Can. J. Math., Vol. XL, No. 5, 1988, pp. 1025-1045

ON THE ASKEY-WILSON AND ROGERS POLYNOMIALS

MOURAD E. H. ISMAIL AND DENNIS STANTON

1. Introduction. The q-shifted factorial $(a)_n$ or $(a; q)_n$ is

$$(a)_n = (a; q)_n := \prod_{j=1}^{n} (1 - aq^{j-1}), \quad n = \infty, 0, 1, 2, \dots,$$

and an empty product is interpreted as 1. Recently, Askey and Wilson [**6**] introduced the polynomials

$$(1.1) \quad p_n(x; a, b, c, d) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, a/z \\ ab, ac, ad \end{matrix}; q, q\right),$$

where

$$(1.2) \quad z = x - \sqrt{x^2 - 1}$$

and

$$(1.3) \quad {}_{r+1}\phi_r\left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_{r+1})_n}{(b_1)_n \dots (b_r)_n} \frac{x^n}{(q)_n}.$$

We shall refer to these polynomials as the Askey-Wilson polynomials or the orthogonal ${}_4\phi_3$ polynomials. They generalize the $6 - j$ symbols and are the most general classical orthogonal polynomials, [**2**]. The only difficult step in proving their orthogonality is the evaluation of the Askey-Wilson integral

$$(1.4) \quad I = I(a, b, c, d)$$
$$= \frac{(q)_\infty}{2\pi} \int_0^\pi \frac{h(\cos 2\theta, 1)d\theta}{h(\cos\theta, a)h(\cos\theta, b)h(\cos\theta, c)h(\cos\theta, d)},$$

where

$$(1.5) \quad h(\cos\theta, \gamma) = (\gamma e^{i\theta})_\infty (\gamma e^{-i\theta})_\infty.$$

Received June 3, 1985 and in revised form February 5, 1986. This research was partially supported by grants from the National Science Foundation. The work of the first author was done while this author was visiting the University of Minnesota on sabbatical from Arizona State University. The generosity and hospitality of the University of Minnesota is gratefully acknowledged.

Askey and Wilson [6] used contour integration and a clever elliptic function argument to evaluate the integral I.

In view of the importance of the orthogonal ${}_4\phi_3$ polynomials, it is desirable to find as many simple evaluations of the integral I as possible. Askey [3] used functional equations to evaluate I. Rahman [15] gave an elementary evaluation of the Askey-Wilson integral. Ismail, Stanton and Viennot [12] gave a combinatorial evaluation of the integral I. We give a new evaluation in Section 2. Our proof uses properties of the continuous q-Hermite polynomials $\{H_n(x|q)\}$

$$(1.6)\qquad \sum_{n=0}^{\infty} H_n(x|q)\frac{t^n}{(q)_n} = 1/h(x, t),$$

where $h(x, t)$ is as in (1.5). We also evaluate a contour integral related to (1.4).

The continuous q-Hermite polynomials, as well as the continuous q-ultraspherical polynomials $\{C_n(x; \beta|q)\}$

$$(1.7)\qquad \sum_{n=0}^{\infty} C_n(x; \beta|q)t^n = h(x, \beta t)/h(x, t),$$

were introduced by L. J. Rogers in his memoirs on expansions of certain infinite products [18], [19], [20]. Rogers solved the connection coefficient problem and computed the coefficients in the linearization of a product of two continuous q-ultraspherical polynomials as a sum. He proved

$$(1.8)\qquad \begin{cases} C_n(x; \beta|q)C_m(x; \beta|q) = \displaystyle\sum_{k=0}^{m\wedge n} a(k, m, n)C_{m+n-2k}(x; \beta|q), \\ a(k, m, n) \\ \quad = \dfrac{(q)_{m+n-2k}(\beta)_{m-k}(\beta)_{n-k}(\beta)_k(\beta^2)_{m+n-k}(1 - \beta q^{m+n-2k})}{(\beta^2)_{m+n-2k}(q)_{m-k}(q)_{n-k}(q)_k(\beta q)_{m+n-k}(1 - \beta)}. \end{cases}$$

In particular

$$(1.9)\qquad H_n(x|q)H_m(x|q) = \sum_{k=0}^{m\wedge n} \frac{(q)_m(q)_n}{(q)_{m-k}(q)_{n-k}(q)_k} H_{m+n-2k}(x|q),$$

holds since

$$(1.10)\qquad H_n(x|q) = (q)_n C_n(x; 0|q).$$

Rogers used his results to prove the Rogers-Ramanujan identities. He realized that $\{C_n(x; \beta|q)\}$ generalize the ultraspherical polynomials but did not investigate their orthogonality. Szegö [23] found the weight function of $\{H_n(x|q)\}$ in 1926. He proved

(1.11) $$\int_0^\pi H_n(\cos\theta|q)H_m(\cos\theta|q)h(\cos 2\theta, 1)d\theta = 2\pi(q)_n\delta_{m,n}/(q)_\infty.$$

The weight function of $\{C_n(x;\beta|q)\}$ was not found till the late seventies, **[4]**, **[5]**, **[6]**. The orthogonality relation of $\{C_n(x;\beta|q)\}$ is

(1.12) $$\int_0^\pi \frac{h(\cos 2\theta, 1)}{h(\cos 2\theta, \beta)} C_n(\cos\theta;\beta|q)C_m(\cos\theta;\beta|q)d\theta = \alpha_n\delta_{m,n},$$

$$\alpha_n = 2\pi(\beta^2)_n(\beta)_\infty^2/[(1-\beta q^n)(q)_n(\beta^2)_\infty(q)_\infty].$$

The purpose of this paper is to investigate the implications of Rogers' formulas (1.8) and (1.9) and study the H_n's and C_n's in some detail. In Section 2 we give an evaluation of the Askey-Wilson integral that uses (1.9) and Szegö's orthogonality relation (1.11). The idea is to observe that the integrand in I is the product of four generating functions of continuous q-Hermite polynomials times their weight function. The integral is then evaluated via repeated applications of (1.9). This led us to consider the integral

(1.13) $$\mathcal{J} = \mathcal{J}(a, b, c, d) = \frac{(q)_\infty(\beta^2)_\infty}{2\pi(\beta)_\infty(\beta)_\infty} \times \int_0^\pi \frac{h(\cos\theta, \beta a)h(\cos\theta, \beta b)h(\cos\theta, \beta c)h(\cos\theta, \beta d)}{h(\cos\theta, a)h(\cos\theta, b)h(\cos\theta, c)h(\cos\theta, d)} \cdot \frac{h(\cos 2\theta, 1)}{h(\cos 2\theta, \beta)}d\theta.$$

When $\beta = 0$ the integral $\mathcal{J}(a, b, c, d)$ reduces to the Askey-Wilson integral $I(a, b\ c, d)$. In Section 3 we prove that $\mathcal{J}$ is a positive symmetric Hilbert-Schmidt kernel in $\cos\theta$ and $\cos\psi$ when

$$a = d\exp(2i\theta), \quad b = c\exp(2i\psi).$$

We also prove that the eigenfunctions are $\{C_n(\cos\theta;\beta|q)\}$ and determine the corresponding eigenvalues. We also find a Poisson-type kernel for the continuous q-ultraspherical polynomials using Rogers' linearization formula (1.8). This also leads to a positive symmetric Hilbert-Schmidt kernel whose eigenfunctions are $\{C_n(x;\beta|q)\}$ and eigenvalues can be found explicitly.

