = 0$. Without lost of generality, we may assume that $h_u = 0$ in the neighborhood where (u, v) is available. Then, since A and II are conformal and $h_u = 0$, $A = \mathbf{E}du^2 + (\mathbf{G} + h_v^2)dv^2 = \frac{1}{\kappa_1}$ II.

First we prove that Σ is invariant under a one parameter group of isometries. Then we show that an orbit of this one parameter group of isometry is a circle.

Proceeding as in Lemma 4.1, we write some compatibility equations with respect to the coordinates (u, v), and we obtain

$$\nu_u = 0, \tag{6.1}$$

$$\frac{\mathbf{E}_v}{2\mathbf{E}}(\kappa_2 - \kappa_1) = \epsilon \nu h_v + (\kappa_1)_v, \tag{6.2}$$

$$\frac{\mathbf{G}_u}{2\mathbf{G}}(\kappa_2 - \kappa_1) = -(\kappa_2)_u, \tag{6.3}$$

$$h_{uv} = 0 = \frac{\mathbf{G}_u}{2\mathbf{G}} h_v. \tag{6.4}$$

From equation (6.1) we obtain that ν does not depend on u. As the extrinsic curvature of Σ is positive, no open neighbourhood of Σ is contained in a slice, therefore, h_v does not vanish in any open set where (u, v) are available, so equation (6.4) implies $G_u = 0$. Thus, by the Codazzi equation (6.3), $(\kappa_2)_u = 0$. Since the extrinsic curvature is $K_e = \frac{c - \epsilon a \nu^2}{a+b}$ and ν does not depend on u, neither does K_e. On the other hand, $K_e = \kappa_1 \kappa_2$ which implies that $(\kappa_1)_u = 0$. The variables (u, v) are available in the interior set of umbilical points and on a neighborhood of non umbilical points. Let us assume for a moment that we are working on a neighbohood free of umbilical points. Then, by the Codazzi equation (6.2), we may write $\mathbf{E} = \mathbf{E}_1(u)\mathbf{E}_2(v)$. Considering the new variables $x := \sqrt{\mathbf{E}_1(u)}du$ and $y := v$, we conclude that the first and second fundamental forms of Σ, h, and ν depend only on y. Then $\varphi(x, y)$ and $\varphi(x + x_0, y)$ only differ by an isometry of the ambient space. In other words, the immersion is invariant under one parameter group of isometries of the ambient space, given by the transformation $(x, y) \mapsto (x + t, y)$, [5]. Once we know that Σ is a topological sphere, we conclude that Σ is invariant by the group of rotations of $M^2(\epsilon)$.

It remains to analyse the case where the coordinates (u, v) are defined on a neighbourhood in the interior of umbilical points. In this case,

$$\mathrm{I} = \mathbf{E}du^2 + \mathbf{G}dv^2, \quad \mathrm{II} = \kappa_1(\mathbf{E}du^2 + \mathbf{G}dv^2),$$
$$A = \mathbf{E}du^2 + (\mathbf{G} + f(1 - \nu^2)h_v^2)dv^2 = \frac{1}{\kappa_1} II.$$

In particular, $\mathbf{G} + f(1 - \nu^2)h_v^2 = \mathbf{G}$ which implies that h_v vanishes identically in this neighborhood. Then, once we are working on a neighborhood of umbilical points, we conclude that the height function is constant, which implies that Σ is contained in a slice. This gives a contradiction, since the extrinsic curvature of the surface is positive. Then there is no such neighborhood of umbilical points. ∎

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References

[1] U. Abresch and H. Rosenberg, *A Hopf differential for constant mean curvature surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$. Acta Math. **193**(2004), 141–174. http://dx.doi.org/10.1007/BF02392562

[2] ———, *Generalized Hopf differentials* Mat. Contemp. **28**(2005), 1–28.

[3] A. Aledo, J. M. Espinar, and J. A. Gálvez, *Complete surfaces of constant curvature in* $\mathbb{H}^2 \times \mathbb{R}$ *and* $\mathbb{S}^2 \times \mathbb{R}$. Calc. Var. Partial Differential Equations **29**(2007), no. 3, 347–363. http://dx.doi.org/10.1007/s00526-006-0067-4

[4] ———, *Surfaces with Constant Curvature in* $\mathbb{S} \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$. *Height Estimates and Representation.* Bull. Braz. Math. Soc., 38, 2007, no. 533–554. http://dx.doi.org/10.1007/s00574-007-0059-9

[5] B. Daniel, *Isometric immersions into 3-dimensional homogeneous manifolds*. Comment. Math. Helv. **82**(2007), 87–131. http://dx.doi.org/10.4171/CMH/86

[6] F. Dillen, J. Fastenakels, J. Van der Veken, and L. Vrancken, *Constant angle surfaces in* $\mathbb{S}^2 \times \mathbb{R}$. Monatsh. Math. **152**(2007), 89–96. http://dx.doi.org/10.1007/s00605-007-0461-9

[7] F. Dillen and M. Munteanu, *Constant angle surfaces in* $\mathbb{H}^2 \times \mathbb{R}$. Bulletin Braz. Math. Soc. **40**(2009), 85–97. http://dx.doi.org/10.1007/s00574-009-0004-1

[8] J. M. Espinar, J. A. Gálvez, and H. Rosenberg, *Complete surfaces with positive extrinsic curvature in product spaces*. Comment. Math. Helv. **84**(2009), 351–386. http://dx.doi.org/10.4171/CMH/165

[9] J. M. Espinar and H. Rosenberg, *When strictly locally convex hypersurfaces are embedded*. Math. Z. **271**(2012), no. 3–4, 1075–1090. http://dx.doi.org/10.1007/s00209-011-0904-9

[10] H. Hopf, *Differential geometry in the large*. Lecture Notes in Mathematics, 1000. Springer-Velarg, Berlin, 1983.

[11] J. Jost, *Two–dimensional geometric variational problems*. JohnWiley & Sons, Chichester, MA, 1991.

[12] T. K. Milnor, *Harmonic maps and classical surface theory in Minkowski 3-space*. Trans. Amer. Math. Soc. **280**(1983), 161–185.

[13] B. Nelli and H. Rosenberg, *Simply connected constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$. Michigan Math. J. **54**(2006), 537–543. http://dx.doi.org/10.1307/mmj/1163789914

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