In Section 4 we study the integral

(1.14) $$K(r, s, t) := \frac{(q)_\infty}{2\pi}\int_0^\pi \frac{h(\cos 2\theta, 1)h(\cos\theta, s\beta)d\theta}{h(\cos\theta, r)h(\cos\theta, t)h(\cos\theta, s)}.$$

This is a variation on the Askey-Wilson integral (1.4) when one of the h's in the denominator is moved to the numerator. It turns out that

$$(1.15)\quad K(r, s, t) = \frac{(\beta)_\infty(\beta s^2)_\infty}{(rs)_\infty(st)_\infty(rt)_\infty}\, {}_2\phi_1\left(\begin{matrix} rs, st \\ \beta s^2 \end{matrix}; q, \beta\right),$$

when $-1 < \beta < 1$, $|r|, |s|, |t| \in [0, 1)$. This is a Mellin-Barnes type integral representation for a ${}_2\phi_1$. The integral $K(r, s, t)$ can be evaluated in certain special cases. This integral representation is due to Nassrallah and Rahman [**15**] but our proof seems to be new.

Mehler's formula (or the Poisson kernel) for the Hermite polynomials is

$$(1.16)\quad \sum_{n=0}^{\infty} H_n\left(\frac{x}{\sqrt{2}}\right)H_n\left(\frac{y}{\sqrt{2}}\right)\frac{(t/2)^n}{n!} = (1 - t^2)^{-1/2} \exp\left\{\frac{xyt - (x^2 + y^2)t/2}{1 - t^2}\right\}$$

[**17**, p. 198]. Kibble [**13**] obtained a multivariable extension of Mehler's formula (1.16). Carlitz [**8**] rediscovered a special case of Kibble's result. Carlitz's work led Slepian [**22**] to derive the full Kibble formula independently. This formula is now known as the "Kibble-Slepian formula". Louck [**14**] used the boson theory to derive the Kibble-Slepian formula. Foata [**10**] found a very interesting combinatorial proof of the same formula.

Two special cases of the Kibble-Slepian formula are

$$(1.17)\quad \sum_{m,n=0}^{\infty} H_{m+n}(a)H_m(b)H_n(c)\frac{x^m y^n}{m!n!} = (1 - 4x^2 - 4y^2)^{-1/2} \times \exp\left\{\frac{-4a^2(x^2 + y^2) + 4a(bx + cy) - 4(bx + cy)^2}{1 - 4x^2 - 4y^2}\right\},$$

and

$$(1.18)\quad \sum_{m,n,p=0}^{\infty} H_{m+n+p}(a)H_m(b)H_n(c)H_p(d)\frac{x^m y^n t^p}{m!n!p!}.$$

In [**5**] Askey and Ismail raised the question of extending the Kibble-Slepian formula to the continuous q-Hermite polynomials. In Section 5 we obtain q-analogues of (1.17) and (1.18) and outline a way to evaluate more general sums.

2. The evaluation of the Askey-Wilson integral. The generating function (1.6) is

$$(2.1)\quad \sum_{n=0}^{\infty} H_n(\cos\theta|q)t^n/(q)_n = 1/\{(te^{i\theta})_\infty(te^{-i\theta})_\infty\}.$$

The Poisson kernel of $\{H_n(x|q)\}$ follows from

$$(2.2)\qquad \sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)H_n(\cos\phi|q)}{(q)_n} t^n = \frac{(t^2)_\infty}{h(\cos(\theta+\phi),t)h(\cos(\theta-\phi),t)},$$

a q-analogue of Mehler's formula (1.16). Our evaluation of I uses the q-binomial theorem

$$(2.3)\qquad \sum_{n=0}^{\infty} (\lambda)_n t^n/(q)_n = (\lambda t)_\infty/(t)_\infty.$$

The generating function (2.1) and the case $\lambda = 0$ of (2.3) lead to the explicit formula, **[5]**

$$(2.4)\qquad H_n(\cos\theta|q) = \sum_{k=0}^{n} \frac{(q)_n e^{i(n-2k)\theta}}{(q)_k (q)_{n-k}}.$$

Since it is not well known that (2.2) and (1.9) are equivalent we first show that they are.

PROPOSITION 2.5. *The q-Mehler's formula* (2.2) *is equivalent to the linearization formula* (1.9).

Proof. We prove that (1.9) implies (2.2). The steps are reversible. Multiply (1.9) by $s^m t^n/(q)_m(q)_n$, replace x by $\cos\theta$ and sum on $m, n \geqq 0$. From (2.1) we obtain

$$(2.6)\qquad \frac{1}{h(\cos\theta,s)h(\cos\theta,t)} = \sum_{k,m,n=0}^{\infty} \frac{s^{k+m}t^{k+n}}{(q)_m(q)_n(q)_k} H_{m+n}(\cos\theta|q).$$

The k-sum is evaluable by (2.3) to $1/(st)_\infty$. Next, replace t by $te^{-i\phi}$, s by $te^{i\phi}$ and n by $l - m$. Then (2.4) implies that the right side of (2.6) is

$$\frac{1}{(t^2)_\infty} \sum_{l=0}^{\infty} \frac{H_l(\cos\theta|q)H_l(\cos\phi|q)}{(q)_l} t^l,$$

which implies the q-Mehler's formula (2.2).

We now give our evaluation of the Askey-Wilson integral (1.4).

PROPOSITION 2.7. *When* $|a| < 1$, $|b| < 1$, $|c| < 1$, $|d| < 1$, *the integral* I *is given by*

$$(2.8)\qquad I(a,b,c,d) = \frac{(abcd)_\infty}{(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty}.$$

https://doi.org/10.4153/CJM-1988-041-0 Published online by Cambridge University Press

Proof. Since the integrand in I involves the product of four continuous q-Hermite generating functions, we must find

$$(2.9)\quad f(j, l, m, n) = \frac{(q)_\infty}{2\pi} \int_0^\pi H_j(\cos\theta|q)H_l(\cos\theta|q)H_m(\cos\theta|q) \times H_n(\cos\theta|q)(e^{2i\theta})_\infty(e^{-2i\theta})_\infty d\theta.$$

Then

$$(2.10)\quad I = \sum_{j,l,m,n=0}^{\infty} \frac{f(j, l, m, n)a^j b^l c^m d^n}{(q)_j(q)_l(q)_m(q)_n}.$$

The linearization formula (1.9) implies that the integral of the product of three continuous q-Hermite polynomials times their weight function is evaluable. We iterate (1.9) to obtain

$$(2.11)\quad H_l(x|q)H_m(x|q)H_n(x|q) = \sum_{k,j} \frac{(q)_l(q)_m(q)_n(q)_{m+n-2k}H_{l+m+n-2k-2j}(x|q)}{(q)_{m-k}(q)_{n-k}(q)_k(q)_{l-j}(q)_{m+n-2k-j}(q)_j}.$$

Clearly (2.11) and (1.11) imply

$$f(l, m, n, l + m + n - 2p) = \sum_k \frac{(q)_l(q)_m(q)_n(q)_{m+n-2k}(q)_{l+m+n-2p}}{(q)_{m-k}(q)_{n-k}(q)_k(q)_{p-k}(q)_{l-p+k}(q)_{m+n-p-k}},$$

and (2.10) and (2.9) give

$$(2.12)\quad I = \sum_{j,k,l,m,n=0} \frac{(q)_{m+n}a^l b^{m+k} c^{n+k} d^j}{(q)_m(q)_n(q)_k(q)_{(l+m+n-j)/2}(q)_{(l+j-m-n)/2}(q)_{(j+m+n-l)/2}}.$$

The k-sum is evaluable to $1/(bc)_\infty$, by the q-binomial theorem (2.3). If we replace (l, n, j) by (α, β, γ) where

$$\alpha = (l + m + n - j)/2,\ \beta = (l + j - m - n)/2,$$
$$\gamma = (j + m + n - l)/2,$$

so that $\alpha + \beta = l$, $\beta + \gamma = j$, $\alpha + \gamma = m + n$; the β-sum contributes $1/(ad)_\infty$, hence

$$I = \frac{1}{(bc)_\infty(ad)_\infty} \sum_{\alpha,\gamma,m=0}^{\infty} \frac{(q)_{\alpha+\gamma}a^\alpha b^m c^{\alpha+\gamma-m} d^\gamma}{(q)_m(q)_{\alpha+\gamma-m}(q)_\alpha(q)_\gamma}.$$

We now replace $\alpha + \gamma$ by p to get

(2.13) $I =$

$$\frac{1}{(bc)_\infty(ad)_\infty} \sum_{p=0}^{\infty} \frac{1}{(q)_p} \left\{ \sum_{\alpha=0}^{p} \frac{(q)_p a^\alpha d^{p-\alpha}}{(q)_\alpha (q)_{p-\alpha}} \right\} \left\{ \sum_{m=0}^{p} \frac{(q)_p b^m c^{p-m}}{(q)_m (q)_{p-m}} \right\}.$$

If $a = a_1 e^{-i\theta}$, $d = a_1 e^{i\theta}$, $b = b_1 e^{-i\phi}$, $c = b_1 e^{i\phi}$ then (2.4) and (2.13) yield

$$(2.14) \quad I = \sum_{p=0}^{\infty} \frac{(a_1 b_1)^p H_p(\cos\theta|q) H_p(\cos\phi|q)}{(q)_p (bc)_\infty (ad)_\infty}.$$

Finally, we obtain the evaluation (2.8) from (2.14) and the q-Mehler formula (2.2). This completes the proof.

We now discuss the cases when the conditions $|a| < 1$, $|b| < 1$, $|c| < 1$ or $|d| < 1$ are violated. In order to do that we first transform the integral defining $I(a, b, c, d)$ to a contour integral. Since the integrand in I is an even function of θ we obtain

$$(2.15) \quad I(a, b, c, d) = \frac{(q)_\infty}{4\pi i} \int_{|z|=1} \frac{(z^2)_\infty (z^{-2})_\infty z^{-1} dz}{(az)_\infty (a/z)_\infty (bz)_\infty (b/z)_\infty (cz)_\infty (c/z)_\infty (dz)_\infty (d/z)_\infty},$$

valid for

$$\max(|a|, |b|, |c|, |d|) < 1.$$

We now analytically continue the above integral as a function of a. As a function of z the integrand in (2.15) has singularities at $z = 0, \lambda q^j, \lambda^{-1} q^{-j}$, $j = 0, 1, \dots, \lambda = a, b, c$ or d. Let

$$(2.16) \quad \begin{cases} A = \{\lambda q^j : \lambda = 0, a, b, c, d, j = 0, 1, 2, \dots\}, \\ B = \{\lambda^{-1} q^{-j} : \lambda = a, b, c, d, j = 0, 1, 2, \dots\}. \end{cases}$$

Now assume that a is allowed to vary in

$$\{a : |a| < q^{-k}, a \neq q^{-j}, j = 0, 1, \dots, k - 1\}$$

but b, c and d are still restricted to

$$\max(|b|, |c|, |d|) < 1.$$

Choose a contour C containing the set A in its interior and B in its exterior and define

$$(2.17) \quad I_1(a, b, c, d) = \frac{(q)_\infty}{4\pi i} \int_C \frac{(z^2)_\infty (z^{-2})_\infty z^{-1} dz}{(az)_\infty (a/z)_\infty (bz)_\infty (b/z)_\infty (cz)_\infty (c/z)_\infty (dz)_\infty (d/z)_\infty}.$$

Clearly, I_1 is an analytic continuation of I. The restrictions $|b| < 1$, $|c| < 1$, $|d| < 1$ can be similarly removed. Thus, the following proposition follows from Proposition 2.7 and analytic continuation of the right-hand side of (2.8). This analytic continuation is possible as long as ab, ac, ad, bc, bd or cd is not of the form q^{-j}, $j = 0, 1, 2, \dots$.

PROPOSITION 2.18. *Assume that the pairwise products of* $\{a, b, c, d\}$ *do not belong to the set* $\{q^j : j = 0, -1, -2, \dots\}$. *Then*

$$(2.19) \quad \frac{(q)_\infty}{2\pi i}\int_C \frac{(z^2)_\infty(z^{-2})_\infty z^{-1}dz}{(az)_\infty(a/z)_\infty(bz)_\infty(b/z)_\infty(cz)_\infty(c/z)_\infty(dz)_\infty(d/z)_\infty}$$
$$= \frac{2(abcd)_\infty}{(ab)_\infty(ac)(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty}$$

where the contour C *is the unit circle with suitable deformations to contain the set* A *in its interior and the set* B *in its exterior.*

Proposition 2.18 is Theorem 2.1 in [**6**] but our approach is new. The relationship (2.19) can be used to prove the orthogonality relation of the ${}_4\phi_3$ orthogonal polynomials when the parameters a, b, c, d are no longer restricted to belong to $(-1, 1)$. The corresponding measure in this case has finitely many discrete masses in addition to the absolutely continuous component. For details, see [**6**].

3. The kernel $\mathscr{I}(a, b, c, d)$. The explicit formula

$$(3.1) \quad C_n(\cos\theta; \beta|q) = \sum_{k=0}^{n} \frac{(\beta)_k(\beta)_{n-k}}{(q)_k(q)_{n-k}} e^{i(n-2k)\theta}$$

follows from the generating function (1.7), [**5**]. The main result of this section is

PROPOSITION 3.2. *The kernel* $\mathscr{I}(a, b, c, d)$ *is given by*

$$(3.3) \quad \mathscr{I}(\rho e^{i\phi}, \sigma e^{i\psi}, \sigma e^{-i\psi}, \rho e^{-i\phi})$$
$$= \sum_{n=0}^{\infty} \frac{(q)_n(\beta^2)_n}{(\beta)_{n+1}(\beta)_n}(\rho\sigma)^n C_n(\cos\phi; \beta|q)C_n(\cos\psi; \beta|q).$$
$$\cdot\, {}_2\phi_1\left(\begin{matrix}\beta^2q^n, \beta\\ \beta q^{n+1}\end{matrix}; q, \rho^2\right){}_2\phi_1\left(\begin{matrix}\beta^2q^n, \beta\\ \beta q^{n+1}\end{matrix}; q, \sigma^2\right)$$

when $|\rho| < 1$, $|\sigma| < 1$, $-1 < \beta < 1$.

Proof. The proof is very similar to our evaluation of the Askey-Wilson integral I; see Proposition 2.7. We first iterate the linearization formula (1.8) to get

$$C_l(x;\ \beta|q)C_m(x;\ \beta|q)C_n(x;\ \beta|q)$$
$$= \sum_{k,j} \frac{(q)_{m+n-2k}(\beta)_{m-k}(\beta)_{n-k}(\beta)_k(\beta^2)_{m+n-k}}{(\beta^2)_{m+n-2k}(q)_{m-k}(q)_{n-k}(q)_k(\beta)_{m+n+1-k}}$$
$$\cdot \frac{(q)_{m+n+l-2k-2j}(\beta)_{l-j}(\beta)_{m+n-2k-j}(\beta)_j(\beta^2)_{l+m+n-2k-j}}{(\beta^2)_{l+m+n-2k-2j}(q)_{l-j}(q)_{m+n-2k-j}(q)_j(\beta)_{l+m+n-2k-j+1}}$$
$$\cdot (1 - \beta q^{m+n-2k})(1 - \beta q^{l+m+n-2k-2j})C_{l+m+n-2k-2j}(x;\ \beta|q).$$

This, (1.7) and the orthogonality relation (1.12) give

$$\mathscr{I}(a, b, c, d) = \frac{(q)_\infty(\beta^2)_\infty}{2\pi(\beta)_\infty(\beta)_\infty} \int_0^\pi \sum_{l,m,n,p=0}^{\infty} C_l(\cos\theta;\ \beta|q)$$
$$\cdot C_m(\cos\theta;\ \beta|q)C_n(\cos\theta;\ \beta|q)$$
$$\cdot C_p(\cos\theta;\ \beta|q)\left\{\frac{h(\cos 2\theta;\ 1)}{h(\cos 2\theta;\ \beta)}\right\}a^l b^m c^n d^p d\theta$$
$$= \sum_{j,k,l,m,n,p} \frac{(q)_{m+n-2k}(\beta)_{m-k}(\beta)_{n-k}(\beta)_k(\beta^2 q^{m+n-2k})_k(\beta^2)_p}{(q)_k(q)_{m-k}(q)_{n-k}(\beta)_{m+n-2k}(\beta q^{m+n+1-2k})_k(\beta)_{p+1}}$$
$$\cdot \frac{(\beta)_{l-j}(\beta)_{m+n-2k-j}(\beta)_j(\beta^2 q^p)_j a^l b^m c^n d^p}{(q)_j(q)_{l-j}(q)_{m+n-2k-j}(\beta q^{p+1})_j},$$

where $l + m + n = p + 2k + 2j$. In the above sum we also have the restrictions $m \geqq k$, $n \geqq k$, $l \geqq j$, $m + n - 2k \geqq j$, $l + m + n \geqq 2k + 2j$. Now replace m, n and l by $m + k$, $n + k$ and $l + j$ respectively, then replace j by $l + m + n - p$ to obtain

$$\mathscr{I}(a, b, c, d)$$
$$= \sum_{k,l,m,n,p} \frac{(q)_{m+n}(\beta)_m(\beta)_n(\beta)_k(\beta^2 q^{m+n})_k(\beta^2)_{l+m+n}(\beta)_l(\beta)_{p-l}}{(q)_k(q)_m(q)_n(\beta)_{m+n}(\beta q^{m+n+1})_k(\beta)_{l+m+n+1}(q)_l(q)_{p-l}};$$
$$\cdot \frac{(\beta)_{l+m+n-p}}{(q)_{l+m+n-p}} a^{l-p} b^{m+k} c^{n+k} d^p a^{l+m+n}.$$

In the above sum $l \leqq p$ so we now replace p by $p + l$ and let $m + n = M$. This leads to

(3.4) $\mathscr{I}(a, b, c, d)$
$$= \sum_{M=0}^{\infty} \frac{(q)_M(\beta^2)_M}{(\beta)_{M+1}(\beta)_M} \sum_{m=0}^{M} \frac{(\beta)_m(\beta)_{M-m}}{(q)_m(q)_{M-m}} b^m c^{M-m} {}_2\phi_1\left(\begin{matrix}\beta,\ \beta^2 q^M \\ \beta q^{M+1}\end{matrix};\ q,\ bc\right)$$
$$\cdot \sum_{p=0}^{M} \frac{(\beta)_p(\beta)_{M-p}}{(q)_p(q)_{M-p}} a^{M-p} d^p {}_2\phi_1\left(\begin{matrix}\beta,\ \beta^2 q^M \\ \beta q^{M+1}\end{matrix};\ q,\ ad\right).$$

The ${}_2\phi_1$'s are the l and k-sums. This and (3.1) prove (3.3).

COROLLARY 3.5. *When* $|\rho|, |\sigma|, |\beta| \in (0, 1)$ *we have*

$$(3.6) \quad \mathscr{J}(\rho e^{i\phi}, \sigma e^{i\psi}, \sigma e^{-i\psi}, \rho e^{-i\phi}) = \frac{(\beta^2)^2_\infty(\beta\rho^2)_\infty(\beta\sigma^2)_\infty}{(\beta)^2_\infty(\rho^2)_\infty(\sigma^2)_\infty} \cdot \sum_{n=0}^{\infty} \frac{(q)_n(1 - \beta q^n)}{(\beta^2)_n}(\rho\sigma)^n C_n(\cos\phi; \beta|q)C_n(\cos\psi; \beta|q) \cdot {}_2\phi_1\left(\begin{matrix}1/\beta, \rho^2\\ \beta\rho^2\end{matrix}; q, \beta^2 q^n\right){}_2\phi_1\left(\begin{matrix}1/\beta, \sigma^2\\ \beta\sigma^2\end{matrix}; q, \beta^2 q^n\right).$$

Proof. We apply the Heine transformation

$$(3.7) \quad {}_2\phi_1\left(\begin{matrix}a, b\\ c\end{matrix}; q, x\right) = \frac{(ax)_\infty(b)_\infty}{(x)_\infty(c)_\infty}\,{}_2\phi_1\left(\begin{matrix}x, c/b\\ ax\end{matrix}; q, b\right)$$

to the ${}_2\phi_1$'s in (3.3). After some simplification we obtain (3.6).

We now investigate the properties of $\mathscr{J}$ viewed as a weighted L^2 kernel on a square, [**25**]. We first consider the case $|\beta| < 1$. Clearly

$$(3.8) \quad \int_0^\pi \mathscr{J}(\rho e^{i\theta}, \sigma e^{i\phi}, \sigma e^{-i\phi}, \rho e^{-i\theta})C_n(\cos\theta; \beta|q)w_\beta(\cos\theta)d\theta = \lambda_n C_n(\cos\phi; \beta|q),$$

where

$$(3.9) \quad \lambda_n = \frac{2\pi(\beta^2)^2_n(\rho\sigma)^n}{(1 - \beta q^n)(\beta^2)_\infty(q)_\infty}\,{}_2\phi_1\left(\begin{matrix}\beta, \beta^2 q^n\\ \beta q^{n+1}\end{matrix}; q, \rho^2\right){}_2\phi_1\left(\begin{matrix}\beta, \beta^2 q^n\\ \beta q^{n+1}\end{matrix}; q, \sigma^2\right)$$

and $w_\beta(\cos\theta)$ is the weight function

$$(e^{2i\theta})_\infty(e^{-2i\theta})_\infty/(\beta e^{2i\theta})_\infty(\beta e^{-2i\theta})_\infty.$$

Observe the $\lambda_n > 0$ when $\rho, \sigma \in (0, 1)$, $-1 < \beta < 1$. The Weierstrass approximation theorem guarantees the completeness of $\{C_n(\cos\theta; \beta|q)\}$ in the space

$$L^2([0, \pi], w_\beta(\cos\theta)d\theta).$$

Therefore, the kernel

$$\mathscr{J}(\rho e^{i\theta}, \sigma e^{i\phi}, \sigma e^{-i\phi}, \rho e^{-i\theta})$$

will be positive on $[0, \pi] \times [0, \pi]$ if and only if

$$\int_0^\pi \int_0^\pi \mathscr{J}(\rho e^{i\theta}, \sigma e^{i\phi}, \sigma e^{-i\phi}, \rho e^{-i\theta})C_n(\cos\theta, \beta|q)C_m(\cos\phi; \beta|q) \cdot w_\beta(\cos\theta)w_\beta(\cos\phi)d\theta d\phi \geqq 0$$

for all m, n. The above double integral is obviously a positive multiple of $\lambda_n \delta_{m,n}$, hence is non-negative.

Recall that when $0 < q < 1$, the continuous q-ultraspherical polynomials are orthogonal with respect to a positive measure if and only if $-1 < \beta < 1$ or $1 < \beta < q^{-1/2}$, [4], so the only case left is the case $1 < \beta < q^{-1/2}$. In this case, the continuous q-ultraspherical polynomials are orthogonal with respect to the measure

$$(3.10)\quad d\psi(x) = \frac{h(\cos 2\theta, 1)}{h(\cos 2\theta, \beta)}\chi[-1, 1]\frac{dx}{\sqrt{1 - x^2}} + \frac{\pi(1/\beta)_\infty(\beta)_\infty}{(q)_\infty(\beta^2)_\infty}\{\delta(x - \xi) + \delta(x + \xi)\}dx,$$

where

$$(3.11)\quad x = \cos\theta, \quad \xi = \frac{1}{2}(\sqrt{\beta} + 1/\sqrt{\beta}), \beta > 1,$$

[4]. The definition of $\mathcal{J}$ when $1 < \beta < q^{-1/2}$ is

$$(3.12)\quad \mathcal{J} = \mathcal{J}(a, b, c, d) := \frac{(q)_\infty(\beta^2)_\infty}{2\pi(\beta)_\infty(\beta)_\infty}\int_{-1}^{1} f(x)d\psi(x), \quad q^{-1/2} > \beta > 1,$$

with

$$(3.13)\quad f(\cos\theta) = \frac{h(\cos\theta, \beta a)h(\cos\theta, \beta b)h(\cos\theta, \beta c)h(\cos\theta, \beta d)}{h(\cos\theta, a)h(\cos\theta, b)h(\cos\theta, c)h(\cos\theta, d)}.$$

In other words, the term

$$\frac{1}{2}\frac{(1/\beta)_\infty}{(\beta)_\infty}[f(\xi) + f(-\xi)]$$

should be added to the right side of (1.13). Here again, $\mathcal{J}$ will be positive if and only if $\lambda_n > 0$. It is clear from (3.9) that $\lambda_0 > 0$. For $n > 0$ the Heine transformation (3.7) enables us to express λ_n as a positive multiple of

$${}_2\phi_1\left(\begin{matrix}1/\beta, \rho^2 \\ \beta\rho^2\end{matrix}; q, \beta^2 q^n\right){}_2\phi_1\left(\begin{matrix}1/\beta, \sigma^2 \\ \beta\sigma^2\end{matrix}; q, \beta^2 q^n\right), n > 0,$$

which implies the positivity of λ_n.

PROPOSITION 3.14. *Let the function* $\mathcal{J}(a, b, c, d)$ *be defined by* (1.13) *when* $\beta \in (-1, 1)$ *and be given by* (3.12) *when* $q^{-1/2} > \beta > 1$. *Set*

$$r = 1, \textit{ if } \beta \in (-1, 1), r := \sqrt{\beta} \textit{ if } 1 < \beta < q^{-1/2},$$

and

https://doi.org/10.4153/CJM-1988-041-0 Published online by Cambridge University Press

$$a: = \frac{1}{2}(r + 1/r).$$

Then for $q \in (0, 1)$, $\rho, \sigma \in (0, r)$ *the kernel*

$$\mathscr{J}(\rho e^{i\theta}, \sigma e^{i\phi}, \sigma e^{-i\phi}, \rho e^{-i\theta})$$

is positive when $x = \cos\theta$, $y = \cos\phi$, $x, y \in [-a, a]$. *The eigenvalues of* $\mathscr{J}$ *are the* λ_n*'s of* (3.9) *and the corresponding eigenfunctions are* $\{C_n(x; \beta|q)\}$.

Proof. We need only to show that $\mathscr{J}$ has no eigenvalues other than the λ_n's of (3.9). But this follows from the completeness of $\{C_n(x; \beta|q)\}$ in the corresponding L^2 space, **[25]**.

PROPOSITION 3.15. *Both Proposition* 3.2 *and Corollary* 3.5 *hold when* $\beta \in (1, q^{-1/2})$ *provided that* $\mathscr{J}$ *is given by* (3.12) *and* $|\rho|, |\sigma| \in (0, r)$.

The key to the results obtained so far in this section has been the linearization formula (1.8). If we multiply (1.8) by $s^m t^n$ and sum over m and n then replace s by $\rho e^{-i\phi}$ and t by $\rho e^{i\phi}$ we obtain the Poisson type kernel

$$(3.16) \quad \frac{(\beta\rho e^{i(\theta+\phi)})_\infty(\beta\rho e^{-i(\theta+\phi)})_\infty(\beta\rho e^{i(\theta-\phi)})_\infty(\beta\rho e^{i(\phi-\theta)})_\infty}{(\rho e^{i(\theta+\phi)})_\infty(\rho e^{-i(\theta+\phi)})_\infty(\rho e^{i(\theta-\phi)})_\infty(\rho e^{i(\phi-\theta)})_\infty}$$

$$= \sum_{n=0}^{\infty} \frac{(q)_n}{(\beta)_n} \rho^n C_n(\cos\theta; \beta|q) C_n(\cos\phi; \beta|q) {}_2\phi_1\left(\begin{matrix}\beta, \beta^2 q^n \\ \beta q^{n+1}\end{matrix}; q, \rho^2\right).$$

This identity is also in **[7]**. Now let $K(\cos\theta, \cos\phi)$ denote the left hand side of (3.17). The kernel $K(x, y)$ can be shown to be positive on $[-a, a] \times [-a, a]$ when $0 < \rho < r$. This also leads to an integral equation satisfied by the continuous q-ultraspherical polynomials.

4. An integral representation. Recall that

$$(4.1) \quad C_n(x; 0|q) = H_n(x|q)/(q)_n.$$

Rogers solved the connection coefficient problem for the continuous q-ultraspherical polynomials. He proved

$$(4.2) \quad C_n(x; \gamma|q) = \sum_{k=0}^{[n/2]} \frac{\beta^k (\gamma\beta^{-1})_k (\gamma)_{n-k} (1 - \beta q^{n-2k})}{(q)_k (\beta q)_{n-k} (1 - \beta)} C_{n-2k}(x; \beta|q),$$

which implies

$$(4.3) \quad C_n(x; \gamma|q) = \sum_{k=0}^{[n/2]} \frac{(-\gamma)^k q^{k(k-1)/2} (\gamma)_{n-k}}{(q)_k (q)_{n-2k}} H_{n-2k}(x|q),$$

in the limiting case $\beta \to 0$. The integral $K(r, s, t)$ has the power series expansion

$$K(r, s, t) = \frac{(q)_\infty}{2\pi} \sum_{m,n,p} K_{m,n,p} r^m s^n t^p,$$

$$K_{m,n,p} = \int_0^\pi \frac{H_m(\cos\theta|q)}{(q)_m} C_n(\cos\theta;\, \beta|q) \frac{H_p(\cos\theta|q)}{(q)_p} \times (e^{2i\theta})_\infty (e^{-2i\theta})_\infty d\theta.$$

It is now clear that evaluating $K_{m,n,p}$ is equivalent to finding the coefficients in the linearization of $H_m(x|q)C_n(x;\, \beta|q)$ in terms of the continuous q-Hermite polynomials since $(e^{2i\theta})_\infty(e^{-2i\theta})_\infty$ is the weight function of the H_n's. So, we multiply (4.3) by $H_m(x|q)/(q)_m$ then use (1.9) to linearize the product

$$H_m(x|q)H_{n-2k}(x|q)$$

as a sum. The result is

$$\frac{H_m(x|q)}{(q)_m} C_n(x;\, \beta|q) = \sum_{k,j} \frac{(-\beta)^k q^{k(k-1)/2} (\beta)_{n-k} H_{m+n-2k-2j}(x|q)}{(q)_k (q)_{m-j} (q)_j (q)_{n-2k-j}}.$$

This and the orthogonality relation (1.11) imply

$$\frac{(q)_\infty}{2\pi} K_{m,n,p} = \sum_{k,j} \frac{(-\beta)^k q^{k(k-1)/2} (\beta)_{n-k}}{(q)_k (q)_j (q)_{m-j} (q)_{n-j-2k}},$$

and the sum is over $k, j \geqq 0$ such that

$$j + k = (m + n - p)/2, j + 2k \leqq n.$$

We replace n by $n + 2k$ then let

$$m + p - n = 2\alpha, m + n - p = 2\gamma, n + p - m = 2\delta.$$

Therefore $m = \alpha + \gamma$, $n = \gamma + \delta$, $p = \alpha + \delta$ and we obtain

$$K(r, s, t) = \sum_{k,\alpha,\gamma,\delta=0}^{\infty} \frac{(-\beta)^k q^{k(k-1)/2} (\beta)_{k+\gamma+\delta}}{(q)_k (q)_\alpha (q)_\gamma (q)_\delta} r^{\alpha+\gamma} s^{2k+\gamma+\delta} t^{\alpha+\delta}.$$

The sum over α is $1/(rt)_\infty$, see (2.3). The above sum becomes

$$K(r, s, t) = \sum_{k,\delta=0}^{\infty} \frac{(-\beta)^k q^{k(k-1)/2} (\beta)_{k+\delta} s^{2k+\delta} t^\delta}{(q)_k (q)_\delta (rt)_\infty} \sum_{\gamma=0}^{\infty} \frac{(\beta q^{k+\delta})_\gamma}{(q)_\gamma} (rs)^\gamma.$$

The γ sum is $(\beta rsq^{k+\delta})_\infty/(rs)_\infty$, by (2.3). We now set $m = k + \delta$, hence

$$K(r, s, t) = \sum_{m=0}^{\infty} \frac{s^m t^m (\beta)_m (\beta rsq^m)_\infty}{(q)_m (rs)_\infty (rt)_\infty} \sum_{k=0}^{m} \frac{(q)_m (-\beta s)^k q^{\binom{k}{2}}}{(q)_k (q)_{m-k} t^k}.$$

The k sum is

$$\sum_{k=0}^{m} \frac{(q^{-m})_k}{(q)_k} (q^m \beta s/t)^k$$

which, in view of (2.3), sums to $(\beta s/t)_m$. Thus, we have

$$\frac{(rs)_\infty (rt)_\infty}{(\beta rs)_\infty} K(r, s, t) = \sum_{m=0}^{\infty} \frac{s^m t^m (\beta)_m (\beta s/t)_m}{(q)_m (\beta rs)_m}.$$

This proves Proposition 4.4 which was obtained first by Nassrallah and Rahman [**15**].

PROPOSITION 4.4. *A basic hypergeometric function has the Mellin-Barnes type integral representation*

$$(4.5) \qquad {}_2\phi_1\left(\begin{matrix}\beta, s\beta/t \\ \beta rs\end{matrix}; q, st\right) = \frac{(rs)_\infty (rt)_\infty}{(rs\beta)_\infty} K(r, s, t),$$

$0 < |r|, |s|, |t| < 1$, *where* $K(r, s, t)$ *is defined in* (1.14).

Observe that $K(r, s, t)$ is symmetric in r, t but the ${}_2\phi_1$ in (4.5) is not a symmetric function of r and t. The application of Heine transformation (3.7) yields the following symmetric form of (4.6)

$${}_2\phi_1\left(\begin{matrix}rs, st \\ \beta s^2\end{matrix}; q, \beta\right) = \frac{(rs)_\infty (rt)_\infty (st)_\infty}{(\beta)_\infty (\beta s^2)_\infty} K(r, s, t),$$

holding for $|r|, |s|, |t|, |\beta| \in [0, 1)$. This is (1.15).

We now consider special cases of (1.15) when the ${}_2\phi_1$ can be evaluated.

PROPOSITION 4.6. *When* $\beta = -q/s^2$ *we have*

$$(4.7) \qquad K(r, s, -r) = \frac{(-q; q)_\infty (-q^2/s^2; q^2)_\infty}{(r^2 s^2; q^2)_\infty (-r^2; q^2)_\infty}, \ |r| < 1, \ 0 < |s| < 1.$$

Proof. When $r = -t$, $\beta = -q/s^2$ the ${}_2\phi_1$ appearing in (1.15) is actually a ${}_1\phi_0$ base q^2. Here we need to impose the restriction $|s| > \sqrt{q}$ since $|\beta| < 1$. Therefore,

$${}_2\phi_1\left(\begin{matrix}rs, -rs \\ -q\end{matrix}; q, -q/s^2\right) = {}_1\phi_0\left(\begin{matrix}r^2 s^2 \\ \rule{2em}{0.4pt}\end{matrix}; q^2, -q/s^2\right)$$

https://doi.org/10.4153/CJM-1988-041-0 Published online by Cambridge University Press

$$= \frac{(-qr^2;\, q^2)_\infty}{\left(-\frac{q}{s^2};\, q^2\right)_\infty},$$

and the integral $K(r, s, -r)$ is

$$\frac{(-q;\, q)_\infty(-q/s^2;\, q)_\infty(-qr^2;\, q^2)_\infty}{(rs;\, q)_\infty(-rs;\, q)_\infty(-r^2;\, q)_\infty(-q/s^2;\, q^2)_\infty}$$

which can be simplified to the right hand side of (4.7). The restriction $|s| > \sqrt{q}$ can be removed by analytic continuation. This completes the proof.

Finally, we consider the special case $t = r\sqrt{q}$, $\beta s^2 = \sqrt{q}$.

PROPOSITION 4.8. *When $t = r\sqrt{q}$, $\beta s^2 = \sqrt{q}$ the integral $K(r, s, t)$ is given by*

$$(4.9) \quad K(r, s, r\sqrt{q}) = \frac{(q^{1/4}r;\, q^{1/2})_\infty}{(q^{1/4}/s;\, q^{1/2})_\infty} + \frac{(-q^{1/4}r;\, q^{1/2})_\infty}{(-q^{1/4}/s;\, q^{1/2})_\infty},$$

$$|r^2q| < 1,\ 0 < |s| < 1.$$

Proof. The ${}_2\phi_1$ on the right hand side of (1.15) gives

$$\begin{aligned} {}_2\phi_1\left(\begin{matrix} rs,\ rs\sqrt{q} \\ \sqrt{q} \end{matrix};\, q,\, s^{-2}\sqrt{q}\right) &= \sum_{n=0}^{\infty} \frac{(r/s;\, \sqrt{q})_{2n}}{(\sqrt{q};\, \sqrt{q})_{2n}} s^{-2n} q^{n/2} \\ &= {}_1\phi_0\left(\begin{matrix} rs \\ \_\_\_ \end{matrix};\, \sqrt{q},\, s^{-1}q^{1/4}\right) \\ &\quad + {}_1\phi_0\left(\begin{matrix} rs \\ \_\_\_ \end{matrix};\, \sqrt{q},\, -s^{-1}q^{1/4}\right), \end{aligned}$$

when $|s^4| > q$. Formula (4.9) follows from the q-binomial theorem. Analytic continuation allows us to weaken the assumption $|s^4| > q$ to $|s| > 0$.

In Propositions 4.6 and 4.7 we could have used (4.5) and avoided the analytic continuation.

5. Multilinear formulas. In this section, we shall give q-analogues of the multilinear Mehler formulas. For convenience, we consider the polynomials

$$(5.1) \quad h_n(x|q) := \sum_{k=0}^{n} \frac{(q)_n}{(q)_k(q)_{n-k}} x^k$$

which are related to $H_n(x|q)$ via

(5.2) $$H_n(\cos\theta|q) = e^{in\theta}h_n(e^{-2i\theta}|q).$$

The multilinear formulas are (5.14) and (5.15).

The key observation is that the polynomials $h_n(a|q)$ are the moments for the Al-Salam-Carlitz [**1**] polynomials

(5.3) $$\int_{-\infty}^{\infty} x^n d\psi_a(x) = h_n(a|q)$$

where the step function $\psi_a(x)$ has jumps

(5.4) $$d\psi_a(q^k) = \frac{q^k}{(a)_\infty(q)_k(q/a)_k},\ d\psi_a(aq^k) = \frac{q^k}{(1/a)_\infty(q)_k(aq)_k},$$

for $a < 0$ and $0 < q < 1$. (We have replaced the normalization constant C in [**1**] and [**9**] by $1 - a$, as in [**11**].) The orthogonal polynomials $U_n^a(x)$ for $d\psi_a(x)$ have the generating function

(5.5) $$\sum_{n=0}^{\infty} U_n^a(x)\frac{t^n}{(q)_n} = \frac{(t)_\infty(at)_\infty}{(xt)_\infty},\quad |xt| < 1,$$

and the orthogonality relation

(5.6) $$\int_{-\infty}^{\infty} U_n^a(x)U_m^a(x)d\psi_a(x) = (-a)^n q^{n(n-1)/2}(q)_n\delta_{nm}.$$

The analogue of the Askey-Wilson integral for $\{U_n^a(x)\}$ is

(5.7) $$E(t_1, t_2, t_3, t_4) = \int_{-\infty}^{\infty}\prod_{j=1}^{4}\frac{(t_j)_\infty(at_j)_\infty}{(xt_j)_\infty}d\psi_a(x).$$

We now evaluate (5.7) in two different ways. From the definition (5.4) it is clear that

(5.8) $$\begin{aligned}E(t_1, t_2, t_3, t_4)\\ &= \left[\left\{\prod_{j=1}^{4}(at_j)_\infty\right\}/(a)_\infty\right]{}_4\phi_3\left(\begin{matrix}t_1, t_2, t_3, t_4\\ q/a, 0, 0\end{matrix}; q, q\right)\\ &+ \left[\left\{\prod_{j=1}^{4}(t_j)_\infty\right\}/(1/a)_\infty\right]{}_4\phi_3\left(\begin{matrix}at_1, at_2, at_3, at_4\\ aq, 0, 0\end{matrix}; q, q\right).\end{aligned}$$

If $a < 0$ and $|t_1| < \min(1, -1/a)$, then (5.5) and (5.3) imply that

(5.9) $$\begin{aligned}E(t_1, t_2, t_3, t_4)\\ &= \sum_{n_1,n_2,n_3,n_4=0}^{\infty} h_{n_1+n_2+n_3+n_4}(a|q)\prod_{j=1}^{4} t_j^{n_j}(at_j)_\infty(t_j)_\infty/(q)_j.\end{aligned}$$

If we let $m = n_1 + n_2$, $n = n_3 + n_4$ and then use (5.1), we obtain the trilinear formula

$$(5.10)\quad \sum_{m,n=0}^{\infty} h_{m+n}(a|q)h_m(t_1/t_2|q)h_n(t_3/t_4|q)\frac{t_2^m t_4^n}{(q)_m(q)_n}$$

$$= {}_4\phi_3\left(\begin{matrix} t_1, t_2, t_3, t_4 \\ q/a, 0, 0 \end{matrix}; q, q\right)\Big/\left\{(a)_\infty \prod_{j=1}^{4} (t_j)_\infty\right\}$$

$$+ {}_4\phi_3\left(\begin{matrix} at_1, at_2, at_3, at_4 \\ aq, 0, 0 \end{matrix}; q, q\right)\Big/\left\{(1/a)_\infty \prod_{j=1}^{4} (at_j)_\infty\right\}.$$

An appropriate change of variables (see (5.2)) will make the left side of (5.10) a q-analogue of the left side of (1.17).

One may ask if it is reasonable for the sum of two ${}_4\phi_3$'s to replace the exponential function in (1.17). We now show that the special case of (5.10) which corresponds to Mehler's formula (2.2) does indeed work. If we put $t_1 = 0$ and then $t_2 = 0$ (5.10) becomes

$$(5.11)\quad \sum_{n=0}^{\infty} h_n(a|q)h_n(t_3/t_4|q)\frac{t_4^n}{(q)_n}$$

$$= {}_2\phi_1\left(\begin{matrix} t_3, t_4 \\ q/a \end{matrix}; q, q\right)/\{ (a)_\infty(t_3)_\infty(t_4)_\infty\}$$

$$+ {}_2\phi_1\left(\begin{matrix} at_3, at_4 \\ aq \end{matrix}; q, q\right)/\{ (1/a)_\infty(at_3)_\infty(at_4)_\infty\}$$

for $a < 0$ and $|t_j| < \min(1, -1/a)$. According to (2.2), this sum of ${}_2\phi_1$'s is a quotient of infinite products. A three-term relation for ${}_2\phi_1$'s due to Sears [**21**, Eq. (4.1)] implies that this sum is

$$\frac{(at_3t_4)_\infty(q/at_3t_4)_\infty(qt_3/t_4)_\infty}{(at_4)_\infty(q/t_4)_\infty(q/at_4)_\infty(t_4)_\infty(t_3)_\infty(at_3)_\infty}\,{}_2\phi_1\left(\begin{matrix} t_3, at_3 \\ qt_3/t_4 \end{matrix}; q, \frac{q}{at_3t_4}\right).$$

Then the q-analogue of Gauss's theorem for a ${}_2\phi_1$ implies that (5.11) is

$$(5.12)\quad \sum_{n=0}^{\infty} h_n(a|q)h_n(t_3/t_4|q)\frac{t_4^n}{(q)_n} = \frac{(at_3t_4)_\infty}{(at_4)_\infty(t_4)_\infty(t_3)_\infty(at_3)_\infty},$$

which is equivalent to (2.2). (A combinatorial proof of (5.12) appears in [**12**].) Similarly, Sears [**21**, Eq. (4.2)] implies that the right side of (5.10) is a sum of three ${}_4\phi_3$'s. However, we have been unable to show that as $q \to 1$ the right side of (1.17) results.

The argument for the trilinear formula (5.10) works for any number of factors (not just four) in (5.7). Put

$$(5.13)\quad H(t_1, \ldots, t_k, a) = {}_k\phi_{k-1}\left(\begin{matrix} t_1, \ldots, t_k \\ q/a, 0, \ldots, 0 \end{matrix}; q, q\right)\Big/\left\{(a)_\infty \prod_{j=1}^{k} (t_j)_\infty\right\}$$

https://doi.org/10.4153/CJM-1988-041-0 Published online by Cambridge University Press

$$+ {}_k\phi_{k-1}\left(\begin{matrix} at_1, \ldots, at_k \\ aq, 0, \ldots, 0 \end{matrix}; q, q\right) \Big/ \left\{(1/a)_\infty \prod_{j=1}^{k} (at_j)_\infty\right\}.$$

Then we find

$$(5.14) \quad \sum_{m_1,\ldots,m_k} h_{m_1+\ldots+m_k}(a|q) h_{m_1}(t_1/t_2|q) \ldots h_{m_k}(t_{2k-1}/t_{2k}|q) \prod_{j=1}^{k} \frac{t_{2j}^{m_j}}{(q)_{m_j}}$$

$$= H(t_1, \ldots, t_{2k}, a)$$

and

$$(5.15) \quad \sum_{\substack{m_1,\ldots,n_k \\ n}} h_{m_1+\ldots+m_k+n}(a|q) h_{m_1}(t_1/t_2|q) \ldots$$

$$h_{m_k}(t_{2k-1}/t_{2k}|q) \prod_{j=1}^{k} \frac{t_{2j}^{m_j}}{(q)_{m_j}} \cdot \frac{t_{2k+1}^n}{(q)_n}$$

$$= H(t_1, \ldots, t_{2k+1}, a).$$

We caution that (5.14) and (5.15) hold for a < 0 and $|t_i| < \min(1, -1/a)$, not as formal power series (as (2.2) does). A combinatorial proof of a formal power series q-analogue to (1.17) is given in [**12**].

Finally, we mention the q-analogue of

$$(5.16) \quad \sum_{n=0}^{\infty} H_{n+k}(x) \frac{t^n}{n!} = \exp(2xt - t^2) H_k(x - t),$$

which Carlitz used for his derivation of the multilinear formulas. It is

$$(5.17) \quad \sum_{n=0}^{\infty} h_{n+k}(x) \frac{t^n}{(q)_n} = \frac{1}{(xt)_\infty (t)_\infty} \sum_{j=0}^{k} \frac{(q)_k x^j}{(q)_j (q)_{k-j}} (t)_j.$$

Equation (5.17) is equivalent to Mehler's formula (2.2). This can be seen by multiplying (5.17) by $u^{n+k}/(q)_k$ and summing on k.

Appendix. Because of the interest in Mehler's formula, we shall indicate how to verify that, as $q \to 1$, the right side of (2.2) approaches the right side of (1.16).

We start with

$$(\text{A1}) \quad \lim_{q \to 1} H_n(\sqrt{1-q}\,x/2|q)/(1-q)^{n/2} = 2^{-n/2} H_n(x/\sqrt{2}).$$

For $\cos\theta = \sqrt{1-q}\,x/2$ and $\cos\phi = \sqrt{1-q}\,y/2$ in (2.2), as $q \to 1$ the left side of (2.2) approaches the left side of (1.16). The right side of R of (2.2) becomes (after the addition formula for $\cos(\theta + \phi)$ and $\cos(\theta - \phi)$)

https://doi.org/10.4153/CJM-1988-041-0 Published online by Cambridge University Press

$$\text{(A2)}\qquad R = (t^2)_\infty (t^2; q^2)_\infty^{-2} \prod_{n=0}^{\infty} \left[1 + \frac{b_n}{1 - 2t^2q^{2n} + t^4q^{4n}}\right]^{-1}$$

where

$$\text{(A3)}\qquad b_n = -tq^n(1-q)xy + t^2q^{2n}(1-q)(x^2+y^2) - t^3q^{3n}(1-q)xy.$$

Since

$$(t^2)_\infty (t^2; q^2)_\infty^{-2} = (t^2q; q^2)_\infty/(t^2; q^2)_\infty,$$

the q-binomial theorem (2.3) implies

$$\text{(A4)}\qquad \lim_{q\to 1} \frac{(t^2)_\infty}{(t^2; q^2)_\infty^2} = (1 - t^2)^{-1/2}$$

which is the first factor of (1.16).

For the exponential factor, note that

$$\text{(A5)}\qquad \log\left(\prod_{n=0}^{\infty}\left(1 + \frac{b_n}{1 - 2t^2q^{2n} + t^4q^{4n}}\right)^{-1}\right) = -\sum_{n=0}^{\infty} \frac{b_n}{1 - 2t^2q^{2n} + t^4q^{4n}} + O(1-q).$$

Thus, we must find the limit of three terms:

$$\text{(A6)}\qquad txy(1-q)\sum_{n=0}^{\infty} \frac{q^n}{1 - 2t^2q^{2n} + t^4q^{4n}} = T_1$$

$$\text{(A7)}\qquad -t^2(x^2+y^2)(1-q)\sum_{n=0}^{\infty} \frac{q^{2n}}{1 - 2t^2q^{2n} + t^4q^{4n}} = T_2$$

and

$$\text{(A8)}\qquad t^3xy(1-q)\sum_{n=0}^{\infty} \frac{q^{3n}}{1 - 2t^2q^{2n} + t^4q^{4n}} = T_3.$$

Each of these three items is a q-integral (see **[2]**), so if $q \to 1$

$$\text{(A6)}'\qquad T_1 \to txy \int_0^1 \frac{dx}{1 - 2t^2x^2 + t^4x^4}$$

$$\text{(A7)}'\qquad T_2 \to -t^2(x^2+y^2) \int_0^1 \frac{xdx}{1 - 2t^2x^2 + t^4x^4}$$

https://doi.org/10.4153/CJM-1988-041-0 Published online by Cambridge University Press

$$\text{(A8)}' \quad T_3 \to t^3xy \int_0^1 \frac{x^2dx}{1 - 2t^2x^2 + t^4x^4}.$$

Clearly

$$T_2 = -t^2(x^2 + y^2)/2(1 - t^2) \quad \text{and} \quad T_1 + T_3 = xyt/(1 - t^2)$$

are the arguments of the exponential function in (1.16).

Acknowledgements. We thank Richard Askey for his help, encouragement, and interest in this work. We also thank the referee for pointing out that the results of Section 4 are special cases of results in Nassrallah and Rahman [**15**]. We are grateful to Dominique Foata for his very useful comments on the first draft of this work.

References

1. W. Al-Salam and L. Carlitz, *Some orthogonal q-polynomials*, Math. Nachr. *30* (1965), 47-61.

2. G. Andrews and R. A. Askey, *Classical orthogonal polynomials*, to appear.

3. R. A. Askey, *An elementary evaluation of a beta type integral*, Indian J. Pure Appl. Math. *14* (1983), 892-895.

4. R. A. Askey and M. E. H. Ismail, *The Rogers q-ultraspherical polynomials*, Approximation Theory III (Academic Press, New York, 1980), 175-182.

5. ——— *A generalization of ultraspherical polynomials*, Studies in Pure Mathematics (Birkhauser, Basel, 1983).

6. R. A. Askey and J. A. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs Amer. Math. Soc. *319* (1985), to appear.

7. D. Bressoud, *Linearization and related formulas for q-ultraspherical polynomials*, SIAM J. Math. Anal. *12* (1981), 161-168.

8. L. Carlitz, *Some extensions of the Mehler formula*, Collectanea Mathematica *21* (1970), 117-130.

9. T. Chihara, *An introduction to orthogonal polynomials* (Gordon and Breach, New York, 1978).

10. D. Foata, *Some Hermite polynomial identities and their combinatorics*, Advances Appl. Math. *2* (1981), 250-259.

11. M. E. H. Ismail, *A queueing model and a set of orthogonal polynomials*, J. Math. Anal. Appl. *130* (1985), to appear.

12. M. E. H. Ismail, D. Stanton and G. Viennot, *The combinatorics of the q-Hermite polynomials and the Askey-Wilson integral*, Eur. J. Comb. *8* (1987), 379-392.

13. W. F. Kibble, *An extension of theorem of Mehler on Hermite polynomials*, Proc. Cambridge Philos. Soc. *41* (1945), 12-15.

14. J. D. Louck, *Extension of the Kibble-Slepian formula for Hermite polynomials using Boson operator methods*, Advances in Appl. Math. *2* (1981), 239-249.

15. B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials*, SIAM J. Math. Anal. *16* (1985), 186-197.

16. M. Rahman, *A simple evaluation of Askey and Wilson's q-integral*, Proc. Amer. Math. Soc. *92* (1984), 413-417.

17. E. D. Rainville, *Special functions* (Chelsea Publishing Co., Bronx, New York, 1971).

18. L. J. Rogers, *On the expansion of some infinite products*, Proc. London Math. Soc. *24* (1893), 337-352.

19. ——— *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. *25* (1894), 318-343.

20. ——— *Third memoir on the expansion of certain infinite products*, Proc. London Math. Soc. *26* (1895), 15-32.

21. D. Sears, *Transformations of basic hypergeometric functions of any order*, Proc. London Math. Soc. *53* (1951), 181-191.

22. D. Slepian, *On the symmetrized Kronecker power of a matrix and extensions of Mehler's formula for Hermite polynomials*, SIAM J. Math. Anal. *3* (1972), 606-616.

23. G. Szegö, *Ein Beitrag zur theorie der Thetafunktionen*, Sitz. Preuss. Akad. Wiss. Phys. Math. K1., XIX (1926), 242-252, Collected Papers (Birkhauser, Boston, 1982), 795-805.

24. ——— *Orthogonal polynomials*, Fourth Edition, Amer. Math. Soc. Colloquium Publications *23* (Amer. Math. Soc., Providence, R.I., 1975).

25. F. G. Tricomi, *Integral equations* (Interscience, New York, 1957).

Arizona State University,
Tempe, Arizona;
University of Minnesota,
Minneapolis, Minnesota